DEVELOPING REALISTIC AGRICULTURAL PRODUCTION FUNCTIONS FOR USE IN UNDERGRADUATE CLASSES

David L. Debertin

Abstract

This note describes an approach for teaching undergraduate students basic properties of cost and production functions with the aid of computer-generated illustrations. Parameters of a third degree polynomial are derived by making use of both the theoretical restrictions and the specific agricultural production process. The function is then used as the basis for the development of a simple computer graphics program which generates illustrations of the corresponding MPP, AC, and MC functions. Such illustrations can be used to make technically accurate visual aids.

Key words: teaching, undergraduate, graphics, computer, production functions, cost functions.

The development of effective examples for illustrating production relationships in agriculture is a problem that has long plagued undergraduate instructors. An effective example should deal with a production process familiar to the student and the units in which the production relationship is expressed should be realistic. Moreover, the example should illustrate accurately the basic marginal relationships. This paper outlines a simple approach for developing functional relationships for depicting agricultural production processes for use as a part of undergraduate instruction and for developing visual aids consistent with these examples.

THE FUNCTIONAL REPRESENTATION

First, a production function is needed that illustrates the familiar marginal relationships with agricultural data and is simple enough to not overwhelm undergraduates whose skills in mathematics are limited. The production function also provides the technical parameters needed to derive the corresponding costs and other functions. A power production function (such as $y = Ax^{\beta}$) does not represent the three stage production function in that it has neither an inflection point nor a maximum. The transcendental production function proposed by Halter et al. (for example, $y = Ax^{\beta} e^{\gamma x}$, where e is the base of the natural log) will represent all three stages of production (when A and β are positive and γ is negative). However, differentiation involves the chain rule, the product rule and $e^{\gamma x}$, and procedures for obtaining the parameters β and γ involve some complicated expressions. Therefore, the transcendental function is not necessarily the best choice for the first presentation to undergraduates, particularly when some students may lack proficiency in calculus.

A better choice to represent the three stage production function for a presentation to undergraduates is a third degree polynomial. The production function:

$$(1)\ y = ax + bx^2 + cx^3$$

is a candidate. The MPP function corresponding to equation (1) is easy to determine even for students with little background in calculus. However, a simple method is needed to estimate the values for a, b, and c in order to meet the requirements of the undergraduate instructor and still reflect the agricultural production process and the appropriate theory. If the requirements are treated as constraints, the solution for the parameters a, b, and c is readily obtainable.

In order to obtain realistic values for the parameters of equation (1), the agricultural economist should elicit the cooperation of plant or animal scientists for basic response

David L. Debertin is Professor of Agricultural Economics, University of Kentucky.
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data from field or feeding trials. Consider the case in which the undergraduate instructor wishes to describe a production relationship between nitrogen application rates and corn yields. Through conversations with agronomists and by studying fertilizer response trial data that links corn yields to nitrogen application rates, the instructor might conclude that a maximum corn yield of 140 bushels per acre is realistic. The yield also appears to be in line with corn yields for the top farmers under nearly ideal weather conditions. University agronomists feel that this yield can be achieved with a nitrogen application rate of 180 pounds per acre. Students could be given access to actual data from nitrogen response trials and could be involved in decisions regarding the yield-maximizing level of nitrogen to be used for the production function. If the production function is to indicate a corn yield of 140 bushels per acre at a nitrogen application level of 180 pounds per acre, then:

$$(2)\ 140 = 180a + 180^2b + 180^3c.$$

Equation (2) must achieve its maximum at 140 bushels per acre and a nitrogen application rate of 180 pounds per acre. The marginal product function from equation (1) is:

$$(3)\ dy/dx = a + 2bx + 3cx^2.$$

Since the necessary condition for a maximum requires that the first derivative be equal to zero at a nitrogen application level of 180 pounds per acre, equation (3) is restated as:

$$(4)\ 0 = a + 2b(180) + 3c(180)^2.$$

The existence and location of the inflection point from the fertilizer response data can be difficult for either the instructor or the students to discern. Instructors might use the inflection point problem as an opportunity to illustrate some of the difficulties in the verification of theoretical concepts with real world data. In order to illustrate the fundamental principles of marginal analysis for the class, a production function with an inflection point where marginal product achieves its maximum is needed. For example, diminishing marginal returns for nitrogen could be assumed to start at an application rate of approximately 70 pounds per acre. If the marginal product function is to achieve a maximum, the necessary condition is that d^2y/dx^2 be zero when 70 pounds of nitrogen are applied per acre:

$$(5)\ 2b + 6cx = 0$$

or

$$(6)\ 2b + 6c(70) = 0.$$

Equations (2), (4), and (6) are three equations in three unknowns. Each equation represents a restriction supplied by the instructor based on marginal principles and the agricultural production process. Equations (2), (4), and (6) are rewritten as:

$$(7)\ 140 = 180a + 32400b + 5832000c,$$

$$(8)\ 0 = 1a + 360b + 97200c,$$

and

$$(9)\ 0 = 0a + 2b + 420c.$$

Equations (7), (8), and (9) can be solved algebraically by the usual substitution methods and the instructor might ask the students to determine the unknown parameter values. However, an easier way is to find the solution with matrix algebra. Let **A** represent the column vector [140 0 0], **B** the column vector of parameters to be estimated [a b c] and **C** represent the matrix:

$$(10)\ \begin{matrix} 180 & 32{,}400 & 5{,}832{,}000 \\ 1 & 360 & 97{,}200 \\ 0 & 2 & 420 \end{matrix}\ .$$

The equations can be stated as:

$$(11)\ \mathbf{A} = \mathbf{B}'\ \mathbf{C}$$

The solution in terms of the parameters is:

$$(12)\ \mathbf{B}' = \mathbf{C}^{-1}\ \mathbf{A}$$

where $\mathbf{C}^{-1}$ is the inverse of the matrix **C**. The advantage of this approach is that the instructor can rely on readily available computer software to invert the **C** matrix and calculate the required parameters. An opportunity exists here to show students uses for simple matrix algebra in solving systems of equations. The solution to these equations gives the parameter values for a as 0.6222, for b as 0.006049, and for c as −0.00002881.

Equations (7), (8), and (9) supply only the necessary conditions for the maximum of marginal and total products. However, the sufficient conditions can be checked once the parameters have been found. Here is an opportunity to introduce to the students the meaning of the terms necessary and sufficient. The sufficient condition for maximum total product is met at x equals 180 in that equation (4) holds and that:

(13) $d^2y/dx^2 = 2b + 6cx$
$= .01210 - .03111$
$= -.01901 < 0.$

The sufficient condition for maximum marginal product is met at x equals 70 in that equation (6) holds and that:

(14) $d^3y/d^3x = 6c = -.0001729 < 0.$

Care must be taken, however, because it is possible to develop a set of restrictions that would not fulfill the sufficient conditions suggested by theory.

This example illustrates but one set of restrictions that could be imposed. Consider the case in which the instructor is unconcerned about the application level at the inflection point, but rather wishes APP to be maximum and stage II to begin at a nitrogen application level of 120 pounds per acre. APP is:

(15) $y/x = a + bx + cx^2.$

The necessary condition for maximum APP occurs at:

(16) $d(y/x)/dx = b + 2cx = 0.$

Therefore,

(17) $0 = 0a + b + 240c.$

Equation (17) is substituted for equation (9), and the three equations again solved for the parameters. The result is:

(18) $y = .3889x + .008642x^2 - .00003601x^3.$

The second order conditions to ensure a maximum APP at x = 120 require that:

(19) $d^2(y/x)/d^2x = 2c = -.00007202 < 0,$

which fulfills the theoretical restriction.

The instructor can choose the level of input use to maximize APP or MPP but not the levels for both maximum MPP and maximum APP, since this will result in two constraints that cannot simultaneously hold. For a polynomial of the form of equation (1), the level of input use that maximizes MPP will be exactly 2/3 the level that maximizes APP, so the maximum MPP in this example will occur at a nitrogen application rate of 80 pounds per acre. In the earlier example, APP was maximum at 70(3/2) or 105 pounds per acre. (For equation (1), maximum MPP occurs at x equals $-b/3c$, and maximum APP occurs at x equals $-b/2c$, so the ratio of maximum MPP to maximum APP is 2/3.)

By expanding the number of terms in the polynomial, additional restrictions could be added that would force the production function to cut through points representing other yield and application rate combinations. However, the function would no longer necessarily illustrate the three stages of production.

APPLICATIONS FOR VISUAL AIDS

Once the parameters of equation (1) have been estimated, they can be used as the basis for a computer program for plotting graphs of production and cost functions with the aid of computer graphics. The graphs generated by the computer are technically accurate with respect to theory and the units are consistent with the specific agricultural production process chosen by the instructor and the students.

Debertin et al. illustrated how computer graphics could be used to generate three dimensional surfaces of various production functions. Bay and Schoney applied similar techniques to actual agricultural data. Computer graphics can also be used to generate cost functions from production functions when the explicit inverse production function cannot be directly obtained. The inverse function for a polynomial production function such as equation (1), representing the cost function expressed in terms of physical units of input use, does not exist for all possible values for a, b, and c. Of course, this problem is not unique to polynomials, but is also true for any production function exhibiting stage III. However, the computer can generate a graphical representation of the corresponding cost function despite the fact that the equation itself cannot be derived, because it uses data points generated from the production function, rather than from the actual inverse cost function.

First, the matrix procedures found in the Statistical Analysis System (SAS Institute) were used to invert the needed C matrix and solve for the parameters of the production function. Assumptions were made with respect to prices for inputs and outputs and a simple program was written to calculate the corresponding marginal physical product (MPP), average physical product (APP), total value product (TVP), and marginal value product (MVP), Exhibit 1, Appendix. Data points were calculated at each quarter pound interval for nitrogen application rates between 0 and 220 pounds per acre. A small interval increases computer time but results in smoother graphics. Total cost (TC) was obtained by multiplying the price of nitrogen times each level of nitrogen use between 0 and 220 pounds, adding a constant representing fixed cost (FC), and plotting the resultant data series on the vertical axis with output, rather than input on the horizontal axis. Average total cost (ATC) is total cost divided by output and plotted against output on the horizontal axis. Marginal cost is the price of the input divided by marginal physical product and is plotted against output on the horizontal axis.

In this paper, publishable pen and ink graphs were needed, so calculated data points were inserted into a graphics package (SAS Graph) capable of providing high quality ink plots. The program was run on a large mainframe computer connected to a drum type plotter. In an actual instructional environment, students would usually rather work with a microcomputer linked to a graphics printer or plotter than with a large mainframe computer. The problem is an excellent example to illustrate the usefulness of a microcomputer for inverting a small matrix and solving for the needed production function parameters. The microcomputer could also be used to generate data points representing the other concepts from theory over the chosen nitrogen application levels. Such a program would allow either the instructor or the student to change the constraints on the problem and observe what happens to parameter values of the production function. While it would not be difficult to write a complete microcomputer program to perform the calculations and construct the graphics, at least one commercial software vendor has a program available that could be modified to meet the specific requirements of the instructor.[1]

Students begin by simply plotting the production function itself. Figure 1 illustrates the production function plotted from the parameters of equation (18). Students then plot the corresponding MPP and APP and observe the resultant relationships that are a part of the three stage production function, Figure 2. Prices are assigned to inputs and outputs.[2] The corresponding total factor cost and total value product, Figure 3, marginal and average value product and marginal factor cost, Figure 4, and profit functions, Figure 5, could then be plotted.

Figure 6 illustrates a total cost function that results from the production function illustrated in Figure 1. This approach is particularly useful in teaching students about the linkages that exist between the production function and the underlying cost functions. The computer graphics approach can also uncover the behavior of average and marginal cost curves in stage III of the production function. Figure 7 illustrates average cost, average variable cost, average fixed cost, and marginal cost based on the parameters of the polynomial and including nitrogen application levels in stage III of the production function. The graphs so far have appeared very similar to those that appear in undergraduate texts. The behavior of cost curves in stage III of the production function is a topic seldom mentioned in undergraduate texts; Figure 7 is different from that found in most introductory texts. The illustration found in Goodwin and Drummond (p. 196) is closest to the cost curve representation generated by the computer. Within stages I and II of the production function, the illustration is also similar to that presented in Doll and Orazem (p. 44). The representation of marginal cost is quite unlike that appearing in beginning economics or agricultural economics textbooks.

Near the start of stage III, marginal cost increases very rapidly. At some arbitrarily small distance to the left of the technical output maximum that marks the dividing line between stage II and stage III, marginal cost is approaching plus infinity. At some arbitrarily small distance to the right of the dividing line between stage II and stage III, marginal cost is approaching negative infinity. Since marginal cost is the input price divided by the marginal physical product, it is undefined at the dividing point between stage II and stage III. In order to plot marginal

[1] A microcomputer program called MATHEMATICS SERIES is capable of finding a solution to a series of simultaneous linear equations via matrix procedures similar to those outlined in this paper. Also, it can make plots of production or cost functions and their derivatives on an IBM PC compatible microcomputer (with a graphics board). Details concerning availability of the software are available from the authors. The program is written in basic and is rudimentary, but it appears to be readily modifiable to include instructor or student-written enhancements.

[2] The price of corn was assumed to be \$4.00, and the price of nitrogen was assumed to be \$0.25 per pound of available N. Total fixed cost was assumed to be \$50 per acre.

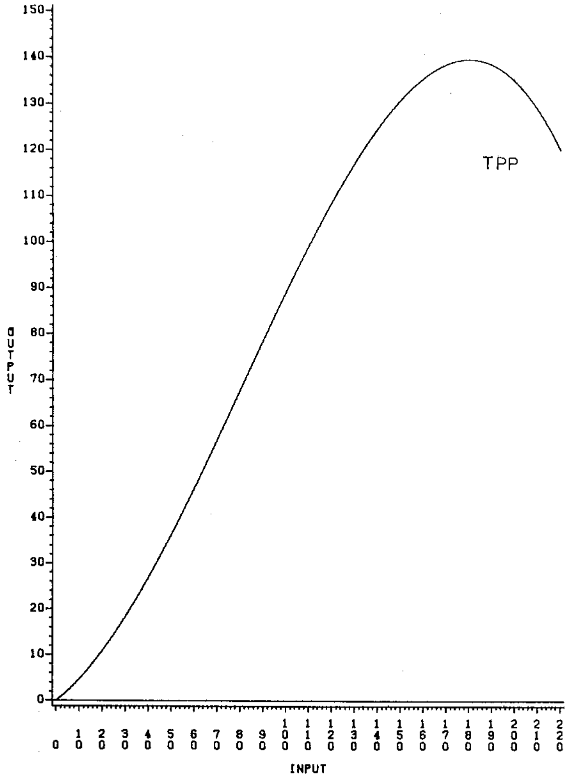

Figure 1. Total Physical Product (TPP).

cost and average fixed cost and yet retain a reasonable scale for the other curves, data points that generated large positive or negative values for these cost curves were not plotted.

CONCLUDING COMMENTS

This paper explained a simple procedure for developing a specific and meaningful polynomial production functions for undergraduate teaching applications with the factor-product model, and illustrated how such a production function could be used as the basis for drawing a series of technically accurate visual aids. Potential applications utilizing a microcomputer linked to a plotter were outlined.

Instructors could also use this approach as a basis for developing a tabular presentation of production concepts. Students might compare the graph of the function with the exact marginal products calculated from the derivative of the production function and with those calculated over a finite range from the tabular data.

While the example presented used the familiar corn response to nitrogen fertilizer, nearly any agricultural production relation-

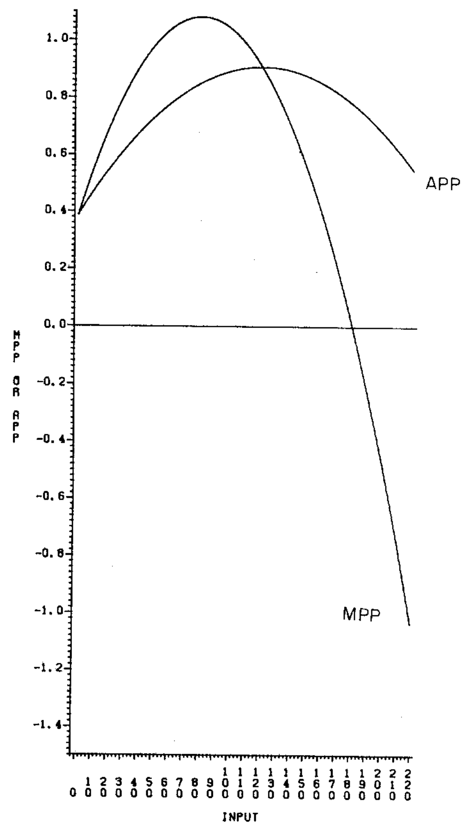

Figure 2. Marginal Physical Product (MPP) and Average Physical Product (APP).

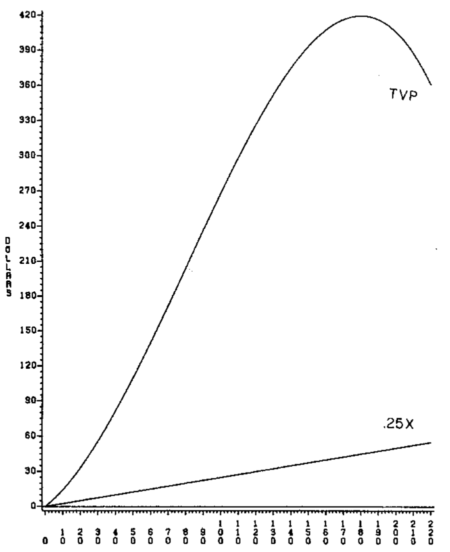

Figure 3. Total Value Product (TVP) and Total Factor Cost (TFC).

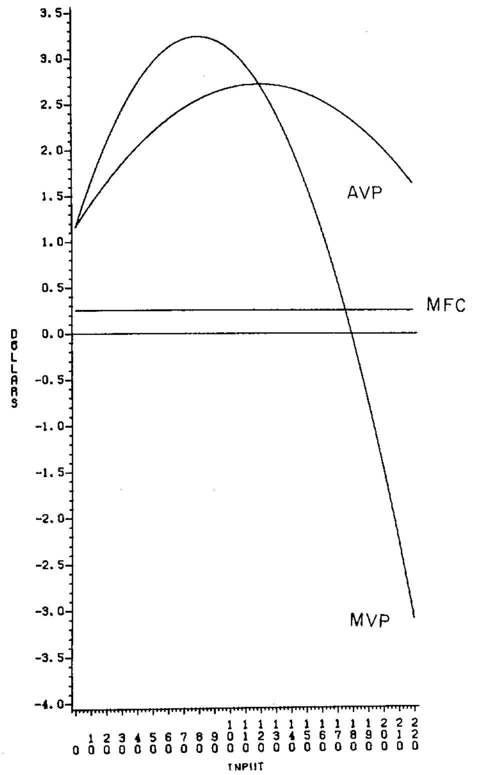

Figure 4. Marginal Value Product (MVP), Average Value Product (AVP) and Marginal Factor Cost (MFC).

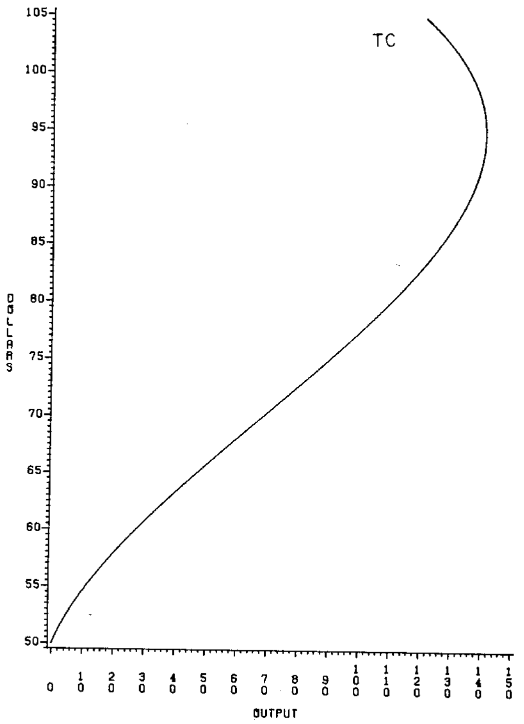

Figure 6. Total Cost Function (TC).

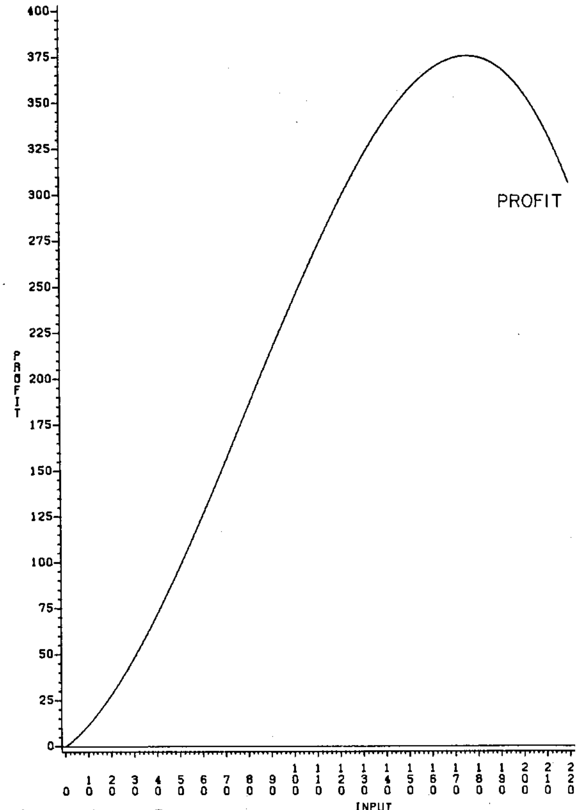

Figure 5. Profit Function.

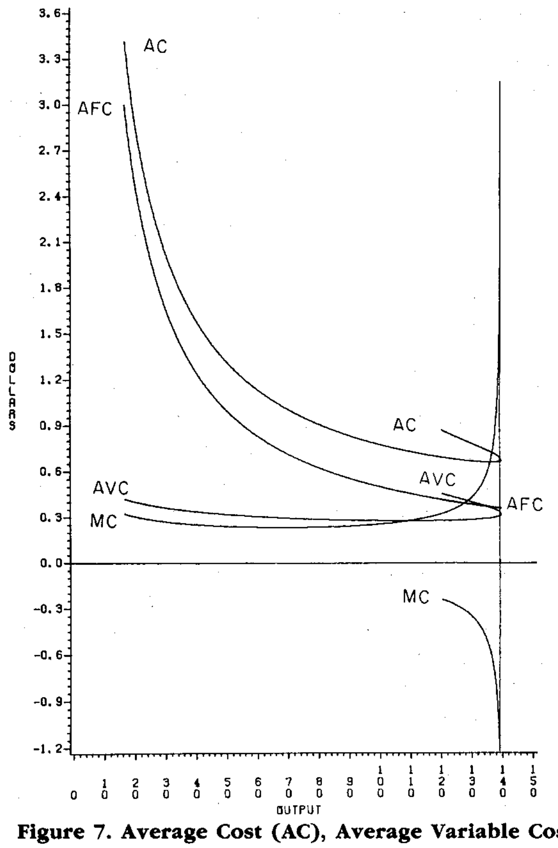

Figure 7. Average Cost (AC), Average Variable Cost (AVC), Average Fixed Cost (AFC), Marginal Cost (MC).

ship of interest to the instructor and students could be used. The approach is capable of developing parameters of production functions for farming enterprises common only to a specific geographic area. Therefore, the approach potentially makes production economics concepts more meaningful and understandable to undergraduates.

The functions thus obtained could readily be inserted into a microcomputer graphics package to obtain plots. Students could then watch what happened to the shape of the function as restrictions were changed. Seldom do opportunities as good as these exist for teaching students basic marginal principles with examples from familiar farm enterprises. Yet, at the same time, students can work with current computer hardware and software on a realistic problem to which they can relate.

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APPENDIX

Exhibit 1. Computer Program for Deriving the Plots.

```
/*SETUP TAPE=(8019,RINGIN)
//S1 EXEC SAS,PLOT=
//PLOTTAPE DD DSN=PLOT,UNIT=PLOT,VOL=SER=8019
//SAS.SYSIN DD *
DATA ONE;
X = 0;
LOOP:;
IF X > 220 THEN STOP;
Y1 = .6222*X + .006049*X**2 -.00002881*X**3;
Y2 = .3889*X+.008642*X**2 -.00003601*X**3;
MPP1 = .6222 + 2*.006049*X -3*.00002881*X**2;
MPP2 = .3889+2*.008642*X - 3* .00003601*X**2;
APP1= .6222 +.006049*X - .00002881*X**2;
APP2 = .3889 + .008642*X - .00003601*X**2;
TVP1=Y1*3;
TVP2=Y2*3;
TFC=25*X;
MFC=.25;
PR1=TVP1-TFC;
PR2=TVP2-TFC;
VMP1=MPP1*3;
VMP2=MPP2*3;
AVP1=APP1*3;
AVP2=APP2*3;
TC=X*.25+50;
TVC=X*.25;
```

```
ATC=TC/Y2;
AVC= (TC-50)/Y2;
AFC=50/Y2;
MC=.25/MPP2;
OUTPUT;
X=X+.25;
GO TO LOOP;
PROC PRINT;VAR X Y2 MPP2 APP2 TC AVC ATC AFC MC;
PROC PRINT;VAR MC AVC ATC AFC;
DATA TWO;SET ONE;
IF _N_ LT 112 THEN DELETE;
IF_N_ GT 706 AND N LT 757 THEN DELETE;
GOPTIONS HSIZE=7 VSIZE=10;
SYMBOL1 I=JOIN C=BLACK;
SYMBOL2 I=JOIN C=BLACK;
SYMBOL3 I=JOIN,C=BLACK;
SYMBOL4 I=JOIN C=BLACK;
PROC GPLOT; PLOT ATC*Y2 AVC*Y2/ VREF=0 OVERLAY HAXIS=0 TO 150 BY 10 ;
LABEL Y2 = OUTPUT ATC = DOLLARS;
PLOT ATC*Y2 AVC*Y2 MC*Y2/ VREF=0 OVERLAY HAXIS=0 TO 150 BY 10 ;
PLOT MC*Y2/ VREF=0 HAXIS=0 TO 150 BY 10;
LABEL MC=DOLLARS;
PLOT ATC*Y2 AVC*Y2 AFC*Y2 MC*Y2/ VREF=0 OVERLAY HAXIS=0 TO 150 BY 10 ;
DATA THR;SET ONE;
GOPTIONS HSIZE=7 VSIZE=10;
SYMBOL1 I=JOIN C=BLACK;
SYMBOL2 I=JOIN C=BLACK;
SYMBOL3 I=JOIN,C=BLACK;
SYMBOL4 I=JOIN C=BLACK;
PROC GPLOT;
PLOT TC*Y2 TVC*Y2/ VREF=0 OVERLAY HAXIS=0 TO 150 BY 10;
LABEL TC = DOLLARS;
PLOT Y1 * X/ VREF=0 VAXIS = 0 TO 150 BY 10 HAXIS = 0 TO 220 BY 10;
LABEL Y1 = OUTPUT X=INPUT;
PLOT Y2 * X/ VREF=0 VAXIS = 0 TO 150 BY 10 HAXIS = 0 TO 220 BY 10;
LABEL Y2 = OUTPUT;
PLOT MPP1*X APP1*X/ VREF=0 OVERLAY HAXIS=0 TO 220 BY 10;
LABEL MPP1=MPP OR APP;
PLOT MPP2*X APP2*X/ VREF=0 OVERLAY HAXIS=0 TO 220 BY 10;
LABEL MPP2=MPP OR APP;
PLOT TVP1*X TFC*X/ VREF=0 OVERLAY HAXIS=0 TO 220 BY 10;
LABEL TVP1 = TVP;
PLOT TVP2*X TFC*X/ VREF=0 OVERLAY HAXIS=0 TO 220 BY 10;
LABEL TVP2 = TVP;
PLOT PR1*X/ VREF=0 HAXIS = 0 TO 220 BY 10;
LABEL PR1=PROFIT;
PLOT PR2*X/ VREF=0 HAXIS = 0 TO 220 BY 10;
LABEL PR2=PROFIT;
PLOT VMP1*X AVP1*X MFC*X/ VREF=0 OVERLAY HAXIS=0 TO 220 BY 10;
LABEL VMP1 = DOLLARS;
PLOT VMP2*X AVP2*X MFC*X/ VREF=0 OVERLAY HAXIS=0 TO 220 BY 10;
LABEL VMP2 = DOLLARS;
/*
```