



AgEcon SEARCH

RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

The World's Largest Open Access Agricultural & Applied Economics Digital Library

This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.

Help ensure our sustainability.

Give to AgEcon Search

AgEcon Search

<http://ageconsearch.umn.edu>

aesearch@umn.edu

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

No endorsement of AgEcon Search or its fundraising activities by the author(s) of the following work or their employer(s) is intended or implied.

AMSTER

N5/84

FACULTY OF
ACTUARIAL SCIENCE
&
ECONOMETRICS

GIANNINI FOUNDATION OF
AGRICULTURAL ECONOMICS
LIBRARY

OCT 10 1985

A & E NOTE

Note N5/84

*The dispersion matrix of $\text{vec } X'AX$, $A'=A$,
when $X'X$ is a Wishart matrix*

H. Neudecker



University of Amsterdam

The dispersion matrix of vec X'AX, A' = A, when X'X is a Wishart matrix

H. Neudecker
Department of Econometrics
University of Amsterdam
Amsterdam, The Netherlands

ABSTRACT

Recently Magnus and Neudecker [3] presented the dispersion matrix of $\text{vec } X'X$, when $X'X$ is a Wishart matrix.

This note is concerned with the matrix quadratic form $X'AX$, $A' = A$, where $X'X$ is a Wishart matrix.

It is not necessary to require A to be idempotent. Cf. Kshirsagar [2, p. 76]. The dispersion matrix of $\text{vec } X'AX$ will then be derived by applying some earlier results of Magnus and Neudecker [3] and Neudecker and Wansbeek [4].

1. INTRODUCTION

Let x_i for $i = 1, \dots, n$ be $p \times 1$ vectors that are jointly independent with densities $N_p(\mu_i, V)$. If we define $X' := (x_1, \dots, x_n)$, then $S := X'X$ is called a Wishart matrix.

The matrix S is said to have density $W_p(n, V, M)$, where $M' := (\mu_1, \dots, \mu_n)$. Magnus and Neudecker [3] presented the dispersion matrix of $\text{vec } S$, viz.

$$D(\text{vec } S) = (I_{p^2} + K_{pp})[n(V \otimes V) + V \otimes M'M + M'M \otimes V], \quad (1.1)$$

where K_{pp} is a commutation matrix.

Although V was taken to be positive definite in their derivation, it is easy to prove that the result generally holds for nonnegative definite V . In this paper we shall consider the matrix quadratic form $S_1 := X'AX$, where $X'X$ is a Wishart matrix and A is symmetric.

We shall apply some earlier results concerning the Kronecker product and

the commutation matrix:

$$(1) \quad \text{vec } ABC = (C' \otimes A) \text{vec } B, \text{ for compatible matrices } A, B \text{ and } C \quad (1.1)$$

$$(2) \quad K_{mn} \text{vec } A = \text{vec } A', \text{ where } A \text{ is an } m \times n \text{ matrix} \quad (1.2)$$

$$(3) \quad K_{pm} (A \otimes B) K_{nq} = B \otimes A, \text{ where } A \text{ and } B \text{ are } m \times n \text{ and } p \times q \text{ matrices} \quad (1.3)$$

$$(4) \quad (\text{vec } A)' \text{vec } B = \text{tr } A'B. \quad (1.4)$$

These results are collected in Magnus and Neudecker [3].

$$(5) \quad \text{vec}(A \otimes B) = (I_n \otimes K_{qm} \otimes I_p) (\text{vec } A \otimes \text{vec } B), \text{ where } A \text{ and } B \text{ are arbitrary } m \times n \text{ and } p \times q \text{ matrices.} \quad (1.5)$$

This is Theorem 3.1(i) of Neudecker and Wansbeek [4].

$$(6) \quad \mathcal{D}(\text{vec}(X \otimes X)) = (I_{m^2n^2} + K_{nn} \otimes K_{mm}) (I_n \otimes K_{nm} \otimes I_m) \times \\ [V \otimes V + V \otimes \text{vec } M(\text{vec } M)' + \text{vec } M(\text{vec } M)' \otimes V] (I_n \otimes K_{mn} \otimes I_m), \quad (1.6)$$

when X is an $m \times n$ matrix and $\text{vec } X$ has density $N_{mn}(\text{vec } M, V)$.

This is application 3 of Neudecker and Wansbeek [4].

The result will be reached in stages. First an intermediate result will be derived.

2. AN INTERMEDIATE RESULT

LEMMA.

Let us have the independent $p \times 1$ vectors x_i for $i = 1, \dots, n$, each with density $N_p(0, V)$. We define $X' := (x_1', \dots, x_n')$. Then $S^* := X'X$ is said to be a Wishart matrix with density $W_p(n, V, 0)$.

Consider the matrix quadratic form

$$S_1^* := X'AX,$$

where A is symmetric. Then

$$\mathcal{D}(\text{vec } S_1^*) = (\text{tr } A^2) (I_{p^2} + K_{pp}) (V \otimes V). \quad (2.1)$$

Proof. We write

$$\text{vec } S_1^* = \text{vec } X'AX = (X' \otimes X') \text{vec } A \quad (2.2)$$

$$= \text{vec}[I_{p^2}(X' \otimes X')\text{vec } A] = (\text{vec } A \otimes I_{p^2})' \text{vec}(X' \otimes X') , \quad (2.3)$$

by means of (1.1).

Using (1.6), (1.2), (1.3) and (1.4), we get

$$\begin{aligned} \mathcal{D}(\text{vec } S_1^*) &= (\text{vec } A \otimes I_{p^2})'(I_{n^2 p^2} + K_{nn} \otimes K_{pp})(I_n \otimes K_{np} \otimes I_p) \times \\ &\quad (I_n \otimes V \otimes I_n \otimes V)(I_n \otimes K_{np} \otimes I_p)(\text{vec } A \otimes I_{p^2}) \end{aligned} \quad (2.4)$$

$$= [\text{vec } A \otimes (I_{p^2} + K_{pp})]'(I_{n^2} \otimes V \otimes V)(\text{vec } A \otimes I_{p^2}) \quad (2.5)$$

$$= (\text{vec } A)' \text{vec } A \cdot (I_{p^2} + K_{pp})(V \otimes V) \quad (2.6)$$

$$= (\text{tr } A^2)(I_{p^2} + K_{pp})(V \otimes V) . \quad \blacksquare \quad (2.7)$$

3. THE MAIN RESULT

THEOREM.

Let us have the independent $p \times 1$ vectors x_i for $i = 1, \dots, n$ with densities $N_p(\mu_i, V)$. We define $X' := (x_1, \dots, x_n)$ and $M' := (\mu_1, \dots, \mu_n)$. Then $S := X'X$ is said to be a Wishart matrix with density $W_p(n, V, M)$. Consider the matrix quadratic form

$$S_1 := X'AX ,$$

where A is symmetric. Then

$$\mathcal{D}(\text{vec } S_1) = (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + M'A^2 M \otimes V + V \otimes M'A^2 M] . \quad (3.1)$$

Proof. We write $Y := X - M$. Hence

$$S_1 = (Y + M)'A(Y + M) \quad (3.2)$$

$$= Y'AY + Y'AM + M'AY + M'AM , \quad (3.3)$$

$$\begin{aligned} \text{and } \text{vec } S_1 &= (\text{vec } A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(\text{vec } Y' \otimes \text{vec } Y') + \\ &\quad + (I_{p^2} + K_{pp})(M'A \otimes I_p)\text{vec } Y' + \text{vec } M'AM . \end{aligned} \quad (3.4)$$

We used (1.1), (1.2), (1.3) and (1.5).

By virtue of a result of Anderson [1, p. 39] we can write:

$$\mathcal{D}(\text{vec } S_1) = \mathcal{D}(\text{vec } Y'AY) + \mathcal{D}\{\text{vec}(Y'AM + M'AY)\} \quad (3.5)$$

$$\begin{aligned} &= (\text{tr } A^2)(I_{p^2} + K_{pp})(V \otimes V) \\ &+ (I_{p^2} + K_{pp})(M'A \otimes I_p)(I \otimes V)(AM \otimes I_p)(I_{p^2} + K_{pp}) \end{aligned} \quad (3.6)$$

$$= (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + (M'A^2 M \otimes V)(I_{p^2} + K_{pp})] , \quad (3.7)$$

where we exploited the fact that $\text{vec } Y'$ has density $N_{np}(0, I \otimes V)$,

$$= (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + M'A^2M \otimes V + V \otimes M'A^2M], \quad (3.8)$$

by virtue of (2.1), (1.2) and (1.3). \square

We are very grateful to Risto Heijmans, George Styan and unknown referees whose suggestions and criticisms have undoubtedly led to a nicer paper.

4. COMMENT

When A is idempotent of rank r say, then formula (3.1) will be simplified to

$$\mathcal{D}(\text{vec } S_1) = (I_{p^2} + K_{pp})[r(V \otimes V) + M'AM \otimes V + V \otimes M'AM]. \quad (4.1)$$

The result of Magnus and Neudecker (1979) is a special case of (4.1) for $r = n$.

Another special case is $A = N$, $N := I_n - \frac{1}{n} s_n s_n'$, where $s_n' := (1, \dots, 1)$ is the n -dimensional summation vector. We then find

$$\mathcal{D}(\text{vec } S_1) = (I_{p^2} + K_{pp})[(n-1)(V \otimes V) + M'NM \otimes V + V \otimes M'NM] \quad (4.2)$$

REFERENCES

- 1 T.W. Anderson, *An Introduction to Multivariate Statistical Analysis*. John Wiley & Sons, New York, 1958
- 2 A.M. Kshirsagar, *Multivariate Analysis*. Marcel Dekker, New York, 1972.
- 3 J.R. Magnus and H. Neudecker, The commutation matrix: some properties and applications. *Ann. Statist.* 7:381 - 394 (1979).
4. H. Neudecker and T.J. Wansbeek, Some results on commutation matrices with statistical applications. *Can. J. Stat.* 11:221 - 231 (1983).

