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A & E NOTE

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The dispersion matrix of vec X'AX, A'=A, when X'X is a Wishart matrix

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ABSTRACT

Recently Magnus and Neudecker [3] presented the dispersion matrix of vec X'X, when X'X is a Wishart matrix.
This note is concerned with the matrix quadratic form X'AX, A' = A, where X'X is a Wishart matrix.
It is not necessary to require A to be idempotent. Cf. Kshirsagar [2, p. 76]. The dispersion matrix of vec X'AX will then be derived by applying some earlier results of Magnus and Neudecker [3] and Neudecker and Wansbeek [4].

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1. INTRODUCTION

Let x_i for $i = 1,\dots,n$ be $p \times 1$ vectors that are jointly independent with densities $N_p(\mu_i, V)$. If we define $X' := (x_1,\dots,x_n)$, then $S := X'X$ is called a Wishart matrix.
The matrix S is said to have density $W_p(n,V,M)$, where $M' := (\mu_1,\dots,\mu_n)$.
Magnus and Neudecker [3] presented the dispersion matrix of vec S, viz.

$$\mathcal{D}(\text{vec } S) = (I_{p^2} + K_{pp})[n(V \otimes V) + V \otimes M'M + M'M \otimes V] , \tag{1.1}$$

where K_{pp} is a commutation matrix.
Although V was taken to be positive definite in their derivation, it is easy to prove that the result generally holds for nonnegative definite V.
In this paper we shall consider the matrix quadratic form $S_1 := X'AX$, where $X'X$ is a Wishart matrix and A is symmetric.
We shall apply some earlier results concerning the Kronecker product and

the commutation matrix:

(1) $\text{vec}\, ABC = (C' \otimes A)\text{vec}\, B$, for compatible matrices A, B and C (1.1)

(2) $K_{mn}\, \text{vec}\, A = \text{vec}\, A'$, where A is an mxn matrix (1.2)

(3) $K_{pm}(A \otimes B)K_{nq} = B \otimes A$, where A and B are mxn and pxq matrices (1.3)

(4) $(\text{vec}\, A)'\, \text{vec}\, B = \text{tr}\, A'B$. (1.4)

These results are collected in Magnus and Neudecker [3].

(5) $\text{vec}(A \otimes B) = (I_n \otimes K_{qm} \otimes I_p)(\text{vec}\, A \otimes \text{vec}\, B)$, where A and B are arbitrary mxn and pxq matrices. (1.5)

This is Theorem 3.1(i) of Neudecker and Wansbeek [4].

(6) $\mathcal{D}(\text{vec}(X \otimes X)) = (I_{m^2n^2} + K_{nn} \otimes K_{mm})(I_n \otimes K_{nm} \otimes I_m)\times$
$[V \otimes V + V \otimes \text{vec}\, M(\text{vec}\, M)' + \text{vec}\, M(\text{vec}\, M)'\otimes V](I_n \otimes K_{mn} \otimes I_m)$, (1.6)
when X is an mxn matrix and $\text{vec}\, X$ has density $N_{mn}(\text{vec}\, M, V)$.

This is application 3 of Neudecker and Wansbeek [4].

The result will be reached in stages. First an intermediate result will be derived.

2. AN INTERMEDIATE RESULT

LEMMA.

Let us have the independent px1 vectors x_i for $i = 1,\dots,n$, each with density $N_p(0,V)$. We define $X' := (x_1,\dots,x_n)$. Then $S^* := X'X$ is said to be a Wishart matrix with density $W_p(n,V,O)$.
Consider the matrix quadratic form

$$S_1^* := X'AX ,$$

where A is symmetric. Then

$$\mathcal{D}(\text{vec}\, S_1^*) = (\text{tr}\, A^2)(I_{p^2} + K_{pp})(V \otimes V). \quad (2.1)$$

Proof. We write

$$\text{vec}\, S_1^* = \text{vec}\, X'AX = (X' \otimes X')\text{vec}\, A \quad (2.2)$$

$$= \text{vec}[I_{p^2}(X' \otimes X')\text{vec } A] = (\text{vec } A \otimes I_{p^2})' \text{ vec}(X' \otimes X') \ , \tag{2.3}$$

by means of (1.1).

Using (1.6), (1.2), (1.3) and (1.4), we get

$$\mathcal{D}(\text{vec } S_1^*) = (\text{vec } A \otimes I_{p^2})'(I_{n^2p^2} + K_{nn} \otimes K_{pp})(I_n \otimes K_{np} \otimes I_p) \times$$

$$(I_n \otimes V \otimes I_n \otimes V)(I_n \otimes K_{np} \otimes I_p)(\text{vec } A \otimes I_{p^2}) \tag{2.4}$$

$$= [\text{vec } A \otimes (I_{p^2} + K_{pp})]'(I_{n^2} \otimes V \otimes V)(\text{vec } A \otimes I_{p^2}) \tag{2.5}$$

$$= (\text{vec } A)' \text{ vec } A \ . \ (I_{p^2} + K_{pp})(V \otimes V) \tag{2.6}$$

$$= (\text{tr } A^2)(I_{p^2} + K_{pp})(V \otimes V) \ . \quad \blacksquare \tag{2.7}$$

3. THE MAIN RESULT

THEOREM.

Let us have the independent px1 vectors x_i for $i = 1,\dots,n$ with densities $N_p(\mu_i,V)$. We define $X' := (x_1,\dots,x_n)$ and $M' := (\mu_1,\dots,\mu_n)$. Then $S := X'X$ is said to be a Wishart matrix with density $W_p(n,V,M)$. Consider the matrix quadratic form

$$S_1 := X'AX \ ,$$

where A is symmetric. Then

$$\mathcal{D}(\text{vec } S_1) = (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + M'A^2M \otimes V + V \otimes M'A^2M] . \tag{3.1}$$

Proof. We write $Y := X - M$. Hence

$$S_1 = (Y + M)'A(Y + M) \tag{3.2}$$

$$= Y'AY + Y'AM + M'AY + M'AM \ , \tag{3.3}$$

and $\text{vec } S_1 = (\text{vec } A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(\text{vec } Y' \otimes \text{vec } Y') +$

$$+ (I_{p^2} + K_{pp})(M'A \otimes I_p)\text{vec } Y' + \text{vec } M'AM \ . \tag{3.4}$$

We used (1.1), (1.2), (1.3) and (1.5).

By virtue of a result of Anderson [1, p. 39] we can write:

$$\mathcal{D}(\text{vec } S_1) = \mathcal{D}(\text{vec } Y'AY) + D\{\text{vec}(Y'AM + M'AY)\} \tag{3.5}$$

$$= (\text{tr } A^2)(I_{p^2} + K_{pp})(V \otimes V)$$

$$+ (I_{p^2} + K_{pp})(M'A \otimes I_p)(I \otimes V)(AM \otimes I_p)(I_{p^2} + K_{pp}) \tag{3.6}$$

$$= (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + (M'A^2M \otimes V)(I_{p^2} + K_{pp}) \ , \tag{3.7}$$

where we exploited the fact that $\text{vec } Y'$ has density $N_{np}(0,I \otimes V)$,

$$= (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + M'A^2M \otimes V + V \otimes M'A^2M] , \qquad (3.8)$$

by virtue of (2.1), (1.2) and (1.3). □

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4. COMMENT

When A is idempotent of rank r say, then formula (3.1) will be simplified to

$$\mathcal{D}(\text{vec } S_1) = (I_{p^2} + K_{pp})[r(V \otimes V) + M'AM \otimes V + V \otimes M'AM] . \qquad (4.1)$$

The result of Magnus and Neudecker (1979) is a special case of (4.1) for $r = n$.

Another special case is $A = N$, $N := I_n - \frac{1}{n} s_n s_n'$, where $s_n' := (1,\ldots,1)$ is the n-dimensional summation vector. We then find

$$\mathcal{D}(\text{vec } S_1) = (I_{p^2} + K_{pp})[(n-1)(V \otimes V) + M'NM \otimes V + V \otimes M'NM] \qquad (4.2)$$

REFERENCES

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