AN INVESTIGATION OF THE RELATIONSHIP BETWEEN CONSTRAINT OMISSION AND RISK AVERSION IN FIRM RISK PROGRAMMING MODELS

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Abstract

A model with omitted resource constraints is suggested as an alternative to a risk aversion model for explaining economic behavior. This paper uses two standard mathematical programming models to further explore this issue. One model is a standard profit maximization linear programming model and the other is a risk averse quadratic programming model with part of the constraints deleted. Theoretical investigation of these models demonstrates that risk aversion can substitute for omitted resource constraints. A small empirical model is then solved under both formulations. With resource constraints deleted, positive risk aversion is necessary to obtain a similar enterprise organization as under profit maximization with complete constraints. These two solutions are then interpreted with the theoretical optimality conditions.

Key words: risk, mathematical programming, optimal enterprise organization.

During the past decade, risk aversion has become a popular explanation for divergence of observed economic behavior from that predicted by profit maximizing models. A major problem with this approach is that differences in observed behavior can also be caused by model specification deficiencies unrelated to risk aversion (Young et al.; Robison et al.). Recently, several alternative explanations of the importance of risk aversion in behavior under uncertainty have been suggested. This literature establishes that risk aversion can be confused with maximization of expected profit because of: (1) response to dynamic conditions (Antle; Gray and Furtan), (2) dispersion of price expectations (Pope), (3) market transaction costs (Raumasset), and (4) incomplete specifications of production systems and resource constraints (Baker and McCarl). The fourth explanation warrants attention because many risk programming studies which support the importance of risk aversion have quite limited constraints. Some examples include Lin et al. with four land, two water, and two rotation constraints; Adams et al. with one labor and one land constraint; and Sengupta and Sfeir with one land and one water constraint. In contrast, Brink and McCarl included 141 constraints for land, labor, and sequencing of machinery operations and found risk neutrality or very low risk aversion best explained the enterprise mix of most farmers.

This paper presents a pedagogical study of the potential confounding of constraint specification and risk aversion in explaining enterprise diversification. Two distinct mathematical programming models serve as the focus of this analysis. One model has a risk averse objective function combined with limited constraints and the second has a risk neutral objective function with more complete constraints. General forms of these models are investigated with mathematical optimization theory to determine the theoretical relationship between risk aversion and resource constraints in explaining optimal enterprise levels. Example empirical forms of these models are then presented and their

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solutions are interpreted with the theoretical optimality conditions. The models have limited constraints and therefore do not necessarily reflect a realistic production situation. However, the size of these models allows a complete interpretations of the empirical solutions and therefore allows illustration of the theoretical conditions associated with confounding risk aversion and deleted resource constraints.

A THEORETICAL VIEW OF RISK AVERSION AND OMITTED CONSTRAINTS

Two standard programming models of firm behavior are utilized in this paper:

(1) Model A:

$$\text{Maximize } c'x \quad \text{subject to: } A_1x \leq b_1,\ A_2x \leq b_2,\ x \geq 0,$$

and

(2) Model B:

$$\text{Maximize } c'x - (\alpha/2)x'Vx \quad \text{subject to: } A_1x \leq b_1,\ x \geq 0,$$

where A_1 is a (m_1xn) constraint matrix, A_2 is a second (m_2xn) constraint matrix, V is a (nxn) variance-covariance matrix of returns for activities, c is a (nx1) vector of expected value of returns for activities, b_1 is a (m_1x1) vector of resource endowments, b_2 is a (m_2x1) vector of resource endowments, x is a (nx1) vector of activity levels, and α is a risk aversion parameter. Model A is the standard profit maximization model of the firm except that the constraint matrix and vector are written as two components. Model B is a standard expected value-variance (E-V) model of the firm as formulated by Freund. These two models are specified to examine the constraints-risk aversion issue. Model A contains a set of constraints which are omitted from Model B, but Model B contains risk terms which are not in Model A. These models allow investigation of the potential confounding of risk aversion in Model B with constraints excluded from it but included in Model A.

The condition for optimal primal (x) as well as dual decision variables (y_1 and y_2) can be specified with the Kuhn-Tucker conditions for the two models:

(3) Model A:

$$\begin{aligned} c - y_1A_1 - y_2A_2 &\leq 0 \\ (c - y_1A_1 - y_2A_2)\, x &= 0 \\ x &\geq 0 \\[4pt] A_1x - b_1 &\leq 0 \\ y_1\,(A_1x - b_1) &= 0 \\ y_1 &\geq 0 \\[4pt] A_2 - b_2 &\leq 0 \\ y_2\,(A_2x - b_2) &= 0 \\ y_2 &\geq 0 \end{aligned}$$

and

(4) Model B:

$$\begin{aligned} c - y_1A_1 - \alpha x'V &\leq 0 \\ (c - y_1A_1 - \alpha x'V)\, x &= 0 \\ x &\geq 0 \\[4pt] A_1x - b_1 &\leq 0 \\ y_1\,(A_1x - b_1) &= 0 \\ y_1 &\geq 0. \end{aligned}$$

The Kuhn-Tucker conditions for these two problems are identical if $y_2 = 0$ and $\alpha = 0$, which would occur if the omitted constraints in Model B are nonbinding in Model A and a risk neutral solution is obtained for Model B. More importantly, the terms involving risk in Model B enter the conditions on x in an equivalent manner to those involving the second set of constraints in Model A: $-\ y_2A_2$ in Model A conditions are simply replaced by $-\ \alpha x'V$ in Model B conditions. The primal optimal conditions would be equivalent, and identical solutions for x would be obtained if $y_2A_2 = \alpha x'V$.

Further understanding of the potential confounding of risk and omitted constraints can be obtained by writing the optimal conditions for one variable x_j which is nonzero in the solution in both models:

(5) Model A:

$$c_j = \sum_{i=1}^{m_1} y_{1i}a_{1ij} + \sum_{k=1}^{m_2} y_{2k}a_{2kj}.$$

and

(6) Model B:

$$c_j = \sum_{i=1}^{m_1} y_{1i}a_{1ij} + \alpha \sum_{p=1}^{n} x_pV_{jp},$$

where c_j, y_{1i}, a_{1ij}, y_{2k}, a_{2kj}, x_p, V_{jp} are scalar components of c, y_1, A_1, y_2, A_2, x, and V respectively. For equations (5) and (6) to both hold, their right-hand-sides must be equal. Three conditions which would allow this equivalence are: (1) all y_{2k} and α equal zero, (2) the second terms on the right side of equations (5) and (6) are equal, and (3) empirical values of A_1, A_2, V, and α simply allow solutions of y_1 and y_2, with y_1 not the same in both equations, such that the primal decision variables are equal. Presumably, many empirical specifications could realistically allow this third condition.

An economic interpretation of equations (5) and (6) is particularly useful in understanding the similarities between the models. The a_{1ij} and a_{2kj} are the marginal amounts of resources i and k required to produce j. Further, y_{1i} and y_{2k} are the opportunity costs of resources i and k for the firm. Thus, $y_{1i}a_{1ij}$ and $y_{2k}a_{2kj}$ are the contributions of resources i and k to the marginal cost of product j. The sum of these values across all resources in equation (5) equals marginal cost of j which must equal marginal revenue (c_j) for products produced (Baumol). Turning to equation (6), the first term on the right again is the marginal costs of resources; the total marginal costs of resources is likely to be understated since all resources are not considered in this model. To interpret the second term, note that α measures the necessary increase in expected revenues to compensate for a unit increase in variance of revenues (Freund), and $\sum_{p=1}^{n} x_p V_{jp}$ is the marginal contribution of a unit of x_j to the variance of total net revenue. Then, the product of these terms is the increase in marginal expected revenue necessary to compensate for the increase in variance of revenue from one unit of x_j. This term is usually identified as the marginal cost of risk of product j. The marginal costs of resources in equation (6) can be less than in equation (5) if any y_{2k} are positive. In this case, the marginal cost of risk can compensate for a resource specification problem rather than reflect risk aversion. Risk aversion in Model B therefore can contribute to an improved explanation of behavior because of failure to model all limiting resources for the firm. More generally, the theoretical analysis in this section implies that an alternative explanation to risk aversion can be that the model does not fully represent resources of the firm.

The third condition above which allows equations (5) and (6) to both hold also suggests another consequence of incomplete resource specification. If components of y_1 vary between equations (5) and (6), shadow values of resources are biased in equation (6). These biases may be important in some applications.

EMPIRICAL DEMONSTRATION

A complete empirical demonstration of the theoretical results in the preceeding section cannot necessarily be based on realistic models. Complete resource constraints under the expected profit maximizing model can require a large model, such as that developed by Brink and McCarl. Use of equations (5) and (6) to interpret the results from such a model is cumbersome if not impossible. This paper therefore uses a simple empirical model with limited constraints and activities, which is not necessarily a full specification of an empirical version of Model A, to illustrate the theoretical properties of the confounding issue. The models were adapted from a quadratic risk programming model of a representative firm in Georgia used in previous research (Musser and Stamoulis; Musser et al.). This model has less time disaggregation in the resource constraints than Brink and McCarl or Baker or McCarl's. However, deletion of some constraints to represent Model B is still possible.

For this study, the model was simplified to facilitate complete reporting of the details of the model and the solutions. The hired labor and tobacco activities were deleted from the original model. Without tobacco, the land acreage was inadequate to fully employ one full time laborer; therefore, labor availability was reduced 50 percent to represent a part-time farm. The tableau for the risk neutrals form of the adapted model is included in Table 1. Cropland and three labor availability constraints are included along with three enterprise restrictions. The peanut restriction reflects a representative acreage allotment under federal peanut price support programs in effect when the model was constructed. The cotton restriction and the cotton-peanut restrictions reflect recommended rotational patterns for insect and nematode control. The first limits cotton to 50 percent of the land on a 2-year rotation with other crops and the second limits cotton and peanuts to two-thirds of the land on a 3-year rotation with other crops. The Model A solution is reported in Table 2.

An empirical form for Model B was created by deleting constraints and adding the variance-covariance matrix from the original study. After some experimentation with constraint deletion, the cotton and cotton-peanut

TABLE 1. MATHEMATICAL PROGRAMMING TABLEAU FOR LINEAR PROGRAMMING MODEL (MODEL A) OF ENTERPRISE ORGANIZATION FOR SOUTH GEORGIA, 1978

	Enterprise							
Item	Corn	Cotton	Peanuts	Wheat	Soybeans	Oats	Grain sorghum	RHS
Objective function	68.29	168.14	432.90	31.15	141.96	10.79	21.16	
Crop land	1	1	1	1	1	1	1	≤182.6
Labor:								
January-April	2.23	0.56	0.56		0.56		2.23	≤437
May-August	0.24	2.00	2.42	1.58	1.91	1.58	0.24	≤412
September-October	1.72	1.00	4.22	2.22	1.58	2.22	1.58	≤407
Cotton restriction		1						≤ 91.3
Peanut restriction			1					≤ 60.9
Cotton-peanut restriction		1	1					≤121.8

rotation restrictions were chosen as the constraints excluded from Model B since this model allowed a solution most similar to that for Model A. This choice was plausible since these constraints could have been omitted if a modeler was unaware of production conditions in this area.

The tableau used for Model B is presented in Table 3. Objective function and primal parametric solutions for this tableau are reported in Table 4. The risk neutral solution only included cotton and peanuts, which represented less diversification than the solution for the linear programming model(reported in Table 2) and included cotton, peanuts, and soybeans. However, the solutions with risk aversion coefficients of .0013 and .0017 did have the same activities as the solution for Model A. Neither of these solutions was exactly equivalent to the Model A solution.

The solution with a risk aversion coefficient .0017 had approximately the same soybean' acreage as the linear programming solution, the cotton acreage was within the omitted restriction, and the sum of the cotton and peanut acreage was 126, which was not much above the peanut-cotton restriction. In contrast, the solution with a risk aversion coefficient of .0013 deviated more from the linear programming solution in all these respects. Thus, the Model B solution with a risk aversion coefficient of .0017 would most likely be judged a nearly equivalent solution to that for Model A. Details of this solution are included in Table 2.

While the primal solutions for Model A and Model B are similar, the dual solutions are quite different. Only land has a nonzero value in the quadratic programming solution, while land, September-October labor, and the cotton-peanut restriction have positive values in the linear programming solution. The September-October labor constraint was not effective because the Model B solution included more cotton and less peanuts and soybeans than the Model A solution and cotton uses less of this resource than the other enterprises.

The solution values, Table 2 and model parameters, (tables 1 and 3) were used to calculate marginal costs of resources and risk in equations (5) and (6), which are reported in Table 5. With only land having a positive dual value in the Model B solution, the total marginal cost of fixed resources equals the marginal cost of land which is $26.25 for all enterprises. In contrast, the total marginal cost of resources in the linear programming solution for Model A equals the sum of the marginal costs of land, September-October labor, and the peanut-cotton restriction and is $141.96 for soybeans, $168.14 for cotton, and $423.90 for peanuts. Risk aversion and a marginal cost of risk was necessary to compensate for the underestimated total marginal costs of resources so that total marginal costs (both of resources and risk) equaled marginal revenues of enterprises in the optimal solution in Model B.

As a final comment, it must be stressed that the marginal costs of individual resources need not be less in the quadratic than in the linear programming solution. In this empirical illustration, land had a dual value of $26.25 in the quadratic programming solution and $12.03 in the linear programming solution. These results are consistent with standard budgeting procedures—returns to a resource are overstated if returns to another limiting resource are not deducted. However,

TABLE 2. PRIMAL AND DUAL SOLUTIONS FOR A LINEAR PROGRAMMING MODEL WITH ALL CONSTRAINTS AND FOR A QUADRATIC SOUTH GEORGIA, 1978

Activity or constraint	Activity level	Expected returns	Variance of returns	Risk Aversion coefficient
		--dol.--	-($1,000)-	
Linear programming:				
Soybeans	60.8 acres	8,631		
Cotton	63.1 acres	10,610		
Peanuts	58.7 acres	25,411		
Total	182.6 acres	44,652	26,745	0.0
Land	$12.03			
Peanut-cotton restriction	$73.88			
Labor: Sept.-Oct.	$82.23			
Quadratic programming:				
Soybeans	56.1 acres	7,964		
Cotton	79.2 acres	13,317		
Peanuts	47.3 acres	20,476		
Total	182.6 acres	41,757	21,332	.0017
Land	$26.15			

TABLE 3. MATHEMATICAL PROGRAMMING TABLEAU FOR QUADRATIC PROGRAMMING MODEL (MODEL B) FOR SOUTH GEORGIA, 1978

Item	Enterprise							RHS
	Corn	Cotton	Peanuts	Wheat	Soybeans	Oats	Grain sorghum	
Linear objective function	68.29	168.14	432.90	31.15	141.96	10.79	21.16	
Crop land	1	1	1	1	1	1	1	≤182.6
Labor:								
January-April	2.23	0.56	0.56		0.56		2.23	≤437
May-August	0.24	2.00	2.42	1.58	1.91	1.58	0.24	≤412
September-October	1.72	1.00	4.22	2.22	1.58	2.22	1.58	≤407
Peanut restriction			1					≤ 60.9
Variance-covariance matrix:								
Corn	3,086	1,410	1,354	−22	924	123	616	
Cotton		2,365	−1,169	174	−895	117	258	
Peanuts			5,482	−182	1,212	51	502	
Wheat				18	−42	7	14	
Soybeans					1,433	−20	167	
Oats						8	28	
Sorghum							138	

TABLE 4. PARAMETRIC SOLUTIONS FOR QUADRATIC PROGRAMMING MODEL (MODEL B) FOR SOUTH GEORGIA, 1978

	Objective function values		Enterprise levels			
Risk aversion coefficient	Expected returns	Variance of return	Cotton	Peanuts	Soybeans	Wheat
	--dol.--	($1,000)	acres			
0.0000	46,826	38,035	122	61		
0.0013	45,743	26,763	80	61	41	
0.0017	41,762	21,332	79	47	56	
0.0039	21,227	4,259	27	24	21	111
0.0100	10,014	507		11	3	161
0.0200	9,124	395		8	2	166

shadow values of included resources could also be biased downward when resources are omitted. This empirical application also supports the plausibility of the third reason for equivalent solutions of the two models discussed in the theoretical section. If values of y_{11} can vary between equations (5) and (6), many empirical situations could have the tendency to have confounding solutions between Models A and B.

SUMMARY AND CONCLUSIONS

Many risk programming studies that indicate risk aversion is necessary to explain firm behavior have limited resource constraints. Among others, Baker and McCarl argue that in such cases, the importance of risk may be overstated. This paper presented a theoretical and empirical analysis of this issue based on two general programming models with characteristics similar to the empirical models in previous research. In the theoretical analysis, optimality conditions for the models had similar structure except that risk aversion substituted for the effect of the deleted resource constraints. An economic interpretation of the optimality conditions is that positive risk aversion could be necessary because the total marginal cost of all resources is understated since all scarce resources are not included. These theoretical results were illustrated with a simple empirical risk programming model. After two constraints were deleted, risk aversion was necessary to obtain the more completely constrained risk neutral solutions. The magnitude of understatement of marginal costs of resources in the risk averse case was also documented. Thus, this paper provides the theoretical foundation and a numerical illustration of the view that omitted resources constraints may be confounded with risk aversion in many firm risk programming studies.

This paper definitively supports the view of Baker and McCarl that more attention to the constraint set in risk programming models is warranted. Risk programming studies have been so involved with specification of matrices of risk parameters that the constraint sets have often received limited attention. For example, Musser et al. devote one short paragraph to the constraint set compared to nearly two pages to the development of risk parameters. These results suggest that the first step in constructing a risk model is to specify a complete constraint set. This set would include disaggregated resources such as labor and machinery capacity and all appropriate rotation and machine operation sequencing requirements. Examination of profit maximization solutions with such a constraint set may still indicate that risk models are necessary. However, use of models with inadequate resource specification can overstate the importance of risk aversion in explaining firm behavior. Risk aversion may be necessary to explain behavior in some situations; however, attention to appropriate constraints is necessary before this conclusion can be substantiated.

TABLE 5. EXPECTED RETURNS AND MARGINAL COST OF FIXED RESOURCES AND RISK FOR LINEAR PROGRAMMING AND QUADRATIC PROGRAMMING MODELS FOR SOUTH GEORGIA, 1978

		Linear programming			Quadratic programming	
		Marginal cost of fixed resources			Marginal cost of fixed resources and risk	
Crop	Expected returns	Land	Peanut-cotton restriction	Labor Sept.-Oct.	Land	Risk
	dollars					
Soybeans	141.96	12.03	—	129.92	26.25	115.71
Cotton	168.14	12.03	73.88	82.23	26.25	141.89
Peanuts	432.90	12.03	73.88	347.01	26.25	406.65

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