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Full Research Article

## Positive Mathematical Programming and Risk Analysis

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**Abstract.** In 1956, Freund introduced the analysis of agricultural price risk in a mathematical programming framework. His discussion admitted only constant absolute risk aversion. This paper generalizes the treatment of risk preference in a mathematical programming approach along the lines suggested by Meyer (1987) who demonstrated the equivalence of expected utility of wealth and a function of mean and standard deviation of wealth for a wide class of probability distributions that differ only by location and scale. This paper extends the definition of calibration under Positive Mathematical Programming (PMP) by considering limiting input prices along with the traditional decision variables. Furthermore, it shows how to formulate an analytical specification for the estimation of the risk preference parameters and calibrates the model to the base data within small deviations. The PMP approach under generalized risk allows also the estimation of output supply elasticities and the response analysis of decoupled farm subsidies that recently has interested policy makers. The approach is applied to a sample of farms that do not produce all the sample commodities.

**Keywords.** Risk analysis, positive mathematical programming, model calibration, chance constraint, policy analysis.

**JEL.** C6.

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### 1. Introduction

This paper accomplishes several objectives:

1. It presents a procedure to estimate generalized risk preferences in combination with Positive Mathematical Programming (PMP).
2. It obtains a unique calibrating solution of a PMP model even with a sample of farms that produce zero levels of some crops.
3. It estimates a complete cost function that can be used in a calibrating model for policy analysis.
4. It shows that Phase I and Phase II of the classical PMP procedure give identical and unique results.
5. It shows how to incorporate exogenously given supply elasticities.

6. It extends the meaning of calibration in PMP by minimizing the distance of optimal solutions from observed output levels and limiting input prices. In this way, it dispenses from the necessity of a user-determined parameter that was originally introduced to guarantee a positive shadow price of binding constraints.

The treatment of agricultural price risk in a mathematical programming framework has dealt mainly with either an exponential utility function and constant absolute risk aversion (CARA) or the minimization of total absolute deviation (MOTAD) of income. The first approach, originally proposed by Freund (1956), appealed to the expected utility (EU) hypothesis and assumed that random prices were normally distributed. These assumptions lead to a mean-variance specification of the certainty equivalent (CE) defined as total expected revenue minus a risk premium. Such a premium corresponds to half the variance of revenue multiplied by a constant absolute risk aversion coefficient. The MOTAD approach was proposed by Hazell (1971) who justified its introduction with the difficult access – at that time – to a quadratic programming computer software necessary to solve a mean-variance model. According to Hazell (1971, p. 56), the MOTAD specification “has an important advantage over the mean-variance criterion in that it leads to a linear programming model in deriving the efficient mean-absolute deviation farm plans.” The MOTAD model approximates a mean-standard deviation (MS) criterion but it says nothing about the economic agent’s risk preference with regard to either decreasing (constant, increasing) absolute or relative risk aversion.

Recently, Cortignani and Severini (2012), Arata *et al.* (2017) and Paris (2018) have combined PMP with a CARA specification of risk preferences. It is difficult, however, to accept the idea that farmers risk behavior does not account for changes in wealth as the CARA approach stipulates. Petsakos and Rozakis (2015) have presented a combination of the traditional PMP specification with a decreasing absolute risk aversion (DARA) parameter. The present paper combines a more encompassing specification of PMP (calibration of output quantities and limiting input prices) with generalized risk preferences where the behavior of the risk-averse farmer can vary over all theoretically possible preferences (CARA, DARA, IARA, constant, decreasing and increasing relative risk aversion). The paper deals with market price risk leaving the treatment of production risk for further research.

The mean-standard deviation approach has a long history [Fisher (1906), Hicks (1933), Tintner (1941), Markowitz (1952), Tobin (1958)]. Meyer (1987) presented a reconciliation between the EU and the MS approaches that may be fruitfully applied in a positive mathematical programming (PMP) analysis of economic behavior under risk. The main objective of Meyer was to find consistency conditions between the EU and the MS approaches in such a way that an agent who ranks the available alternatives according to the value of some function defined over the first two moments of the random payoff would rank those alternatives in the same way by means of the expected value of some utility function defined over the same payoffs. It turns out that the location and scale condition is the crucial link to establish the consistency between the EU and the MS approaches. We reproduce here Meyer’s argument (1987, p. 423):

“Assume a choice set in which all random variables  $Y_i$  (with finite means and variances) differ from one another only by location and scale parameters. Let  $X$  be the random variable obtained from one of the  $Y_i$  using the normalizing transformation  $X_i =$

$(Y_i - \mu_i)/\sigma_i$  where  $\mu_i$  and  $\sigma_i$  are the mean and standard deviation of  $Y_i$ . All  $Y_i$ , no matter which was selected to define  $X$ , are equal *in distribution* to  $\mu_i + \sigma_i X$ . Hence, the expected utility from  $Y_i$  for any agent with utility function  $u(\cdot)$  can be written as

$$EU(Y_i) = \int_a^b u(\mu_i + \sigma_i x) dF(x) \equiv V(\mu_i, \sigma_i) \quad (1)$$

where  $a$  and  $b$  define the interval containing the support of the normalized random variable  $X$ . "... under the location and scale condition, various popular and interesting hypotheses concerning absolute and relative risk-aversion measures in the EU setting can be translated into equivalent properties concerning  $V(\mu, \sigma)$ ." Given the assumptions made by Meyer about first and second derivatives,  $V(\mu, \sigma)$  is a concave function of  $\mu$  and  $\sigma$ . Concavity is established when second derivatives  $V_{\mu\mu}$  and  $V_{\sigma\sigma}$  are non-positive and  $V_{\mu\mu}V_{\sigma\sigma} - V_{\mu\sigma}^2 \geq 0$ .

The structure of absolute risk (AR) is measured by the slope of the indifference curves in the  $(\mu, \sigma)$  space that is represented as

$$AR(\mu, \sigma) = \frac{-V_{\sigma}(\mu, \sigma)}{V_{\mu}(\mu, \sigma)} \quad (2)$$

where  $V_{\mu}(\mu, \sigma)$  and  $V_{\sigma}(\mu, \sigma)$  are first partial derivatives of the  $V(\mu, \sigma)$  function. Some properties of this risk measure are:

1. Risk aversion is associated with  $AR(\mu, \sigma) > 0$ , risk neutrality with  $AR(\mu, \sigma) = 0$  and risk propensity with  $AR(\mu, \sigma) < 0$ .
2. If  $u(\mu + \sigma x)$  displays decreasing (constant, increasing) absolute risk aversion for all  $\mu + \sigma x$ , then

$$\frac{\partial AR(\mu, \sigma)}{\partial \mu} < (=, >) 0 \text{ for all } \mu \text{ and } \sigma > 0.$$

3. If  $u(\mu + \sigma x)$  displays increasing (constant, decreasing) relative risk aversion for all  $\mu + \sigma x$ , then

$$\frac{\partial AR(t\mu, t\sigma)}{\partial t} > (=, <) 0 \text{ for } t > 0.$$

Saha (1997) proposed a two-parameter MS utility function that conforms to Meyer's specification:

$$V(\mu, \sigma) = \mu^{\theta} - \sigma^{\gamma} \quad (3)$$

and assumed that  $\theta > 0$ . According to this MS utility function, the absolute risk measure (AR) is specified as

$$AR(\mu, \sigma) = \frac{-V_{\sigma}(\mu, \sigma)}{V_{\mu}(\mu, \sigma)} = \frac{\gamma}{\theta} \mu^{(1-\theta)} \sigma^{(\gamma-1)}. \quad (4)$$

Hence, risk aversion, risk neutrality and risk propensity are specified by  $\gamma > 0$ ,  $\gamma = 0$  and  $\gamma < 0$ , respectively. As economic agents do not, in general, operate directly upon expected wealth and its standard deviation but, rather, upon a string of decision variables such as output and input levels, it is important to analyze the behavior of the absolute risk measure (AR) under risk aversion and risk propensity. The justification for this requirement is due to the fact that knowledge of parameters  $\theta$  and  $\gamma$  is obtained only by empirical estimation of economic relations involving entrepreneur's decisions. The sign of these parameters, therefore, is an empirical question.

For  $\gamma > 0$ , (risk aversion), decreasing, constant and increasing absolute risk aversion is defined by

$$\frac{\partial AR(\mu, \sigma)}{\partial \mu} = \frac{(1-\theta)\gamma}{\theta} \mu^{-\theta} \sigma^{(\gamma-1)} < (=, >) 0 \quad (5)$$

and, therefore, by  $\theta > 1$ ,  $\theta = 1$ ,  $\theta < 1$ , respectively. For  $\gamma > 0$ , (risk propensity), decreasing, constant and increasing absolute risk propensity is defined by  $\theta < 1$ ,  $\theta = 1$ ,  $\theta > 1$ , respectively.

For  $\gamma > 0$ , (risk aversion), decreasing, constant and increasing relative risk aversion is defined by

$$\left. \frac{\partial AR(t\mu, t\sigma)}{\partial t} \right|_{t=1} = (\gamma - \theta)AR < (=, >) 0 \quad (6)$$

and, therefore, by  $\theta > \gamma$ ,  $\theta = \gamma$ ,  $\theta < \gamma$  respectively. For  $\gamma < 0$ , (risk propensity), neither decreasing nor constant relative risk propensity are applicable because the combination of parameters' signs produces always a positive derivative. Increasing relative risk propensity is defined by any value of  $\theta > 0$ .

The meaning of decreasing absolute risk aversion relates to an economic agent who experiences a wealth increase and chooses to augment his investment – measured in absolute terms – in the risky asset. Decreasing relative risk aversion relates to an economic agent who experiences a wealth increase and chooses to increase the share of his investment in the risky asset. It is possible, therefore, for an economic agent to behave according to a decreasing absolute risk aversion framework and an increasing relative risk aversion scenario if the absolute amount of increase in the risky asset is not sufficient to increase also the share of that asset. In any given sample of economic agents' performances, therefore, the prevailing combination of risk preference is an empirical question. The risk analysis of Meyer (1987) admits all possible combinations of risk behavior (risk aversion and risk propensity). Saha (1997) listed the risk aversion combinations for the MS utility function specified in relation (3) when  $\gamma > 0$ . Table 1, for example, admits absolute risk aversion behavior that may be decreasing, when  $\theta > 1$  and  $\gamma > 0$ , in association with either increasing relative risk aversion when  $\gamma > \theta > 0$  or decreasing relative risk aversion when  $\theta > \gamma$ . Decreasing, constant and increasing absolute risk aversion are denoted by DARA, CARA and IARA, respectively. Decreasing, constant and increasing relative risk aversion are denoted by DRRA, CRRA and IRRA, respectively.

**Table 1.** Possible risk preferences under risk aversion ( $\theta > 0, \gamma > 0$ )

	DRRA	CRRA	IRRA
DARA	$\theta > 1, \theta > \gamma$	$\theta > 1, \theta = \gamma$	$\theta > 1, \theta < \gamma$
CARA	$\theta = 1, \theta > \gamma$	$\theta = 1, \theta = \gamma$	$\theta = 1, \theta < \gamma$
IARA	$\theta < 1, \theta > \gamma$	$\theta < 1, \theta = \gamma$	$\theta < 1, \theta < \gamma$

**Table 2.** Possible risk preferences under risk propensity ( $\theta > 0, \gamma < 0$ )

	DRRP	CRRP	IRRP
DARP	$\theta < 1, NA$	$\theta < 1, NA$	$\theta < 1, YES$
CARP	$\theta = 1, NA$	$\theta = 1, NA$	$\theta = 1, YES$
IARP	$\theta > 1, NA$	$\theta > 1, NA$	$\theta > 1, YES$

"NA" stands for "Not Applicable" because the combination of parameters' signs produces always a positive value of the derivative (6).

When  $\theta > 0$  and  $\gamma < 0$ , risk propensity is active and the behavior of the risk measure  $AR$ , under the given MS utility, assumes the specification reported in Table 2. Decreasing, constant and increasing absolute risk propensity are denoted by DARP, CARP and IARP, respectively. Decreasing, constant and increasing relative risk propensity are denoted by DRRP, CRRP and IRRP, respectively.

The  $V(\mu, \sigma) = \mu^\theta - \sigma^\gamma$  function is concave with respect to  $\mu$  and  $\sigma$  when  $\theta < 1$  and  $\gamma > 1$ . The same function  $V[\mu(\mathbf{x}), \sigma(\mathbf{x})] = \mu(\mathbf{x})^\theta - \sigma(\mathbf{x})^\gamma$ , however, exhibits a flexible behavior with respect to entrepreneur's decisions,  $\mathbf{x}$ . This behavior depends on the relative values of parameters  $\theta$  and  $\gamma$ . In other words, the upper contour sets of  $V[\mu(\mathbf{x}), \sigma(\mathbf{x})] = \mu(\mathbf{x})^\theta - \sigma(\mathbf{x})^\gamma$  are convex for a wide range of values of parameters  $\theta$  and  $\gamma$ . A few examples illustrate the function's graph and the associated upper contour sets in the appendix.

The rest of the paper is organized as follows. Section 2 discusses a PMP model that combines a generalized risk analysis with an extension of calibration constraints involving observed prices of limiting inputs. This extension integrates the traditional PMP specification of calibration constraints dealing only with observed levels of realized outputs. In particular, the extension provides a unique estimate of the optimal decision variables and avoids the user-determined perturbation parameters introduced by Howitt (1995a, 1995b) to guarantee that the dual variables of binding structural constraints will assume positive values. Section 3 discusses a chance-constrained relation that anchors the  $\theta$  and  $\gamma$  parameters to the decision quantities and, therefore, provides an independent relation for their estimation. Section 4 assembles a Phase-I estimation model of the novel PMP approach. Section 5 defines and estimates a complete cost function involving output quantities and limiting input prices. The derivatives of the cost function are used in calibrating models that are suitable for policy analysis. Section 6 discusses how to obtain endogenous (to a farm sample) output supply elasticities. This section matches exogenous (to the farm sam-

ple) supply elasticities (available through econometric estimation, for example) with the endogenous supply elasticities. Section 7 states that optimal decision variables are identical whether estimated as solution of the Phase I model or solution of Phase I and Phase II models combined. Section 8 defines two alternative calibrating equilibrium models which reproduce calibrating solutions that are identical to those ones obtained in section 4. Section 9 presents the empirical results of the more elaborate PMP and risky model applied to a sample of 14 farms when not all farms produce all commodities. Conclusions follow.

## 2. Generalized Risk Preference in a PMP Framework

A Positive Mathematical Programming approach has been adopted frequently to analyze agricultural policy scenarios ever since Howitt proposed the methodology (1995a, 1995b). In this section, we extend the PMP methodology to deal with generalized risk preference and risky market output prices. Furthermore, we extend the PMP methodology to deal with calibration constraints involving observed prices of limiting inputs, say land. This extension modifies the traditional specification of calibration constraints and the notion of calibrating solution, as explained further on.

Suppose  $N$  farmers produce  $J$  crops using  $I$  limiting inputs and a linear technology. Let us assume that, for each farmer, the  $(J \times 1)$  vector of crops' market prices is a random variable  $\tilde{\mathbf{p}}$  with mean  $E(\tilde{\mathbf{p}})$  and variance-covariance matrix  $\Sigma_p$ . A  $(J \times 1)$  vector  $\mathbf{c}$  of accounting unit costs is also known. The  $(I \times 1)$  vector  $\mathbf{b}$  indicates farmer's availability of limiting resources. The matrix  $A$  of dimensions  $(I \times J, I < J)$  specifies a linear technology. The  $(J \times 1)$  vector  $\mathbf{x}$  symbolizes the unknown output levels to be optimized. Furthermore, farmer has knowledge of previously realized levels of outputs that are observed (by the econometrician) as  $\mathbf{x}_{obs}$ . Random wealth is defined by previously accumulated wealth,  $\bar{w}$ , augmented by the current random net revenue. Assuming a MS utility function under this scenario, mean wealth is defined as  $\mu = [\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]$  with standard deviation equal to  $\sigma = (\mathbf{x}' \Sigma_p \mathbf{x})^{1/2}$ .

Then, a primal PMP-MS model is specified as follows:

$$\max_{\mathbf{x}, \mathbf{h}, \theta, \gamma} V(\mu, \sigma) = \mu^\theta - \sigma^\gamma = [\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]^\theta - (\mathbf{x}' \Sigma_p \mathbf{x})^{\gamma/2} \quad (7)$$

$$\begin{array}{ll} \text{subject to} & A\mathbf{x} \leq \mathbf{b} \quad \text{dual variable } \mathbf{y} \\ & \mathbf{x} = \mathbf{x}_{obs} + \mathbf{h} \quad \text{dual variable } \boldsymbol{\lambda} \end{array}$$

where  $\mathbf{h}$  is a vector of deviations from the realized and observed output levels,  $\mathbf{x}_{obs}$ . The first set of constraints forms the structural (technological) relations while the second set constitutes the calibration constraints. This specification of the calibration constraints differs from the traditional statement  $\mathbf{x} \leq \mathbf{x}_{obs}(1 + \varepsilon)$  where  $\varepsilon$  is a user-determined, small positive number whose purpose is to allow the dual variables of binding structural constraints to take on positive values. In Howitt's words (1995a, p. 151): "The  $\varepsilon$  perturbation on the calibration constraints decouples the true resource constraints from the calibration constraints and ensures that the dual values on the allocable resources represent the marginal values of the resource constraints." The present paper avoids the user-determined parameter  $\varepsilon$  of the traditional PMP methodology and allows the empirical data to reveal the com-

ponents of the vector of deviations  $\mathbf{h}$ . Such deviations can take on either positive or negative values. To justify the specification of the calibration constraints  $\mathbf{x}=\mathbf{x}_{obs}+\mathbf{h}$ , we note that the vector of realized output levels,  $\mathbf{x}_{obs}$ , has been “observed”, that is measured, by persons other than the economic entrepreneur, say by an econometrician. It is likely, therefore, that the measured  $\mathbf{x}_{obs}$  vector may either overstate or understate the true levels of realizable optimal outputs. The deviation vector  $\mathbf{h}$  captures these likely measurement errors.

The dual constraints of problem (7) – derived by Lagrange methods – turn out to be .

$$\gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x}+A'\mathbf{y}+\boldsymbol{\lambda}\geq\theta[\bar{\mathbf{w}}+(E(\tilde{\mathbf{p}})-\mathbf{c})'\mathbf{x}]^{(\theta-1)}[E(\tilde{\mathbf{p}})-\mathbf{c}] \quad (8)$$

Parameters  $\theta$  and  $\gamma$  are unknown as are the output levels,  $\mathbf{x}$ , the deviations,  $\mathbf{h}$ , the dual variables,  $\mathbf{y}$ , and the Lagrange multipliers,  $\boldsymbol{\lambda}$ . Appropriate initial values of the unknown variables are of great importance to achieve an admissible solution. Furthermore, it is often the case that also the (approximate) market price of some input – say land – is known for the region of the sample farms or even for a single farm. The PMP methodology of this paper, therefore, uses also information  $\mathbf{y}_{obs}$  while the unknown dual variable  $\mathbf{y}$  is treated as

$$\mathbf{y}=\mathbf{y}_{obs}+\mathbf{u} \quad (9)$$

with  $\mathbf{u}$  as an  $(I \times 1)$  vector of deviations from the observed input prices.

Let  $W$  be a nonsingular diagonal matrix of dimensions  $(J \times J)$  with positive diagonal terms equal to observed expected price  $E(\tilde{\mathbf{p}}_j) > 0$ . And let  $V$  be a nonsingular diagonal matrix of dimensions  $(I \times I)$  with positive diagonal terms  $b_i/\gamma_{obs,i} > 0$ . The purpose of matrices  $W$  and  $V$  is twofold. First, to render homogeneous the units of measurement of all terms in the objective function of models defined below. Second, to weigh the deviations  $\mathbf{h}$  and  $\mathbf{u}$  according to the scale of the corresponding expected price and input size, respectively. Using a least-squares approach for the estimation of deviations  $\mathbf{h}$  and  $\mathbf{u}$ , it turns out that, by the self-duality of least squares (LS),  $\boldsymbol{\lambda}=W\mathbf{h}$  and  $\boldsymbol{\psi}=V\mathbf{u}$ , where  $\boldsymbol{\psi}$  is the vector of Lagrange multipliers associated with constraints (9): see Paris (2015). To show this result, consider the following weighted LS problem

$$\min LS = \mathbf{h}'W\mathbf{h}/2 + \mathbf{u}'V\mathbf{u}/2$$

$$\begin{array}{ll} \text{subject to} & \mathbf{x}=\mathbf{x}_{obs}+\mathbf{h} & \text{dual variable } \boldsymbol{\lambda} \\ & \mathbf{y}=\mathbf{y}_{obs}+\mathbf{u} & \text{dual variable } \boldsymbol{\psi}. \end{array}$$

The corresponding Lagrange function and first-order-necessary conditions with respect to  $\mathbf{h}$  and  $\mathbf{u}$  are

$$L=\mathbf{h}'W\mathbf{h}/2+\mathbf{u}'V\mathbf{u}/2+\boldsymbol{\lambda}'(\mathbf{x}-\mathbf{x}_{obs}-\mathbf{h})+\boldsymbol{\psi}'(\mathbf{y}-\mathbf{y}_{obs}-\mathbf{u})$$

$$\frac{\partial L}{\partial \mathbf{h}} = W\mathbf{h} - \boldsymbol{\lambda} = 0$$



$$\frac{\partial L}{\partial \mathbf{u}} = V\mathbf{u} - \boldsymbol{\psi} = 0$$

with the result that  $\boldsymbol{\lambda} = W\mathbf{h}$  and  $\boldsymbol{\psi} = V\mathbf{u}$  as asserted.

A crucial issue concerns parameters  $\theta$  and  $\gamma$ . On the one hand, an economic entrepreneur wishes to maximize her utility of random wealth while minimizing the disutility of its risk. On the other hand, it is a fact that high levels of current income (a component of wealth) are associated with high risk of losses. Another fact is that this entrepreneur has already made her choice and executed a production plan,  $\mathbf{x}_{obs}$ , in the face of output price risk. It is also likely that she does not know (or that she is not even aware of) parameters  $\theta$  and  $\gamma$ . The challenge, therefore, is to infer – from her decisions – the values of parameters  $\theta$  and  $\gamma$  that could explain the behavior of this entrepreneur in a rational fashion.

### 3. A Chance-Constrained Relation for $\theta$ and $\gamma$

Charnes and Cooper (1959) proposed a very interesting approach to deal with risky prospects based upon the notion of chance-constrained programming. This idea is particularly useful within the context of this paper because it establishes an independent link between the  $\theta$  and  $\gamma$  parameters, on one side, and the entrepreneur's decisions,  $\mathbf{x}$ , on the other side. Consider the following scenario. With some probability, a farmer may survive unfavorable events such as total revenue being less than total cost. In terms of the chance-constrained methodology this risky scenario is expressed by the following probabilistic proposition:

$$Prob\{\tilde{\mathbf{p}}'\mathbf{x} \leq \mathbf{y}'\mathbf{A}\mathbf{x} + (\mathbf{c} + \boldsymbol{\lambda})'\mathbf{x}\} \leq 1 - \beta \quad (10)$$

where the probability that uncertain (random) total revenue  $\tilde{\mathbf{p}}'\mathbf{x}$  be less than or equal to certain total cost  $\mathbf{y}'\mathbf{A}\mathbf{x} + (\mathbf{c} + \boldsymbol{\lambda})'\mathbf{x}$  should be smaller than or equal to  $1 - \beta$ . Intuitively, for how many years could a farmer survive while operating in the red? As an example, say once every ten years. In this case, the estimated probability equals to  $1 - \beta = 1/10 = 0.10$ . The  $\mathbf{y}'\mathbf{A}\mathbf{x}$  term is total cost associated with fixed limiting inputs ( $\mathbf{y}'\mathbf{A}\mathbf{x} = \mathbf{y}'\mathbf{x}$ ). The  $(\mathbf{c} + \boldsymbol{\lambda})'\mathbf{x}$  term is total variable cost associated directly with output levels.

To derive a deterministic equivalent of relation (10) it is convenient to standardize the random variable  $\tilde{\mathbf{p}}'\mathbf{x}$  by subtracting its expected value  $E(\tilde{\mathbf{p}})\mathbf{x}$  and dividing it by the corresponding standard deviation  $(\mathbf{x}'\Sigma_p\mathbf{x})^{1/2}$ :

$$Prob\left(\frac{\tilde{\mathbf{p}}'\mathbf{x} - E(\tilde{\mathbf{p}})\mathbf{x}}{(\mathbf{x}'\Sigma_p\mathbf{x})^{1/2}} \leq \frac{\mathbf{y}'\mathbf{A}\mathbf{x} + (\mathbf{c} + \boldsymbol{\lambda})'\mathbf{x} - E(\tilde{\mathbf{p}})\mathbf{x}}{(\mathbf{x}'\Sigma_p\mathbf{x})^{1/2}}\right) \leq 1 - \beta$$

$$Prob\left(\tau \leq \frac{\mathbf{y}'\mathbf{A}\mathbf{x} + (\mathbf{c} + \boldsymbol{\lambda})'\mathbf{x} - E(\tilde{\mathbf{p}})\mathbf{x}}{(\mathbf{x}'\Sigma_p\mathbf{x})^{1/2}}\right) \leq 1 - \beta \quad (11)$$

$$Prob[E(\tilde{\mathbf{p}})\mathbf{x} + \tau(\mathbf{x}'\Sigma_p\mathbf{x})^{1/2} \leq \mathbf{y}'\mathbf{A}\mathbf{x} + (\mathbf{c} + \boldsymbol{\lambda})'\mathbf{x}] \leq 1 - \beta.$$

By assuming that  $\tau$  is a standard normal random variable and choosing a value, say  $\tau = \bar{\tau}$ , that corresponds to probability  $1-\beta$ , the deterministic equivalent of relation (11) assumes the specification

$$E(\tilde{\mathbf{p}})' \mathbf{x} + \bar{\tau}(\mathbf{x}' \Sigma_p \mathbf{x})^{1/2} \leq \mathbf{y}' \mathbf{A} \mathbf{x} + \mathbf{c}' \mathbf{x} + \boldsymbol{\lambda}' \mathbf{x} \quad (12)$$

To establish the relation between the  $\bar{\tau}$  parameter and the MS coefficients  $\theta$  and  $\gamma$  the dual complementary slackness condition of constraint (8) is subtracted from the deterministic equivalent (12) (recall that  $\boldsymbol{\lambda} = W\mathbf{h}$ ):

$$\begin{aligned} E(\tilde{\mathbf{p}})' \mathbf{x} + \bar{\tau}(\mathbf{x}' \Sigma_p \mathbf{x})^{1/2} &\leq \mathbf{y}' \mathbf{A} \mathbf{x} + \mathbf{c}' \mathbf{x} + \mathbf{h}' W \mathbf{x} \\ -\theta[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]^{\theta-1} (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x} &= -\mathbf{y}' \mathbf{A} \mathbf{x} - \mathbf{h}' W \mathbf{x} - \gamma(\mathbf{x}' \Sigma_p \mathbf{x})^{\gamma/2}. \end{aligned} \quad (13)$$

With simplification, relation (13) corresponds to

$$E(\tilde{\mathbf{p}})' \mathbf{x} - \mathbf{c}' \mathbf{x} + \bar{\tau}(\mathbf{x}' \Sigma_p \mathbf{x})^{1/2} - \theta[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]^{\theta-1} (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x} + \gamma(\mathbf{x}' \Sigma_p \mathbf{x})^{\gamma/2} \leq 0 \quad (14)$$

Relation (14) establishes a simultaneous and independent link between the risk parameters  $\theta$ ,  $\gamma$  and the decision variables  $\mathbf{x}$ , once the value of  $\bar{\tau}$  is selected by the researcher. As an example, if the survival probability is determined to be  $1-\beta=0.10$ , the one tail value of the standard normal random variable is  $\bar{\tau}=-1.285$ .

#### 4. Phase I PMP Model – Estimation of Calibrating Primal and Dual Solutions

The components of Phase I PMP model are ready to be assembled. For estimation purposes, deviations  $\mathbf{h}$  and  $\mathbf{u}$  will be minimized in a weighted least-squares objective function subject to relevant primal and dual constraints, their associated complementary slackness conditions and relation (14). This task leads to the following Phase I model

$$\min LS = \mathbf{h}' W \mathbf{h} / 2 + \mathbf{u}' V \mathbf{u} / 2 \quad (15)$$

subject to

$$\mathbf{A} \mathbf{x} \leq \mathbf{b} + V \mathbf{u} \quad (16)$$

$$\theta[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]^{\theta-1} [E(\tilde{\mathbf{p}}) - \mathbf{c}] \leq \mathbf{A}' \mathbf{y} + W \mathbf{h} + \gamma(\mathbf{x}' \Sigma_p \mathbf{x})^{(\gamma/2-1)} \Sigma_p \mathbf{x} \quad (17)$$

$$\mathbf{x} = \mathbf{x}_{obs} + \mathbf{h} \quad (18)$$

$$\mathbf{y} = \mathbf{y}_{obs} + \mathbf{u} \quad (19)$$

$$\mathbf{y}' (\mathbf{b} + V \mathbf{u} - \mathbf{A} \mathbf{x}) = 0 \quad (20)$$

$$\mathbf{x}'\{A'\mathbf{y} + W\mathbf{h} + \gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x} - \theta[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(\theta-1)}[E(\tilde{\mathbf{p}}) - \mathbf{c}]\} = 0 \quad (21)$$

$$E(\tilde{\mathbf{p}})'\mathbf{x} - \mathbf{c}'\mathbf{x} + \bar{c}(\mathbf{x}'\Sigma_p\mathbf{x})^{1/2} - \theta[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{\theta-1}(E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x} + \gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{\gamma/2} = 0 \quad (22)$$

with  $\mathbf{x} \geq 0, \gamma \geq 0, \theta > 0, \gamma, \mathbf{h}$  and  $\mathbf{u}$  free.

With the specification of the calibration constraints as in relations (18) and (19), the notion of a PMP calibrating solution differs from the traditional concept according to which the optimal calibrating solution is equal to the observed output levels, that is,  $\mathbf{x}^* \equiv \mathbf{x}_{obs}$ , as the perturbation results in a very small (user-determined) positive number. With the methodology proposed in this paper, a calibrating solution  $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$  will not, in general, be exactly equal to the corresponding vectors of the observed production plan and input prices  $(\mathbf{x}_{obs}, \mathbf{y}_{obs})$ . The objective of model (15)-(22), therefore, is to minimize the deviations  $\mathbf{h}$  and  $\mathbf{u}$  in the amount allowed by the technological and risky environment facing farmers.

Constraints (16) represent the structural (technological) relations of input demand being less-than-or-equal to the effective input supply. Constraints (17) represent the dual relations with marginal utility of the production plan being less-than-or-equal to its marginal cost. Here marginal cost has two parts: the marginal cost due to limiting and variable inputs,  $A'\mathbf{y} + W\mathbf{h}$ , and the marginal cost of output price risk,  $\gamma(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x}$ . Constraints (18) and (19) are the calibration relations. Constraints (20) and (21) are complementary slackness conditions of constraints (16) and (17). Constraint (22) results from the chance-constrained specification (10). Because constraints (16)-(22) represent primal and dual relations and their complementary slackness conditions, any feasible solution of relations (16)-(22) constitutes an admissible economic equilibrium that is consistent with the behavior of decision making under price risk. Furthermore, the calibrating solution  $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$  is unique because the least-squares solution of  $(\hat{\mathbf{h}}, \hat{\mathbf{u}})$  is also unique.

## 5. Phase II PMP Model – Estimation of the Cost Function

Phase II of the PMP methodology deals with the estimation of a cost function that embodies all the technological and behavioral information revealed in Phase I. Typically, a marginal cost function expresses a portion of the dual constraints in a Phase I PMP model. In the absence of risk, PMP marginal cost is defined as  $A'\mathbf{y} + W\mathbf{h} + \mathbf{c}$ , where  $A'\mathbf{y}$  stands for the marginal cost due to limiting inputs and  $W\mathbf{h} + \mathbf{c}$  for the effective marginal cost due to variable outputs. In the risky price case, marginal cost is given by the right-hand-side of relation (17) where all the elements are measured in utility units. It is crucial to obtain a dollar expression of marginal cost, as in the familiar relation  $MC \geq E(\tilde{\mathbf{p}})$ . To achieve this result, the elements of relation (17) will be divided by the term  $\theta[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(\theta-1)}$  to write

$$MC \geq E(\tilde{\mathbf{p}}) \quad (23)$$

$$\mathbf{c} + \frac{1}{\theta}[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(1-\theta)}[A'\mathbf{y} + W\mathbf{h}] + \frac{\gamma}{\theta}[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(1-\theta)}(\mathbf{x}'\Sigma_p\mathbf{x})^{(\gamma/2-1)}\Sigma_p\mathbf{x} \geq E(\tilde{\mathbf{p}})$$

In relation (23), all the terms are measured in dollars. The marginal cost due to limiting and variable inputs is given by

$$\left\{ \mathbf{c} + \frac{1}{\theta} [\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]^{(1-\theta)} [A' \mathbf{y} + W \mathbf{h}] \right\}.$$

The marginal cost due to risky output prices is given by

$$\left\{ \frac{\gamma}{\theta} [\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \mathbf{x}]^{(1-\theta)} (\mathbf{x}' \Sigma_p \mathbf{x})^{(\gamma/2-1)} \Sigma_p \mathbf{x} \right\}.$$

The cost function selected to synthesize the technological and behavioral relations of Phase I is expressed as a modified Leontief cost function such as

$$C(\mathbf{x}, \mathbf{y}) = (\mathbf{f}' \mathbf{x})(\mathbf{g}' \mathbf{y}) + (\mathbf{g}' \mathbf{y})(\mathbf{x}' Q \mathbf{x}) / 2 + (\mathbf{f}' \mathbf{x})[(\mathbf{y}^{1/2})' G \mathbf{y}^{1/2}] \quad (24)$$

A cost function is non-decreasing in output quantities and input prices. It is linearly homogeneous and concave in input prices,  $\mathbf{y}$ . The  $(I \times I)$  matrix  $G$  has elements  $G_{i,ii} = G_{ii,i} \geq 0, i \neq ii, i, ii = 1, \dots, I$ . The diagonal elements  $G_{i,i}$  can take on either positive or negative values. The  $(J \times J)$  matrix  $Q$  is symmetric positive semidefinite. The components of vectors  $\mathbf{f}$  and  $\mathbf{g}$  are free to take on any value as long as  $\mathbf{f}' \mathbf{x} > 0$  and  $\mathbf{g}' \mathbf{y} > 0$ . The reason for introducing a term like  $(\mathbf{f}' \mathbf{x})(\mathbf{g}' \mathbf{y})$  is to add flexibility to the cost function.

The marginal cost function associated with cost function (24) is given by

$$MC_x = \frac{\partial C}{\partial \mathbf{x}} = \mathbf{f}(\mathbf{g}' \mathbf{y}) + (\mathbf{g}' \mathbf{y}) Q \mathbf{x} + \mathbf{f}[(\mathbf{y}^{1/2})' G \mathbf{y}^{1/2}] \quad (25)$$

The derivative of the cost function with respect to input prices corresponds to Shephard's lemma that produces the demand function for inputs:

$$\frac{\partial C}{\partial \mathbf{y}} = (\mathbf{f}' \mathbf{x}) \mathbf{g} + \mathbf{g}(\mathbf{x}' Q \mathbf{x}) / 2 + (\mathbf{f}' \mathbf{x}) [\Delta(\mathbf{y}^{-1/2})' G \mathbf{y}^{1/2}] = A \mathbf{x} \quad (26)$$

where  $\Delta(\mathbf{y}^{-1/2})$  represents a diagonal matrix with elements  $y_i^{-1/2}$  on the main diagonal.

With knowledge of the solution components resulting from the Phase I model (15)-(22),  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{h}}, \hat{\mathbf{u}}, \hat{\theta}, \hat{\gamma}$ , a Phase II model's goal is to estimate the parameters of the cost function,  $\mathbf{f}, \mathbf{g}, Q, G$ . This task is accomplished by means of the following specification

$$\min Aux = \mathbf{d}' \mathbf{d} / 2 + \mathbf{r}' \mathbf{r} / 2 \quad (27)$$

subject to

$$\mathbf{f}(\mathbf{g}' \hat{\mathbf{y}}) + (\mathbf{g}' \hat{\mathbf{y}}) Q \hat{\mathbf{x}} + \mathbf{f}[(\hat{\mathbf{y}}^{1/2})' G \hat{\mathbf{y}}^{1/2}] = \quad (28)$$

$$\mathbf{c} + \frac{1}{\hat{\theta}} [\bar{\mathbf{w}} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \hat{\mathbf{x}}]^{(1-\hat{\theta})} [A' \hat{\mathbf{y}} + W \hat{\mathbf{h}}] + \frac{\hat{\gamma}}{\hat{\theta}} [\bar{\mathbf{w}} + (E(\tilde{\mathbf{p}}) - \mathbf{c})' \hat{\mathbf{x}}]^{(1-\hat{\theta})} (\hat{\mathbf{x}}' \Sigma_p \hat{\mathbf{x}})^{(\hat{\gamma}/2-1)} \Sigma_p \hat{\mathbf{x}} + \mathbf{d} \quad (29)$$

$$(\mathbf{f}' \hat{\mathbf{x}}) \mathbf{g} + \mathbf{g} (\hat{\mathbf{x}}' Q \hat{\mathbf{x}}) / 2 + (\mathbf{f}' \hat{\mathbf{x}}) [\Delta(\hat{\mathbf{y}}^{-1/2})' G \hat{\mathbf{y}}^{1/2}] = A \hat{\mathbf{x}} + \mathbf{r}$$

$$Q = LDL' \quad (30)$$

$$QQ^{-1} = I \quad (31)$$

with  $\mathbf{f}' \hat{\mathbf{x}} > 0, \mathbf{g}' \hat{\mathbf{y}} > 0, D \geq 0$ ,  $\mathbf{f}$  and  $\mathbf{g}$  free. The GAMS software requires an objective function. The vector variables  $\mathbf{d}, \mathbf{r}$  perform the role of slack variables in the estimation of the marginal cost function and Shephard's lemma, respectively.

The objective function (27) is a typical least-squares specification. Relation (28) represents the marginal cost function. Relation (29) is Shephard's lemma. Relation (30) is the Cholesky factorization of the  $Q$  matrix with  $D$  as a diagonal matrix with nonnegative elements on the main diagonal and  $L$  is a unit lower triangular matrix. The Cholesky factorization guarantees symmetry and positive semidefiniteness of the  $Q$  matrix. Relation (31) defines the inverse of the  $Q$  matrix and, thus, guarantees the positive definiteness of that matrix. This constraint assumes relevance for computing the supply elasticities of the various outputs. Any feasible solution of model (27)-(31) is an admissible cost function for representing the economic agent's decisions under price risk.

## 6. PMP and Output-Supply Elasticities

It may be of interest to estimate price supply elasticities for the various commodity outputs involved in a PMP-MS approach. The supply function for outputs is derivable from relation (25) by equating it to the expected market output prices,  $E(\tilde{\mathbf{p}})$ , and inverting the marginal cost function:

$$\mathbf{x} = -Q^{-1} \mathbf{f} - Q^{-1} \mathbf{f} [(y^{1/2}) G y^{1/2}] / (\mathbf{g}' \mathbf{y}) + [1 / (\mathbf{g}' \mathbf{y})] Q^{-1} E(\tilde{\mathbf{p}}) \quad (32)$$

that leads to the supply elasticity matrix

$$\Xi = \Delta[E(\tilde{\mathbf{p}})] \frac{\partial \mathbf{x}}{\partial E(\tilde{\mathbf{p}})} \Delta[\mathbf{x}^{-1}] = \Delta[E(\tilde{\mathbf{p}})] Q^{-1} \Delta[\mathbf{x}^{-1}] / (\mathbf{g}' \mathbf{y}) \quad (33)$$

where matrices  $\Delta[E(\tilde{\mathbf{p}})]$  and  $\Delta[\mathbf{x}^{-1}]$  are diagonal with elements  $E(\tilde{p}_j)$  and  $x_j^{-1}$  on the main diagonals, respectively. Relation (33) includes all the own- and cross-price elasticities for all the output commodities admitted in the model.

PMP has been applied frequently to analyze farmers' behavior to changes in agricultural policies. A typical empirical setting is to map out several areas in a region (or state) and to assemble a representative farm for each area (or to treat each area as a large farm). When supply elasticities are exogenously available (say the own-price elasticities of crops) at the regional (or state) level (via econometric estimation or other means), a connection of

all area models can be specified by establishing a weighted sum of all the areas endogenous own-price elasticities and the given regional elasticities. The weights are the share of each area's expected revenue over the total expected revenue of the region. The advantage of using exogenously supply elasticities has been asserted by Mérel and Bucharam (2010) and Petsakos and Rozakis (2015) in order to account for second-order conditions' information.

Let us suppose that exogenous own-price elasticities of supply are available at the regional level for all the  $J$  crops, say  $\bar{\eta}_j, j=1, \dots, J$ . Then, the relation among these exogenous own-price elasticities and the corresponding areas' endogenous elasticities can be established as a weighted sum such as

$$\bar{\eta}_j = \sum_{n=1}^N w_{nj} \eta_{nj}$$

where the weights are the areas' expected revenue shares in the region (state)

$$w_{nj} = \frac{E(\tilde{p}_{nj})x_{nj}}{\sum_{t=1}^N E(\tilde{p}_{tj})x_{tj}} \quad (34)$$

$$\eta_{nj} = E(\tilde{p}_{nj})Q^{jj}x_{nj}^{-1} / (\mathbf{g}'_n \mathbf{y}_n) \quad (35)$$

where  $Q^{jj}$  is the  $j$ th element on the main diagonal in the inverse of the  $Q$  matrix.

The Phase II model that executes the estimation of the cost function parameters and the disaggregated (endogenous) output supply elasticities for a region (state) that is divided into  $N$  areas takes on the following specification:

$$\min Aux = \sum_{n=1}^N \mathbf{d}'_n \mathbf{d}_n / 2 + \sum_{n=1}^N \mathbf{r}'_n \mathbf{r}_n / 2 \quad (36)$$

subject to

$$\mathbf{f}_n(\mathbf{g}'_n \hat{\mathbf{y}}_n) + (\mathbf{g}'_n \hat{\mathbf{y}}_n)Q\hat{\mathbf{x}}_n + \mathbf{f}_n[(\hat{\mathbf{y}}_n^{1/2})' G \hat{\mathbf{y}}_n^{1/2}] = \quad (37)$$

$$\begin{aligned} \mathbf{c}_n + \frac{1}{\hat{\theta}_n} [\bar{w}_n + (E(\tilde{\mathbf{p}}_n) - \mathbf{c}_n)' \hat{\mathbf{x}}_n]^{(1-\hat{\theta}_n)} [A'_n \hat{\mathbf{y}}_n + W_n \hat{\mathbf{h}}_n] \\ + \frac{\hat{\gamma}_n}{\hat{\theta}_n} [\bar{w}_n + (E(\tilde{\mathbf{p}}_n) - \mathbf{c}_n)' \hat{\mathbf{x}}_n]^{(1-\hat{\theta}_n)} (\hat{\mathbf{x}}'_n \Sigma_p \hat{\mathbf{x}}_n)^{(\hat{\gamma}_n/2-1)} \Sigma_p \hat{\mathbf{x}}_n + \mathbf{d}_n \geq E(\tilde{\mathbf{p}}_n) \end{aligned}$$

$$(\mathbf{f}'_n \hat{\mathbf{x}}_n) \mathbf{g}_n + \mathbf{g}_n (\hat{\mathbf{x}}'_n Q \hat{\mathbf{x}}_n) / 2 + (\mathbf{f}'_n \hat{\mathbf{x}}_n) [\Delta(\hat{\mathbf{y}}_n^{-1/2})' G \hat{\mathbf{y}}_n] = A_n \hat{\mathbf{x}}_n + \mathbf{r}_n \quad (38)$$

$$Q = LDL' \quad \text{positive semidefiniteness} \quad (39)$$

$$QQ^{-1} = I \quad \text{positive definiteness} \quad (40)$$

$$\Xi_n = \Delta[E(\tilde{\mathbf{p}}_n)]Q^{-1}\Delta[(\mathbf{x}_n^{-1})]/(\mathbf{g}'_n\mathbf{y}_n) \text{ endogenous own- and cross-price elasticities} \quad (41)$$

$$w_{nj} = \frac{E(\tilde{p}_{nj})\hat{x}_{nj}}{\sum_{i=1}^N E(\tilde{p}_{ij})\hat{x}_{ij}} \quad \text{expected revenue weights} \quad (42)$$

$$\eta_{nj} = E(\tilde{p}_{nj})Q^{jj}\hat{x}_{nj}^{-1}/(\mathbf{g}'_n\hat{\mathbf{y}}_n) \quad \text{own-price elasticities} \quad (43)$$

$$\bar{\eta}_j = \sum_{n=1}^N w_{nj}\eta_{nj} \quad \text{disaggregation of exogenous elasticities} \quad (44)$$

with  $D_n \geq 0$ ,  $\mathbf{g}_n$  and  $\mathbf{f}_n$  free and  $\mathbf{f}'_n\hat{\mathbf{x}}_n > 0$ ,  $\mathbf{g}'_n\hat{\mathbf{y}}_n > 0$ .

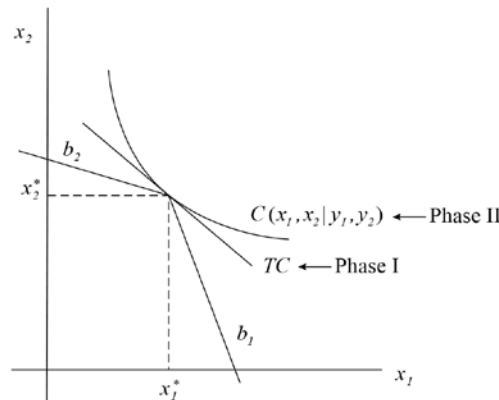
The GAMS software requires an objective function. The objective function *Aux* minimizes the pseudo slack variables,  $\mathbf{r}_n$  and  $\mathbf{d}_n$ , of the primal and dual constraints.

## 7. Phase I Versus Phase II Estimates of the Calibrating Solution

A strand of the PMP literature has discussed the issue of whether the Phase I estimates of decision variables and input shadow prices,  $\mathbf{x}, \mathbf{y}$ , are consistent with the corresponding Phase II estimates where the cost function parameters are estimated simultaneously with them. The short answer is positive because the amount of information is the same in the two Phases. With the limitations of a two-dimensional diagram, Figure 1 illustrates the issue. In Phase I, total cost is a linear function of the decision variables while in Phase II total cost is a nonlinear function of the same variables. Hence, the calibrating optimal solution,  $\mathbf{x}^*$ , is the same in the two Phases.

In the context of this paper, Phase I model is stated as a LS specification of relations (15) through (22). This model results in a unique Least-Squares solution of deviations  $\mathbf{h}$  and  $\mathbf{u}$  and, therefore, of the decision variables  $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ . The Phase II model that estimates

**Figure 1.** Phase I and Phase II estimates of decision variables  $\mathbf{x}$  and input shadow prices  $\mathbf{y}$ .



simultaneously the cost function parameters and the optimal decision variables is stated as the LS specification in Phase I combined with constraints (28) through (31) (where the “ $\hat{\cdot}$ ” symbol is removed from the decision variables). The original information is identical in the two models and, therefore, the LS methodology guarantees the unique and identical solution for the two sets of estimates.

## 8. Phase III PMP Model – Calibrating Models

With the parameter estimates of the cost function,  $\hat{\mathbf{f}}_n, \hat{\mathbf{g}}_n, \hat{Q}, \hat{G}$ , derived from either Phase II model (27)-(31) or model (36)-(44), it is possible to set up a calibrating equilibrium model to be used for policy analysis. Such a model takes on the following economic equilibrium specification

$$\min \text{CSC} = \mathbf{y}'\mathbf{z}_p + \mathbf{x}'\mathbf{z}_d = 0 \quad (45)$$

subject to

$$(\hat{\mathbf{f}}'\mathbf{x})\hat{\mathbf{g}} + \hat{\mathbf{g}}(\mathbf{x}'\hat{Q}\mathbf{x})/2 + (\hat{\mathbf{f}}'\mathbf{x})[\Delta(\mathbf{y}^{-1/2})'\hat{G}\mathbf{y}^{1/2}] + \mathbf{z}_p = \mathbf{b} + \mathbf{V}\hat{\mathbf{u}} \quad (46)$$

$$\hat{\mathbf{f}}(\hat{\mathbf{g}}'\mathbf{y}) + (\hat{\mathbf{g}}'\mathbf{y})\hat{Q}\mathbf{x} + \hat{\mathbf{f}}[(\mathbf{y}^{1/2})'\hat{G}\mathbf{y}^{1/2}] = E(\tilde{\mathbf{p}}) + \hat{\mathbf{z}}_d \quad (47)$$

with  $\mathbf{x} \geq 0, \mathbf{y} \geq 0, \mathbf{z}_p \geq 0, \mathbf{z}_d \geq 0$ . The objective function represents the complementary slackness conditions (CSC) of constraints (46) and (47) with an optimal value of zero. The variables  $\mathbf{z}_p$  and  $\mathbf{z}_d$  are surplus variables of the primal and the dual constraints, respectively. The solution of model (45)-(47) calibrates precisely the solution obtained from the Phase I model (15)-(22), that is,  $\hat{\mathbf{x}}_{LS} = \hat{\mathbf{x}}_{CSC}$  and  $\hat{\mathbf{y}}_{LS} = \hat{\mathbf{y}}_{CSC}$ . This remarkable result is due simply to the fact that all the information of the Phase I model has been transferred to the cost function. Note that the matrix of fixed technical coefficients  $A$  does not appear in either constraint (46) or (47). The calibrating model, then, can be used to trace the production and revenue response to changes in the expected output prices, subsidies and the supply of limiting inputs in a more flexible technical framework.

An alternative calibrating equilibrium model is suitable for dealing with a crucial aspect of a risky policy scenario. Wealth is the anchoring measure of risk preference of an economic agent. As illustrated above, wealth is composed of accumulated income (or exogenous income) and net revenue derived from the current production cycle as in  $[\bar{w} + (E(\tilde{\mathbf{p}}) - \mathbf{c}'\mathbf{x})]$  where  $\bar{w}$  measures the amount of exogenous income. Agricultural policies in many countries deal with subsidies to farmers for cultivating (or not cultivating) crops. These subsidies may or may not be coupled to the level of crop production. Subsidies that are decoupled from the crop production decisions of farmers constitute exogenous income and end up in the term of wealth that becomes an important target of policy makers. The  $\bar{w}$  term, then, must appear in the calibrating model to allow the representation of decoupled subsidies as in the following specification

$$\min \text{CSC} = \mathbf{y}'\mathbf{z}_p + \mathbf{x}'\mathbf{z}_d = 0 \quad (48)$$



subject to

$$(\hat{\mathbf{f}}'\mathbf{x})\hat{\mathbf{g}} + \hat{\mathbf{g}}(\mathbf{x}'\hat{\mathbf{Q}}\mathbf{x})/2 + (\hat{\mathbf{f}}'\mathbf{x})[\Delta(\mathbf{y}^{-1/2})\hat{\mathbf{G}}\mathbf{y}^{1/2}] + \mathbf{z}_p = \mathbf{b} + \mathbf{V}\hat{\mathbf{u}} \quad (49)$$

$$\begin{aligned} \mathbf{c} + \frac{1}{\hat{\theta}}[\bar{\mathbf{w}} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(1-\hat{\theta})}[A'\mathbf{y} + \mathbf{W}\hat{\mathbf{h}}] \\ + \frac{\hat{\gamma}}{\hat{\theta}}[\bar{\mathbf{w}} + (E(\tilde{\mathbf{p}}) - \mathbf{c})'\mathbf{x}]^{(1-\hat{\theta})}(\mathbf{x}'\Sigma_p\mathbf{x})^{(\hat{\gamma}/2-1)}\Sigma_p\mathbf{x} = E(\tilde{\mathbf{p}}) + \mathbf{z}_d \end{aligned} \quad (50)$$

with  $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{z}_p \geq \mathbf{0}, \mathbf{z}_d \geq \mathbf{0}$ . Also the solution of model (48)-(50) calibrates precisely the solution obtained from the Phase I model (15)-(22), that is,  $\hat{\mathbf{x}}_{LS} = \hat{\mathbf{x}}_{CSC}$  and  $\hat{\mathbf{y}}_{LS} = \hat{\mathbf{y}}_{CSC}$ .

### 9. Empirical Implementation of PMP-MS With Supply Elasticities

The PMP-MS approach described in previous sections was applied to a sample of  $N = 14$  representative farms of the Emilia-Romagna region of Italy. There are four crops: sugar beets, soft wheat, corn and barley. There is only one limiting input: land. Empirical reality compels a further consideration of the above methodology in order to deal with farm samples where not all farms produce all commodities. It turns out that very little must be changed for obtaining a calibrating solution in the presence of missing commodity levels, their prices and the corresponding technical coefficients. Using the GAMS software, it is sufficient to condition the various constraints of Phase I, Phase II and Phase III models by the nonzero observations of the output levels. To exemplify, the available farm sample displays the following Table 3 of observed crop levels while Table 4 presents the variance-covariance matrix of the market output prices.

**Table 3.** Observed output levels,  $\mathbf{x}_{obs}$ , with non produced commodities.

Farm	Sugar Beets	Soft Wheat	Corn	Barley
1	1133.4240	0	341.3693	18.2398
2	3103.7830	841.7445	0	59.8025
3	0	450.7937	881.9748	0
4	3488.3540	821.3934	1493.3320	51.1247
5	959.1102	468.2848	0	28.2406
6	942.2039	801.1288	1283.5910	152.5810
7	1600.7310	0	899.4739	66.9718
8	0	1212.8550	1237.5840	98.0497
9	1050.5370	332.3773	0	63.6696
10	3473.6780	952.5199	774.7402	0
11	0	765.1689	501.9673	59.5366
12	3276.1450	1100.1680	0	177.9740
13	877.0970	380.9171	564.6091	76.2122
14	1430.9460	0	1309.3920	0

Other missing information deals with prices and unit accounting costs associated with the zero-levels of crops. Furthermore, the technical coefficients of farms not producing the observed crops also equal to zero. Hence, we can state that, for  $n=1,\dots,N$ , the number of farms, and  $j=1,\dots,J$ , the number of crops, if  $x_{nj}^{obs}=0$ , also  $p_{nj}=0$ ,  $c_{nj}=0$  and  $A_{nij}=0$ . Furthermore, suppose that only one input, land, is involved in this farm sample. Let us assume also that the land price is observed for all farms. The procedure to deal with this type of sample data consists in conditioning the relevant constraints on the positive values of the output levels. In GAMS, this procedure requires a conditional statement using the \$ sign option.

**Table 4.** Variance-covariance matrix of the market output prices.

	Sugar Beets	Soft Wheat	Corn	Barley
Sugar Beets	0.0024719	-0.0164391	-0.0117184	-0.0121996
Soft Wheat	-0.0164391	0.2386034	0.1821288	0.2049011
Corn	-0.0117184	0.1821288	0.1530464	0.1610119
Barley	-0.0121996	0.2049011	0.1610119	0.1830829

Tables 5 and 6 present the estimated output levels and input prices  $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ . They also exhibit the percent deviation of the solution  $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$  of model (15)-(22) from the corresponding targets  $(\mathbf{x}_{obs}, \mathbf{y}_{obs})$ . It is of interest to report that the same identical solution was obtained in three different ways. All the estimations were performed with the GAMS software. The first round of estimates were obtained by solving model (15)-(22) one farm at a time. The second round of estimates were obtained by solving model (15)-(22) using the entire sample of observations. This means that the objective function was specified as

$$\min LS = \sum_{n=1}^N \mathbf{h}_n W_n \mathbf{h}_n + \sum_{n=1}^N \mathbf{u}_n V_n \mathbf{u}_n$$

subject to constraints (16)-(22) specified for each single farm observation. The third round of estimates of the optimal decision variables were obtained by solving model (36)-(44) with the “^” symbol removed from the variables.

Table 7 presents the estimates of the parameters  $\theta$  and  $\gamma$  of the MS utility function.

The sample is composed of relatively homogeneous farms. Hence, the limited numerical range of variation of the MS utility parameters is not a surprise. Within that range, however, a wide variety of risk preferences is detected. Seven farmers exhibit decreasing absolute risk aversion accompanied by increasing relative risk aversion. This result matches a statement of Tsiang (1972, p. 357): “...the most commonly observed pattern of behavior toward risk of a risk-averting individual is probably decreasing absolute risk-aversion coupled with increasing relative risk-aversion when his wealth increases...” Two farmers exhibit increasing absolute risk aversion associated with decreasing relative risk aversion. Four farmers exhibit decreasing absolute risk propensity and increasing relative risk propensity. It should be noted that the negative gamma coefficients of these four farmers are

**Table 5.** Estimated LS solution,  $\hat{\mathbf{x}}$ , and percent deviation from the observed levels,  $\mathbf{x}_{obs}$  with zero levels for some crops and some farms.

Farm	Optimal Decisions $\hat{\mathbf{x}}$				Percent deviation from $\mathbf{x}_{obs}$			
	Sugar Beets	Soft Wheat	Corn	Barley	Sugar Beets	Soft Wheat	Corn	Barley
1	1133.851	0	341.622	18.156	0.0377	0	0.0741	-0.4587
2	3104.392	861.829	0	52.923	0.0196	0.0098	0	0.2021
3	0	450.794	881.975	0	0	-0.0000	0.0000	0
4	3488.400	821.340	1493.477	51.165	0.0013	-0.0065	0.0097	0.0791
5	959.234	468.140	0	28.308	0.0129	-0.0310	0	0.2399
6	942.488	801.394	1283.947	152.923	0.0301	0.0331	0.0278	0.2238
7	1601.381	0	899.724	67.104	0.0406	0	0.0278	0.1975
8	0	1213.157	1237.937	98.080	0	0.0249	0.0285	0.0307
9	1051.373	332.592	0	63.767	0.0796	0.0645	0	0.1528
10	3474.183	952.606	774.966	0	0.0145	0.0085	0.0291	0
11	0	765.267	502.186	59.659	0	0.0128	0.0436	0.2052
12	3276.657	1100.245	0	178.324	0.0156	0.0070	0	0.1964
13	877.324	380.970	564.926	76.467	0.0258	0.0138	0.0561	0.3347
14	1431.231	0	1309.653	0	0.0199	0	0.0199	0

**Table 6.** Deviation of  $\hat{\mathbf{y}}$  from  $\mathbf{y}_{obs}$ .

Farm	Observed Land Prices $\mathbf{y}_{obs}$	Estimated Land Prices $\hat{\mathbf{y}}$	Percent Deviation
1	4.42	4.4213	0.0287
2	4.38	4.3810	0.0219
3	6.98	6.9800	0.0000
4	5.73	5.7302	0.0036
5	4.40	4.3995	-0.0111
6	1.86	1.8609	0.0458
7	3.65	3.6517	0.0454
8	3.36	3.3609	0.0266
9	2.75	2.7521	0.0780
10	4.28	4.2807	0.0158
11	3.28	3.2810	0.0318
12	1.93	1.9305	0.0281
13	2.32	2.3213	0.0579
14	4.03	4.0308	0.0199

rather small, suggesting that risk neutrality may – probably – be a better risk-preference representation of these farmers. The approach does not allow for a statistical testing of this conjecture. Finally, one farmer exhibits increasing absolute and relative risk aversion. This

**Table 7.** Estimates of  $\theta$  and  $\gamma$ .

Farm	Parameter $\theta$	Parameter $\gamma$	Risk Preference
1	1.0131215	1.1397862	DARA, IRRA
2	1.0050568	1.0766995	DARA, IRRA
3	1.1313873	1.2841485	DARA, IRRA
4	0.9836798	0.9273945	IARA, DRRA
5	0.9578977	-0.1867746	DARP, IRRP
6	0.9645178	-0.1465580	DARP, IRRP
7	1.0183502	1.1310367	DARA, IRRA
8	1.0562629	1.1969911	DARA, IRRA
9	1.0277583	1.1992494	DARA, IRRA
10	1.0043433	1.0570120	DARA, IRRA
11	0.9503372	-0.1567640	DARP, IRRP
12	0.9986044	1.0263446	IARA, IRRA
13	0.9577406	-0.1663556	DARP, IRRP
14	0.9797649	0.8443536	IARA, DRRA

empirical result is a clear illustration of the flexible structure of risk preferences as stated by the theoretical analysis. The corresponding meaning of the various acronyms is derived from Table 1 and Table 2.

The estimated parameters of the cost function are reported in Tables 8 and 9. In this numerical example, the  $G$  matrix contains only one parameter whose value is  $G_{i,i} = -11.39904$ .

Regional, exogenous own-price supply elasticities were available in the magnitude of 0.6 for sugar beets, 0.5 for soft wheat, 0.7 for corn and 0.4 for barley. The endogenous own-price elasticities of all farms were aggregated to be consistent with the regional exogenous elasticities according to relation (44). Table 10 presents the farms' own-price supply elasticities used in the aggregation relation.

## 10. Conclusion

This paper accomplished several objectives. First, it extended the treatment of risk in a mathematical programming framework to include any combination of risk preferences represented by absolute risk aversion (or absolute risk propensity) and relative risk aversion (or relative risk propensity). Second, it modified the traditional PMP approach to deal with calibration constraints regarding observed output levels and observed input prices by eliminating the user-determined perturbation parameter. The combination of these two approaches provides suitable models for agricultural policy analysis that take into consideration farmers' risk preferences associated with the randomness of output prices. Third, this paper integrated the use of exogenous supply elasticities observed for, say, an entire region with the endogenous elasticities derived from the supply functions of the sample farms. This objective is achieved by specifying a complete and flexible total cost function that fulfills all the theoretical requirements. Fourth, it resolves in a positive

**Table 8.** Intercepts  $\hat{\mathbf{f}}$  and  $\hat{\mathbf{g}}$  of the marginal cost and input demand functions.

Farm	$\hat{\mathbf{f}}$				$\hat{\mathbf{g}}$	$\hat{\mathbf{f}}'\hat{\mathbf{x}}$	$\hat{\mathbf{g}}'\hat{\mathbf{y}}$
	Sugar Beets	Soft Wheat	Corn	Barley			
1	0.00949	0	0.00666	-0.00320	0.00465	12.923	0.02055
2	0.00364	0.03154	0	-0.06940	0.00149	34.378	0.00654
3	0	-0.00290	0.00374	0	0.00129	1.965	0.00901
4	0.00734	-0.00284	0.00902	-0.06489	0.00132	33.448	0.00756
5	-0.00307	0.02349	0	-0.05730	0.00202	6.426	0.00888
6	-0.02082	0.08018	0.08193	0.26459	0.00658	190.289	0.01224
7	0.00473	0	0.02687	0.05681	0.00271	35.568	0.00992
8	0	0.08121	-0.00668	-0.05092	0.00220	85.254	0.00738
9	0.00408	0.04408	0	0.07081	0.00610	23.462	0.01679
10	0.00905	0.03772	-0.02931	0	0.00164	44.673	0.00703
11	0	0.08005	-0.04152	-0.05365	0.00300	37.213	0.00985
12	0.00395	0.11439	0	0.06448	0.00329	150.291	0.00635
13	0.00041	0.05078	0.03585	0.19131	0.00950	54.584	0.02205
14	-0.00159	0	0.03287	0	0.00192	40.770	0.00773

**Table 9.** Estimated matrices  $\hat{\mathbf{Q}}$  and  $\hat{\mathbf{D}}$ .

Matrix $\hat{\mathbf{Q}}$				
	Sugar Beets	Soft Wheat	Corn	Barley
Sugar Beets	0.0408842	-0.0269584	-0.0084418	-0.0405819
Soft Wheat	-0.0269584	0.8509183	-0.1850005	-0.5925655
Corn	-0.0084418	-0.1850005	0.3698877	-0.0130721
Barley	-0.0405819	-0.5925655	-0.0130721	7.7008830

Matrix $\hat{\mathbf{D}}$				
	Sugar Beets	Soft Wheat	Corn	Barley
Sugar Beets	0.0408842			
Soft Wheat		0.8331423		
Corn			0.324558	
Barley				7.1182454

way the dispute debated in the PMP literature whether Phase I calibrating estimates are consistent with Phase II estimates. Fifth, a calibrating model resulting from the PMP-MS framework described here allows for the analysis of policy scenarios dealing with farm subsidies that are decoupled from the current crop production. Consider the parameter

**Table 10.** Disaggregation/aggregation of the regional, exogenous own-supply elasticities with zero observations of some output levels.

Farm	Exogenous Sugar Beets: 0.6	Exogenous Soft Wheat: 0.5	Exogenous Corn: 0.7	Exogenous Barley: 0.4
1	0.4422	0	0.9144	0.7529
2	0.6093	0.6344	0	0.8320
3	0	0.8010	0.7415	0
4	0.3126	0.5757	0.6101	0.8427
5	1.1497	0.6980	0	1.1181
6	0.9824	0.3464	0.4243	0.1741
7	0.5677	0	0.7124	0.4221
8	0	0.3910	0.6719	0.4075
9	0.5837	0.5928	0	0.2624
10	0.3482	0.4793	1.1446	0
11	0	0.4196	1.2469	0.4779
12	0.6557	0.4610	0	0.2876
13	0.5061	0.3453	0.4804	0.1667
14	1.0249	0	0.6590	0

$\bar{w}$  in the measure of wealth that may represent exogenous income subsidy. With a Freund approach to risk based upon a constant absolute risk aversion utility function, the wealth parameter disappears from the programming model. On the contrary, one version of the calibrating equilibrium model presented in this paper allows for the analysis of decoupled farm subsidies that are more frequently the target of policy makers. This general model has been tested on different farm samples with satisfactory results including a data sample where not all farms produce all the commodities.

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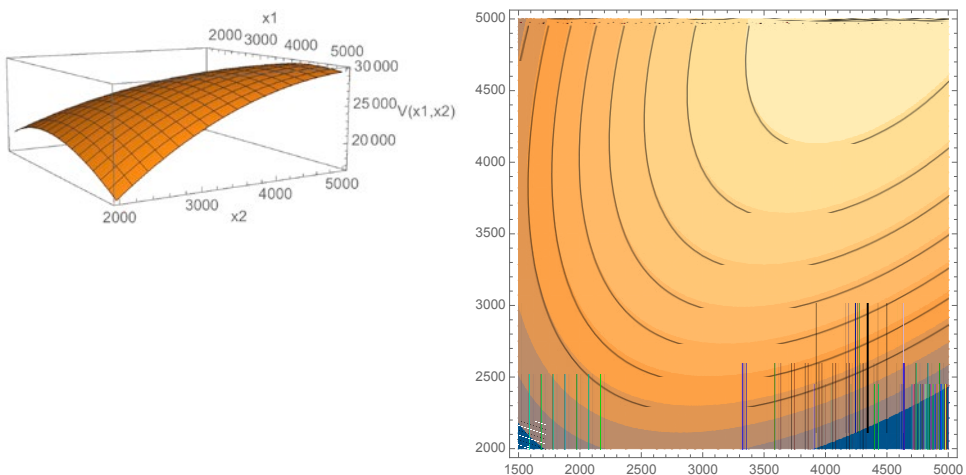
## Appendix

The function  $V(\mu, \sigma) = \mu^\theta - \sigma^\gamma$  is concave in  $\mu$  and  $\sigma$  when the corresponding Hessian matrix is negative definite. This event occurs when  $\theta < 1$  and  $\gamma > 1$ . When the mean and standard deviation of wealth,  $\mu$  and  $\sigma$ , are expressed in terms of decision variables,  $\mathbf{x}$ ,  $\mu(\mathbf{x})$  and  $\sigma(\mathbf{x})$ , the resulting function assumes a flexible structure whose concavity depends on different values of parameters  $\theta$  and  $\gamma$ . This appendix illustrates the possible shapes of the MS utility function (as a function of decision variables) by means of simple graphs and the associated upper contour sets that are conditional upon the magnitude of the  $\theta$  and  $\gamma$  parameters. The value of  $\theta$  and  $\gamma$  are chosen to reflect the estimates of Table 7. The MS utility function is simplified to show two decision variables,  $x_1$  and  $x_2$ . The expected prices are chosen as  $E(\tilde{p}_1) = 4$  and  $E(\tilde{p}_2) = 6$  with standard deviation  $\sigma_{p_1} = 0.5$ ,  $\sigma_{p_2} = 0.7$  and  $\sigma_{p_1 p_2} = 0.1$ . With these stipulations, all the figures' functional forms and the upper contour sets exhibit the following specification

$$\begin{aligned} V[\mu(\mathbf{x}), \sigma(\mathbf{x})] &= \mu(\mathbf{x})^\theta - \sigma(\mathbf{x})^\gamma = [E(\tilde{p}_1)x_1 + E(\tilde{p}_2)x_2]^\theta - [\sigma_{p_1}^2 x_1^2 + \sigma_{p_2}^2 x_2^2 + 2\sigma_{p_1 p_2} x_1 x_2]^\gamma / 2 \\ &= [4x_1 + 6x_2]^\theta - [0.5^2 x_1^2 + 0.7^2 x_2^2 + 0.2x_1 x_2]^\gamma / 2 \end{aligned}$$

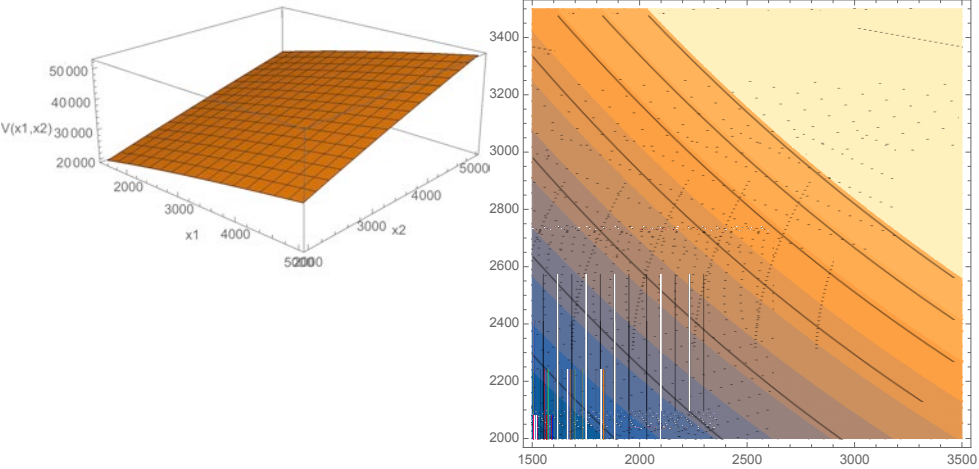
In all the figures, the upper contour sets appear to be convex even though the contour levels appear rather flat in some figures. The convexity of the upper contour sets is a crucial reason for obtaining an optimal solution. The flatness of the contour levels may make it more laborious for the algorithm to converge to an optimal solution. The figures were drawn using Mathematica.

$\theta = 1.1$ ,  $\gamma = 1.36$  DARA, IRRA

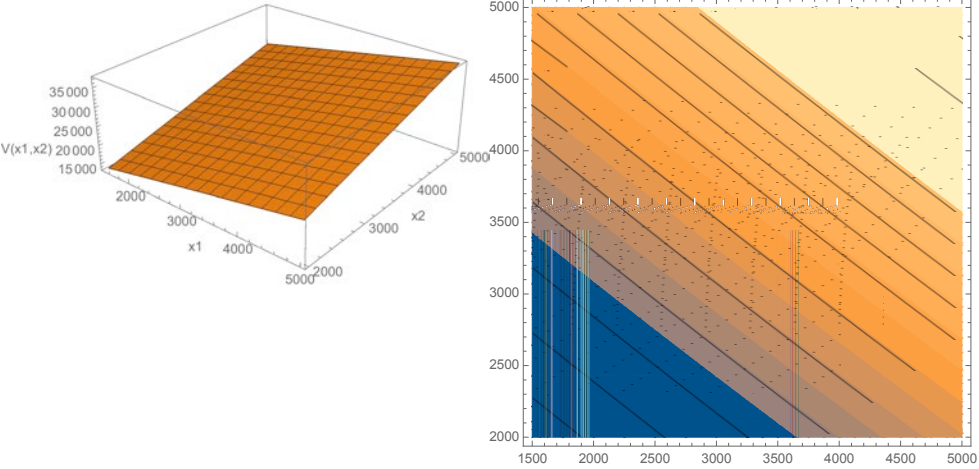




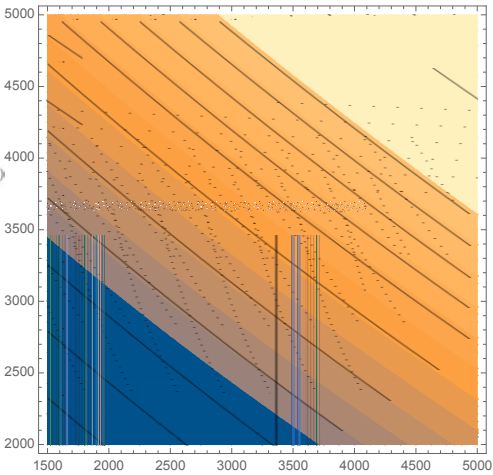
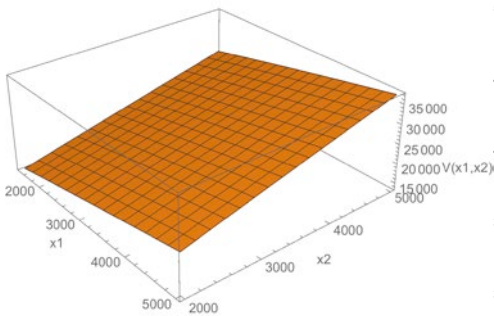
$\theta=1.05, \gamma=1.20$  DARA, IRRA



$\theta=0.98, \gamma=0.92$  IARA, DRRA



$\theta=0.99, \gamma=1.03$  IARA, IRRR



$\theta=0.95, \gamma=-0.15$  DARP, IRRP

