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Homogeneity of Degree 1 Production Functions and the Development of Dual Equilibrium Displacement Models

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Introduction

Equilibrium Displacement Models (EDMs) are frequently used by applied economists for policy analyses because they allow for the quantification of changes in prices and quantities among vertically- and/or horizontally-related markets. Such models are also widely used to estimate both short-term and long-term impacts of exogenous economic shocks and regulatory actions on multiple markets. Because complex interactions often exist among markets, EDMs provide a comprehensive approach to modeling changes in market equilibria.

The use of EDMs has a long, prominent role in applied economic analyses because they provide both modeling flexibility and consistency with basic economic concepts. For example, EDMs are derived from, and represent a set of, comparative static results expressed in elasticity form. One major advantage of such models is that they allow researchers to use elasticities and factor input shares to estimate the relative importance of various supply/demand shifters on market equilibria (i.e., generally price and quantity) outcomes. Moreover, researchers can use elasticity estimates from the extant literature without the need to estimate (often) large systems of supply and demand equations. This frees researchers from concerns related to functional forms, data availability, and other estimation issues.

The flexibility of EDMs is evident in the broad array of issues to which they have been applied. EDMs have been used to evaluate exogenous economic shocks and regulatory actions on the demand for factor inputs (e.g., Allen, 1938; Buse, 1958; Hicks, 1957; Muth, 1964; Wohlgenant, 1989). The models have been widely used to quantify changes in market outcomes that result from a host of market interventions such as trade, policy, and tax legislation (e.g., Alston and Scobie, 1983; Brester and Wohlgenant, 1997; Freebairn, Davis, and Edwards, 1982; Holloway, 1989; Lemieux and Wohlgenant, 1989; Mullen, Alston, and Wohlgenant, 1989; Pendell, et al., 2013; Perrin and Scobie, 1981). The modeling approach has also been used to evaluate the impacts of technological change and research and development investments on market equilibria as well as the efficacy of commodity advertising programs (e.g., Kinnucan, 1999; Lusk and Anderson, 2004; Mullen and Alston, 1994; Perrin, 1980; Piggott, Piggott, and Wright, 1995; Wohlgenant, 1993; Zhao, Anderson, and Wittwer, 2003).

Although hundreds of EDMs have been developed and used, many suffer from a fundamental flaw. Specifically, economic theory suggests that EDMs be homogeneous of degree zero in all input and output prices. That is, homogeneity of degree zero in all prices implies that no output response would occur if all input prices were, say doubled, along with all output prices. The condition is analogous to the concept of a lack of "money illusion" in consumer theory. Homogeneity means that only changes in relative prices (rather than price levels) influence economic choices. In the absence of this requirement, accurate results from EDMs cannot be obtained. It is apparent that many EDMs do not meet this criterion. In fact, the usual source for EDMs is Gardner (1990). However, personal communication with Bruce Gardner in the early 2000s resulted in his concern over the fact that his EDM did not meet this criterion. He encouraged us to remedy the problem through a publication so that researchers who refer to his model could alter it so that homogeneity of degree zero in all prices would be maintained in the system.

EDMs can be developed in a variety of ways. However, most approaches do not result in models that are homogeneous of degree zero in all prices. Although Allen (1932) presents a partial derivation, we have not found a published derivation of internally consistent EDMs in which the primal problem of using underlying, but unobservable, production functions are converted into a dual EDM. The dual version of EDMs are preferred because they obviate the need for specifying production functions.

This paper presents the mathematical development of EDMs that are homogeneous of degree zero in all prices. The approach uses the specification of production technologies that are homogeneous of degree 1 in *inputs*. That is, a dual EDM that is homogeneous of degree zero in all prices can be developed if production technologies are assumed to homogeneous of degree 1 in inputs. This latter homogeneity condition implies that long run production function technologies are constant-returns-to-scale. The condition means that if one, say doubles the quantities of all inputs used in a production process, output would likewise double. In a broad sense, the assumption of long-run production function homogeneity is a logical issue. The general notion of decreasing or increasing returns-to-scale can only happen if one or more fixed inputs cannot be duplicated in the long-run. If such a situation arises, then increasing or decreasing returns-to-scale do not result from the underlying production technology. Rather, they are a function of a factor that is "fixed" in supply to the extent that an increase in its use must increase (or decrease) the input's price in the long-run. EDM models can, however, consider such situations by including highly inelastic own-price factor supply elasticities while still maintaining production technologies that are homogeneous of degree 1 in inputs.

The Primal Model

If the long run production function $q = f(x)$ is homogenous if degree 1 in prices, then the following expressions hold:

(1) $q = f(tx) = tf(x)$

(2) $\boldsymbol{H} = \left[\frac{\partial^2 f}{\partial x_i \partial x_i}\right] = [f_{i,j}]$

(3) $\sum_j f_j(x)x_j = f_x \bullet x = q$

(4) $\sum_j f_{i,j}(x)x_j = \sum_i f_{i,j}(x)x_i = 0 \Rightarrow H(x) \bullet x = 0; x \bullet H(x) = 0; |\boldsymbol{H}| = 0$

A consequence of equations (1)-(4) is that the unconstrained competitive first order conditions (FOC) given by:

(5) $pf_x - w = 0$

have an infinite number of solutions. While the FOCs do not have a unique solution, we assume that they will be satisfied. In this case expressions (1)–(5) imply zero profits for the competitive firm and industry such that

(6) $pf_x \bullet x = w \bullet x \Leftrightarrow pq = w \bullet x.$

Gardner (1990) suggested treating output (q) in equation (1) as exogenous and output price (p) in equation (5) as endogenous as a means for arriving at a unique solution

(7) $a\colon q = f(x)$

$b\colon pf_x - w = 0\ .$

The use of equation (7) can be intuitively justified by considering it as part of a larger simultaneous system:

(8) $a\colon q = q^D(p)$

$b\colon q = f(x)$

$c\colon pf_x - w = 0$

$d\colon x = x^S(w)$

where q is output, p is output price, x is a n-vector of inputs, w is a vector of input prices, q^D is a demand equation, and x^S is a set of factor or input supply functions. This system consists of 2n+2 equations and 2n+2 endogenous variables (q, p, x, and w). While the FOCs of equations (8a-8c) do not have a unique solution when examined in isolation, the implicit function theorem can be used to derive an implicit local solution. For example, begin by taking the total differential of system equation (8)

(9) $a\colon dq = \left(\frac{\partial q^D}{\partial p}\right) dp$

$b\colon dq = f_x dx$

$c\colon f_x dp + pf_{xx'} dx - I\ dw = 0$

$d\colon dx = \left(\frac{\partial x^S}{\partial x}\right) dw\ .$

After matrix row operations and some algebraic manipulation, this system can be shown to be equivalent to the following EDM system

(10) $a\colon E(q) = \eta_D E(p)$

$b\colon E(p) = \sum_j K_j E(w_j)$

$$c\colon\ E(x_i) = E(q) + \sum_j K_j \sigma_{i,j} E(w_j)\ for\ i = 1,2, \dots, n$$

$$d\colon\ E(x_i) = \sum_j K_j \varepsilon_{i,j} E(w_j)\ for\ i = 1,2, \dots, n$$

where $E(\bullet)$ represents percentage changes, η_D is consumers' own-price elasticity of demand, K_j is the expenditure factor share of the j^{th} input, $\sigma_{i,j}$ is the Allen elasticity of factor substitution between inputs *i* and *j*, and $\varepsilon_{i,j}$ is the own- and cross-price elasticities of supply for inputs *i* and *j*.

Allen elasticities of factor substitution are the appropriate metric for such models because (as shown below and by Christev and Featherstone (2007)) a **set** of Allen elasticities contain the complete curvature information of the associated constrained Hessian. Allen elasticities of substitution are inaccurate measures of the substitution between two inputs when those two inputs are considered in isolation. However, when more than one pair of prices change, Allen elasticities account for all simultaneous sets of relative price changes. As demonstrated below, an EDM is obtained using row operations of a differential system of equations which implies that all "signable" results from a comparative statics outcome will be identical to that of an EDM.

Expressions (10a) and (10d) are derived by converting equations (9a) and (9d) into elasticity form. Equation (10a) is obtained by first multiplying equation (9a) through by *1/q*, and then the right-hand side is multiplied by *p/p*. Equation (10d) is obtained by multiplying equation (9d) through by *1/x*, and then the right-hand side is multiplied by *w/w*. The derivation of equations (10b) and (10c) are described below.

Developing an EDM for the Production Sector

Using Cramer's rule, we expand (9b) and (9c) to obtain

$$\text{(11)} \quad \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & pf_{11} & pf_{12} & \cdots & pf_{1n} \\ f_2 & pf_{21} & pf_{22} & \cdots & pf_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & pf_{n1} & pf_{n2} & \cdots & pf_{nn} \end{bmatrix} \begin{bmatrix} dp \\ dx_1 \\ dx_2 \\ \cdot \\ \cdot \\ \cdot \\ dx_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} dq \\ dw_1 \\ dw_2 \\ \cdot \\ \cdot \\ \cdot \\ dw_n \end{bmatrix}.$$

If the cost minimization problem (discussed below) has a solution, the bordered matrix in (11) is nonsingular and, by the implicit function theorem, we can find implicit solutions

(12) $\quad p^* = p^*(q, w)$

and

(13) $\quad x^* = x^*(q, w)$

which implies the following *dual differential system:*

$$\text{(14)} \quad dp^* = \left(\frac{\partial p^*}{\partial q}\right) dq + \sum_j \left(\frac{\partial p^*}{\partial w_j}\right) dw_j$$

$$\text{(15)} \quad dx_i^* = \left(\frac{\partial x_i^*}{\partial q}\right) dq + \sum_j \left(\frac{\partial x_i^*}{\partial w_j}\right) dw_j \,.$$

Because the dual system reduces mathematical complications, we use equation (11) and Cramer's rule to identify expressions for $\left(\frac{\partial p^*}{\partial q}\right)$, $\left(\frac{\partial p^*}{\partial w_j}\right)$, $\left(\frac{\partial x_i^*}{\partial q}\right)$, and $\left(\frac{\partial x_i^*}{\partial w_j}\right)$. The results are used to construct equations (10b) and (10c) in the EDM model.

To identify expressions for $\left(\frac{\partial p^*}{\partial q}\right)$, $\left(\frac{\partial p^*}{\partial w_j}\right)$, $\left(\frac{\partial x_i^*}{\partial q}\right)$, and $\left(\frac{\partial x_i^*}{\partial w_j}\right)$, we will use several results from Allen (1938). Let

$$(16) \qquad \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} = \begin{bmatrix} 0 & f_x' \\ f_x & \boldsymbol{H} \end{bmatrix}.$$

In the following, we denote the borders as row or column 0 and refer to the remaining rows and columns using the indexes 1, ..., n which results in the following definitions

F_0 ≡ the cofactor of the 0 element in matrix ***F***.

$F_{0,ij}$ ≡ the cofactor of the *i,j*'th element in ***F*** after the borders have first been deleted.

$\boldsymbol{F}_{ij}$ ≡ the matrix remaining after the *i*'th row and *j*'th column in ***F*** have been deleted.

Given these definitions, the following results are obtained

$$(17) \qquad \boldsymbol{F}_0 = 0$$

$$(18) \qquad F_{0,ij} = (-1)^{i+j+1} \frac{x_i x_j |F|}{q^2}$$

$$(19) \qquad \frac{|\boldsymbol{F}_{ij}|}{|\boldsymbol{F}|} = (-1)^{i+j} \frac{\sigma_{ij} x_i x_j}{q} = (-1)^{i+j} \frac{K_j \sigma_{ij} x_i}{f_j}.$$

Recall that $\sum_j K_j = 1$ and $\sum_j K_j \sigma_{ij} = 0$.

Proofs of (17), (18) and (19)

<u>Proof of (17):</u>

$\boldsymbol{F}_0 = |\boldsymbol{H}| = 0$ from (4) . *Q.E.D.*

Proof of (18):

(20) $$|\boldsymbol{F}| = \frac{1}{x_i}\begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_i x_i & f_{i1}x_i & f_{i2}x_i & \cdots & f_{in}x_i \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}.$$

Multiplying all rows $h > 0$ *and* $h \neq i$ by x_h and adding the results to row *i* and using equations (3) and (4)

(21) $$|\boldsymbol{F}| = \frac{1}{x_i}\begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ q & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} = (-1)^{i+2}\frac{q}{x_i}\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_{11} & f_{12} & \cdots & f_{1n} \\ \sim_i & \sim_i & \cdots & \sim_i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} =$$

$$(-1)^{i}\frac{q}{x_i}\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_{11} & f_{12} & \cdots & f_{1n} \\ \sim_i & \sim_i & \cdots & \sim_i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

where the notation $\sim_i$ or $\sim_j$ denotes that row i (or column j) has been deleted.

Multiplying column *j* by x_j gives:

(22) $$|\boldsymbol{F}| = (-1)^{i}\frac{q}{x_i x_j}\begin{vmatrix} f_1 & f_j x_j & \cdots & f_n \\ f_{11} & f_{1j}x_j & \cdots & f_{1n} \\ \sim_i & \sim_i & \cdots & \sim_i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{nj}x_j & \cdots & f_{nn} \end{vmatrix}.$$

Multiplying all columns $h > 0$ *and* $h \neq j$ by x_h and adding the results to column *j* using equations (3) and (4)

$$(23) \qquad |\boldsymbol{F}| = (-1)^i \frac{q}{x_i x_i} \begin{vmatrix} f_1 & q & \cdots & f_n \\ f_{11} & 0 & \cdots & f_{1n} \\ \sim_i & \sim_i & \cdots & \sim_i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{n1} & 0 & \cdots & f_{nn} \end{vmatrix} = (-1)^i(-1)^{j+1}\left(\frac{q^2}{x_i x_j}\right) \boldsymbol{F}_{0,ij}\, \lambda$$

giving

$$(24) \qquad \boldsymbol{F}_{0,ij} = (-1)^{i+j+1}\left(\frac{x_i x_j}{q^2}\right)|\boldsymbol{F}| \qquad \textit{Q.E.D.}$$

Proof of (19):

To prove (19) we use the following cost minimization Lagrangian:

$$(25) \qquad L = w \bullet x + \lambda[q - f(x)]$$

with first order conditions:

$$(26) \qquad a\colon\ q - f(x) = 0$$

$$b\colon\ w - \lambda_c f_x = 0\,.$$

Totally differentiating and rearranging gives

$$(27) \qquad \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & \lambda f_{11} & \lambda f_{12} & \cdots & \lambda f_{1n} \\ f_2 & \lambda f_{21} & \lambda f_{22} & \cdots & \lambda f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & \lambda f_{n1} & \lambda f_{n2} & \cdots & \lambda f_{nn} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1^c \\ dx_2^c \\ \cdot \\ \cdot \\ \cdot \\ dx_n^c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} dq \\ dw_1 \\ dw_2 \\ \cdot \\ \cdot \\ \cdot \\ dw_n \end{bmatrix}.$$

In the following, we denote the bordered Hessian in (27) as L^c and the matrix after deleting the i^{th} row and j^{th} column of L^c as L^c_{ij}. Using Cramer's rule we obtain

(28) $$\frac{\partial x_i^c}{\partial w_i} = \frac{(-1)^{i+j+1}\left|L_{ij}^c\right|}{|L_c|} = \frac{(-1)^{i+j}\lambda_c^{n-2}}{\lambda_c^{n-1}}\frac{|F_{ij}|}{|F|} = \frac{(-1)^{i+j}|F_{ij}|}{\lambda_c|F|} .$$

Allen (1938, p. 504) defines the Allen elasticity of substitution, which in our example, as

(29) $$\sigma_{ij} = \frac{\partial x_i^c}{\partial w_j}\frac{w_j}{x_i}\frac{w \bullet x}{w_j x_j} = \frac{(-1)^{i+j}|F_{ij}|}{\lambda_c|F|}\frac{\lambda_c f_x \bullet x}{x_i x_j} = (-1)^{i+j}\frac{f_x \bullet x}{x_i x_j}\frac{|F_{ij}|}{|F|} .$$

If the production function is homogenous of degree 1 in prices (i.e., $f_x \bullet x = q$), then (29) reduces to

(30) $$\sigma_{ij} = (-1)^{i+j}\frac{q}{x_i x_j}\frac{|F_{ij}|}{|F|}$$

or

(31) $$\frac{|F_{ij}|}{|F|} =_{ij}= (-1)^{i+j}\frac{\sigma_{ij} x_i x_j}{q} .$$

Note that:

(32) $$K_j \sigma_{ij} = \frac{\partial x_i^c}{\partial w_j}\frac{w_j}{x_i} = (-1)^{i+j}\frac{|F_{ij}|}{\lambda_c|F|}\frac{\lambda_c f_j}{x_i} = (-1)^{i+j}\frac{f_j}{x_i}\frac{|F_{ij}|}{|F|}$$

giving

(33) $$\frac{|F_{ij}|}{|F|} = (-1)^{i+j}\frac{K_j \sigma_{ij} x_i}{f_j} \qquad Q.E.D.$$

Deriving Optimal Solutions for the Endogenous Variables

Given equations (17), (18), and (19), we can now derive expressions for $\left(\frac{\partial p^*}{\partial q}\right)$, $\left(\frac{\partial p^*}{\partial w_j}\right)$, $\left(\frac{\partial x_i^*}{\partial q}\right)$, and $\left(\frac{\partial x_i^*}{\partial w_j}\right)$. To facilitate the discussion, we restate system (11) as

$$
(34) \qquad \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & pf_{11} & pf_{12} & \cdots & pf_{1n} \\ f_2 & pf_{21} & pf_{22} & \cdots & pf_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & pf_{n1} & pf_{n2} & \cdots & pf_{nn} \end{bmatrix} \begin{bmatrix} dp \\ dx_1 \\ dx_2 \\ \cdot \\ \cdot \\ \cdot \\ dx_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} dq \\ dw_1 \\ dw_2 \\ \cdot \\ \cdot \\ \cdot \\ dw_n \end{bmatrix}.
$$

Denoting the bordered matrix in (34) as $\boldsymbol{H}_p$ and using Cramer's rule we obtain

$$
(35) \qquad \left(\frac{\partial p^*}{\partial q}\right) = \frac{\begin{vmatrix} 1 & f_1 & f_2 & \cdots & f_n \\ 0 & pf_{11} & pf_{12} & \cdots & pf_{1n} \\ 0 & pf_{21} & pf_{22} & \cdots & pf_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & pf_{n1} & pf_{n2} & \cdots & pf_{nn} \end{vmatrix}}{|\boldsymbol{H}_p|} = \frac{p^{n-1}\begin{vmatrix} 1 & f_1 & f_2 & \cdots & f_n \\ 0 & f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}}{p^{n-1}|\boldsymbol{F}|} = \frac{\boldsymbol{F}_0}{|\boldsymbol{F}|} = 0
$$

or

$$
(36) \qquad \left(\frac{\partial p^*}{\partial q}\right) = 0\,.
$$

The symmetry and non-singularity of $\boldsymbol{H}_p$ imply that $\left(\frac{\partial p^*}{\partial w_j}\right) = \left(\frac{\partial x_i^*}{\partial q}\right)$. From (34), we obtain

$$
(37) \qquad \left(\frac{\partial x_j^*}{\partial q}\right) = \frac{\begin{vmatrix} 1 & f_1 & f_2 & \cdots & f_n \\ 0 & pf_{11} & pf_{12} & \cdots & pf_{1n} \\ 0 & pf_{21} & pf_{22} & \cdots & pf_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & pf_{n1} & pf_{n2} & \cdots & pf_{nn} \end{vmatrix}}{|\boldsymbol{H}_p|} = \frac{(-1)^{j+2}\,\frac{p^{n-1}}{x_j}\begin{vmatrix} f_1 & f_{11} & \sim_j & \cdots & f_{1n} \\ f_j x_j & f_{j1} x_j & \sim_j & \cdots & f_{j1n} x_j \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{n1} & \sim_j & \cdots & f_{nn} \end{vmatrix}}{p^{n-1}|\boldsymbol{F}|}.
$$

Adding x_i time rows i to row j for $i \neq j$ and using results (3) and (4) gives

$$
(38) \qquad \left(\frac{\partial x_j^*}{\partial q}\right) = \frac{(-1)^{j+2}\,\frac{p^{n-1}}{x_j}\begin{vmatrix} f_1 & f_{11} & \sim_j & \cdots & f_{1n} \\ q & 0 & \sim_j & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{n1} & f_{n1} & \sim_j & \cdots & f_{nn} \end{vmatrix}}{p^{n-1}|\boldsymbol{F}|} = (-1)^{j+2}\,(-1)^{j+1}\,\frac{q\boldsymbol{F}_{0,jj}}{x_j|\boldsymbol{F}|} =
$$

$$
(-1)^{2j+1}\,\frac{q\boldsymbol{F}_{0,jj}}{x_j|\boldsymbol{F}|}\,.
$$

Upon substituting for $\boldsymbol{F}_{0,jj}$, we obtain

(39) $\left(\frac{\partial x_j^*}{\partial q}\right) = (-1)^{2j+1}\,(-1)^{2j+1}\,\frac{q}{x_j}\frac{x_j x_j}{q^2}\frac{|F|}{|F|} = \frac{x_j}{q}$

which results in

(40) $\left(\frac{\partial x_j^*}{\partial q}\right) = \frac{x_j}{q}$

and, by symmetry

(41) $\left(\frac{\partial p^*}{\partial w_j}\right) = \frac{x_j}{q}$.

From expression (34)

(42) $$\left(\frac{\partial x_i^*}{\partial w_j}\right) = \frac{\begin{vmatrix} 0 & f_1 & f_2 & 0 & \cdots & f_n \\ f_1 & pf_{11} & pf_{12} & 0 & \cdots & pf_{1n} \\ f_j & pf_{j1} & pf_{22} & 1 & \cdots & pf_{2n} \\ f_3 & pf_{31} & pf_{32} & 0 & \cdot & pf_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & pf_{n1} & pf_{n2} & 0 & \cdots & pf_{nn} \end{vmatrix}}{p^{(n-1)}\,|F|} = \frac{(-1)^{i+j+2}\begin{vmatrix} 0 & f_1 & f_2 & \sim_i & \cdots & f_n \\ f_1 & pf_{11} & pf_{12} & \sim_i & \cdots & pf_{1n} \\ \sim_j & \sim_j & \sim_j & \sim_{ji} & \cdots & pf_{2n} \\ f_3 & pf_{31} & pf_{32} & \sim_i & \cdot & pf_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & pf_{n1} & pf_{n2} & \sim_i & \cdots & pf_{nn} \end{vmatrix}}{p^{n-1}|F|} =$$

$$(-1)^{i+j}\,\frac{p^{n-2}\,|F_{ij}|}{p^{n-1}|F|}\ .$$

Substituting using (33) and (5) gives

(43) $\left(\frac{\partial x_i^*}{\partial w_j}\right) = (-1)^{i+j}\,(-1)^{i+j}\,\frac{K_j \sigma_{ij} x_i}{pf_1}$

or

(44) $\left(\frac{\partial x_i^*}{\partial w_j}\right) = \frac{K_j \sigma_{ij} x_i}{w_j}$.

Substituting (36), (40), (41), and (44) into (14) and (15) gives

(45) $dp = \sum_j \frac{x_j}{q} dw_j$

and

(46) $dx_i = \frac{x_i}{q} dq + \sum_j \frac{K_j \sigma_{ij} x_i}{w_j} dw_j$.

Expressions (45) and (46) can be converted to an EDM as

(47) $\frac{dp}{p} = \sum_j \frac{w_j x_j}{pq}\frac{dw_j}{w_j} = \sum_j \frac{w_j x_j}{w \bullet x}\frac{dw_j}{w_j} = \sum_j K_j \frac{dw_j}{w_j}$

or

(48) $E(p) = \sum_j K_j E(w_j)$

and

(49) $\frac{dx_i}{x_i} = \frac{1}{q}dq + \sum_j \frac{K_j \sigma_{ij}}{w_j} dw_j = \frac{dq}{q} + \sum_j K_j \sigma_{ij} \frac{dw_j}{w_j}$

or

(50) $E(x_i) = E(q) + \sum_j K_j \sigma_{i,j} E(w_j)$.

The result of this exercise is that the EDM approximation of a "jointly enforced" system (7) can be written as

(51) $q = f(x) \approx E(p) = \sum_j K_j E(w_j)$

$pf_x - w = 0 \approx E(x_i) = E(q) + \sum_j K_j \sigma_{i,j} E(w_j)$ for $i = 1, \ldots, n$.

Consequently, the comparative statics and effects of policy changes on a market that includes consumers, producers, and factor suppliers can be approximated using an EDM of the form

(52) $a\!: \; q = q^D(p) \approx \quad E(q) = \eta_D E(p)$

$b\!: \; p = f(x) \approx \quad E(p) = \sum_j K_j E(w_j)$

$c\!: \; pf_x - w = 0 \approx E(x_i) = E(q) + \sum_j K_j \sigma_{i,j} E(w_j) \;\; for\ i = 1,2, \ldots, n$

$d\!: \; x = x^S(w) \approx \quad E(x_i) = \sum_j K_j \varepsilon_{i,j} E(w_j) \;\; for\ i = 1,2, \ldots, n$.

Comparing Models with Respect to Homogeneity of Degree Zero in All Prices

Many authors have developed EDM models of market relationships by directly considering the demand and supply for a good or service at one market level, and the demand and supply of a major input that is produced at another market level (Brester, Marsh, and Atwood; Pendell, et al., 2010; Pendell, et al., 2013). However, most of these models do meet the requirement that the resulting system of equations is homogeneous of degree zero in all input and output prices. To illustrate the errors caused by ignoring this condition, we construct a "traditional" EDM model and estimate impacts from an exogenous shock. Then, we develop an EDM based on the above derivation that is homogeneous of degree zero in all prices. The exercise clearly indicates the importance of maintaining this theoretically consistent condition.

A Traditional Supply/Demand Function Approach

Although EDMs have been developed using various approaches, a common method is to specify general demand and supply functions, which are then totally differentiated and converted to an EDM form. For example, consider the demand and supply of finished retail beef products. The consumer demand for the product could be represented by

(53) $q_d = f_1(p^d) + \varphi_1$

where q_d is the quantity of retail beef consumed, and p^d is the consumer price of retail beef. Thus, equation (53) represents the primary demand level.

Food processors slaughter and convert live cattle into finished beef products. The supply of which could be presented by:

(54) $q_s = f_2(p^s, x_d) + \varphi_2$

where q_s is the quantity of retail beef produced, and p^s is the producer price of retail beef. Thus, equation (53) represents the derived supply level (Tomek and Robinson, 1990).

These same processors demand live cattle (the major factor input into producing retail beef) that are produced by feedlot enterprises

(55) $x_d = f_3(w^d, q_d) + \varphi_3$

where x_d is the quantity of live cattle demanded by processors, and w^d is the factor demand price of live cattle. Equation (55) is referred to as the derived demand for beef.

Finally, feedlot enterprises supply live cattle to the processing sector as indicated by

(56) $x_s = f_4(w^s) + \varphi_4$

where x_s is the quantity of live cattle supplied by feedlots, and w^s is the factor supply price of live cattle. This supply function is often termed the primary supply function. In addition, the following equilibrium conditions are assumed

(57) $q_d = q_s = q$

(58) $p^d = p^s = p$

(59) $x_d = x_s = x$

(60) $w^d = w^s = w$

so that equations (53)-(56) can be written as

(61) $q = f_1(p) + \varphi_1$

(62) $q = f_2(p, x) + \varphi_2$

(63) $x = f_3(w, q) + \varphi_3$

(64) $x = f_4(w) + \varphi_4$.

Totally differentiating equations (61)-(64) yields

(65) $$dq = \frac{dq}{dp}dp + d\varphi_1$$

(66) $$dq = \frac{dq}{dp}dp + \frac{dq}{dx}dx + d\varphi_2$$

(67) $$dx = \frac{dx}{dw}dw + \frac{dx}{dq}dq + d\varphi_3$$

(68) $$dx = \frac{dz}{dw}dw + d\varphi_4 .$$

Dividing equations (65) and (68) by q, and equations (66) and (67) by x yields

(69) $$\frac{dq}{q} = \left(\frac{1}{q}\right)\frac{dq}{dp}dp + \left(\frac{1}{q}\right)d\varphi_1$$

(70) $$\frac{dq}{q} = \left(\frac{1}{q}\right)\frac{dq}{dp}dp + \left(\frac{1}{q}\right)\frac{dq}{dx}dx + \left(\frac{1}{q}\right)d\varphi_2$$

(71) $$\frac{dx}{x} = \left(\frac{1}{x}\right)\frac{dx}{dw}dw + \left(\frac{1}{x}\right)\frac{dx}{dq}dq + \left(\frac{1}{x}\right)d\varphi_3$$

(72) $$\frac{dx}{x} = \left(\frac{1}{x}\right)\frac{dx}{dw}dw + \left(\frac{1}{x}\right)d\varphi_4 .$$

Multiplying the first term on the right-hand side of equation (69) by $\frac{p}{p}$ results in

(73) $$\frac{dq}{q} = \left(\frac{1}{q}\right)\left(\frac{p}{p}\right)\frac{dq}{dp}dp + \left(\frac{1}{q}\right)d\varphi_1 .$$

Rearranging yields

(74) $$\frac{dq}{q} = \left(\frac{dq}{dp}\frac{p}{q}\right)\left(\frac{dp}{p}\right) + \frac{d\varphi_1}{q}$$

or

(75) $$E(q) = \eta^q E(p) + E(\varphi_1)$$

where η^q is the own-price elasticity of demand for the retail product.

Next, multiply the first term on the right-hand side of equation (70) by $\frac{p}{p}$ and the second term by $\frac{x}{x}$ to yield

(76) $\frac{dq}{q} = \left(\frac{1}{q}\right)\left(\frac{p}{p}\right)\frac{dq}{dp}dp + \left(\frac{1}{q}\right)\left(\frac{x}{x}\right)\frac{dq}{dx}dx + \left(\frac{1}{q}\right)d\varphi_2$.

Rearranging yields

(77) $\frac{dq}{q} = \left(\frac{dq}{dp}\frac{p}{q}\right)\frac{dp}{p} + \left(\frac{dq}{dx}\frac{x}{q}\right)\frac{dx}{x} + \frac{d\varphi_2}{q}$

or

(78) $E(q) = \varepsilon^q E(p) + \tau E(x) + E(\varphi_2)$.

where ε^q is the own-price derived supply elasticity of retail beef products, and τ represents a elasticity of quantity transmission between the live cattle sector and the finished beef product sector.

The first term on the right-hand side of equation (71) is then multiplied by $\frac{w}{w}$ and the second term by $\frac{q}{q}$

(79) $\frac{dx}{x} = \left(\frac{1}{x}\right)\left(\frac{w}{w}\right)\frac{dx}{dw}dw + \left(\frac{1}{x}\right)\left(\frac{q}{q}\right)\frac{dx}{dq}dq + \left(\frac{1}{x}\right)d\varphi_3$.

Rearranging yields

(80) $\frac{dx}{x} = \left(\frac{dx}{dw}\frac{w}{x}\right)\frac{dw}{w} + \left(\frac{dx}{dq}\frac{q}{x}\right)\frac{dq}{q} + \frac{d\varphi_3}{x}$

or

(81) $E(x) = \eta^{x_1} E(w) + \gamma E(q) + E(\varphi_3)$

where $\eta^{x_1}$is the own-price derived demand elasticity for live cattle and γ represents a elasticity of quantity transmission between the retail finished beef product sector and the live cattle sector.

Finally, multiply the first term on the right-hand side of equation (72) by $\frac{w}{w}$ yields

(82) $\frac{dx}{x} = \left(\frac{1}{x}\right)\left(\frac{w}{w}\right)\left(\frac{dx}{dw}\right)dw + \frac{d\varphi_4}{x}$.

Rearranging yields

(83) $\frac{dx}{x} = \left(\frac{dx}{dw}\frac{w}{x}\right)\frac{dw}{w} + \frac{d\varphi_4}{x}$

Or

(84) $E(x) = \varepsilon^{x_1} E(w) + E(\varphi_4)$

where ε^{x_1} is the own-price primary supply elasticity of live cattle.

Collecting equations (75), (78), (81), and (84) results in an EDM model of the form

(85)	$E(q) = \eta^q E(p) + E(\varphi_1)$	Primary Retail Demand
(86)	$E(q) = \varepsilon^q E(p) + \tau E(x) + E(\varphi_2)$	Derived Retail Supply
(87)	$E(x) = \eta^{x_1} E(w) + \gamma E(q) + E(\varphi_3)$	Derived Feedlot Demand
(88)	$E(x) = \varepsilon^{x_1} E(w) + E(\varphi_4)$	Primary Feedlot Supply

Moving the endogenous variables to the left-hand side yields

(89) $E(q) - \eta^q E(p) = E(\varphi_1)$

(90) $E(q) - \varepsilon^q E(p) - \tau E(x) = E(\varphi_2)$

(91) $E(x) - \eta^{x_1} E(w) - \gamma E(q) = E(\varphi_3)$

(92) $E(x) - \varepsilon^{x_1} E(w) = E(\varphi_4)$.

In matrix notation, equations (89)-(92) are written as:

(93) $$\begin{bmatrix} 1 & -\eta^q & 0 & 0 \\ 1 & -\varepsilon^q & -\tau & 0 \\ -\gamma & 0 & 1 & -\eta^{x_1} \\ 0 & 0 & 1 & -\varepsilon^{x_1} \end{bmatrix} \begin{bmatrix} E(q) \\ E(p) \\ E(x) \\ E(w) \end{bmatrix} = \begin{bmatrix} E(\varphi_1) \\ E(\varphi_2) \\ E(\varphi_3) \\ E(\varphi_4) \end{bmatrix}.$$

The parametrization of equation (93) requires estimates of η^q, ε^q, η^{x_1}, ε^{x_1}, τ, and γ. However, to illustrate some of the shortcomings of this approach, assume that the values are $\eta^q = \eta^{x_1} = -1.0$, and $\varepsilon^q = \varepsilon^{x_1} = \tau = \gamma = 1.0$ such that

$$(94) \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E(q) \\ E(p) \\ E(x) \\ E(w) \end{bmatrix} = \begin{bmatrix} E(\varphi_1) \\ E(\varphi_2) \\ E(\varphi_3) \\ E(\varphi_4) \end{bmatrix}.$$

Note that there are no behavioral equations in the EDM that specify input price (w) or output price (p). Hence, a direct test of the homogeneity of degree 0 in all prices cannot be conducted. While it is possible to add additional equations to the EDM to conduct such a test, it is sufficient to simply show the shortcomings of the model caused by the lack of homogeneity of degree zero in all prices.

Assume that an exogenous shock in feed markets increases the costs of feeding cattle by 10%. This would be represented by $E(\varphi_4) = -0.10$. Solving equation (94) for the endogenous variables results in

$$(95) \quad \begin{bmatrix} E(q) \\ E(p) \\ E(x) \\ E(w) \end{bmatrix} = \begin{bmatrix} 0.667 & 0.667 & 0.333 & 0.333 \\ 0.333 & -0.667 & -0.333 & -0.333 \\ 0.333 & 0.333 & 0.667 & 0.667 \\ 0.333 & 0.333 & 0.667 & -0.333 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.10 \end{bmatrix}$$

or

$$(96) \quad \begin{bmatrix} E(q) \\ E(p) \\ E(x) \\ E(w) \end{bmatrix} = \begin{bmatrix} -0.033 \\ 0.033 \\ -0.067 \\ 0.033 \end{bmatrix}.$$

The results indicate that retail quantity has declined by 3.3%, while retail prices have increased by 3.3%. This is a consistent result given the assumed demand and supply elasticities being -1.0 and 1.0, respectively. The model also indicates that the price of live cattle inputs has increased by 3.3%, but that the quantity of live cattle being produced declines by 6.7%. However, the assumed price and quantity transmission elasticities of 1.0 should cause the system to add up. That is, the quantity changes at the feedlot level should be equivalent to the quantity changes occurring at the consumer level given that the model does not allow for variable input proportions between live cattle and other inputs (which do not appear in the model) in the production of retail beef. Clearly, the result cannot be correct given that amount of live cattle being produced has declined by 6.7%, while the quantity of retail beef has declined by only 3.3%.

An EDM that is Homogeneous of Degree Zero in All Prices

The EDM developed in equation (52) for a single output (retail beef) and two inputs (live cattle, x_1, and all other processing inputs, x_2, can be written as:

(97) $$E(q) = \eta^q E(p) + E(\theta_1)$$

(98) $$E(p) = K_1 E(w_1) + K_2 E(w_2) + E(\theta_2)$$

(99) $$E(x_1) = E(q) + K_1\sigma_{11}E(w_1) + K_2\sigma_{12}E(w_2) + E(\theta_3)$$

(100) $$E(x_2) = E(q) + K_1\sigma_{21}E(w_1) + K_2\sigma_{22}E(w_2) + E(\theta_4)$$

(101) $$E(x_1) = \varepsilon^{x_1} E(w_1) + E(\theta_5)$$

(102) $$E(x_2) = \varepsilon^{x_2} E(w_2) + E(\theta_6)$$

where q is the quantity demanded of retail beef, p is the retail price of beef, $w_1$is the input factor price of live cattle, w_2 is the input factor price of all other processing inputs cattle, η^q is the own-price elasticity of demand for retail beef, ε^{x_1} is the own-price elasticity of supply of factor 1, ε^{x_2} is the own-price elasticity of supply of factor 2, K_j represent factor cost shares ($K_j = \left(\frac{w_j x_j}{wx}\right)$) such that $\sum_j K_j = 1$, and σ_{ij} is the Allen elasticity of substitution between factors *i* and *j*. Silberberg (1990) notes that $\sum_j K_j \sigma_{ij} = 0$ is necessary to ensure that the system of equations is homogeneous of degree 0 in input and output prices.

For the following, we use the simplifying assumptions that factor input supply quantities are functions of only their own-factor prices rather than influenced by the price of the other factor in the system. It seems reasonable to assume that the impact of the price of all processing inputs (w_2) would have a *de minimis* influence on the supply of live cattle (x_1) and vice versa.

Equation (97) represents retail demand, while equations (101) and (102) represent input supply functions. Equations (98)-(100) represent the production technologies derived from the first order conditions of profit maximization.

To test if the model presented in equations (97)-(102) is homogeneous of degree zero in input and output prices, we must first alter the model to allow for price wedges to exist between demand and supply output and input prices. Hence the model is rewritten as

(103) $$E(q) = \eta^q E(p^d) + E(\theta_1)$$

(104) $$E(p^s) = K_1 E(w_1^d) + K_2 E(w_2^d) + E(\theta_2)$$

(105) $$E(x_1) = E(q) + K_1\sigma_{11} E(w_1^d) + K_2\sigma_{12} E(w_2^d) + E(\theta_3)$$

(106) $$E(x_2) = E(q) + K_1\sigma_{21} E(w_1^d) + K_2\sigma_{22} E(w_2^d) + E(\theta_4)$$

(107) $$E(x_1) = \varepsilon^{x_1} E(w_1^s) + E(\theta_5)$$

(108) $$E(x_2) = \varepsilon^{x_2} E(w_2^s) + E(\theta_6)$$

(109) $$E(p^d) = E(p^s) + E(\theta_7)$$

(110) $$E(w_1^d) = E(w_1^s) + E(\theta_8)$$

(111) $$E(w_2^d) = E(w_2^s) + E(\theta_9)$$

where the superscript *d* represents the demand price for retail beef or the factor inputs, and the superscript *s* represents the supply price for retail beef or the factor inputs.

Equation (109) allows for a price wedge to exist between the demand and supply price of retail beef, while equations (110) and (111) allows for a price wedge between the demand and supply prices of the two factor inputs. In equilibrium and assuming the absence of taxes or subsidies, there would be no difference between demand and supply prices which would make these equations superfluous. However, equations (109)-(111) allow for the testing of homogeneity of degree zero across input and output prices.

The model is operationalized by moving the endogenous variables of equations (103)-(111) to the left-hand side

(112) $$E(q) - \eta^q E(p^d) = E(\theta_1)$$

(113) $$E(p^s) - K_1 E(w_1^d) - K_2 E(w_2^d) = E(\theta_2)$$

(114) $$E(x_1) - E(q) - K_1\sigma_{11} E(w_1^d) - K_2\sigma_{12} E(w_2^d) = E(\theta_3)$$

(115) $$E(x_2) - E(q) - K_1\sigma_{21} E(w_1^d) - K_2\sigma_{22} E(w_2^d) = E(\theta_4)$$

(116) $$E(x_1) - \varepsilon^{x_1} E(w_1^s) = E(\theta_5)$$

(117) $$E(x_2) - \varepsilon^{x_2} E(w_2^s) = E(\theta_6)$$

(118) $$E(p^d) - E(p^s) = E(\theta_7)$$

(119) $$E(w_1^d) - E(w_1^s) = E(\theta_8)$$

(120) $$E(w_2^d) - E(w_2^s) = E(\theta_9)\ .$$

Putting equations (112)-(120) into matrix notation yields

(121) $$\boldsymbol{Ay} = \boldsymbol{x}$$

where $\boldsymbol{A}$ is a *9x9* matrix of parameters, $\boldsymbol{y}$ is a *9x1* vector of endogenous variables, and $\boldsymbol{x}$ is a *9x1* vector of exogenous shocks such that

$$
(122)\quad \begin{bmatrix} 1 & -\eta^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -K_1 & -K_2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -K_1\sigma_{11} & -K_2\sigma_{12} & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -K_1\sigma_{21} & -K_2\sigma_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\varepsilon_2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} E(q) \\ E(p^D) \\ E(p^S) \\ E(x_1) \\ E(x_2) \\ E(w_1^D) \\ E(w_2^D) \\ E(w_1^S) \\ E(w_2^S) \end{bmatrix}
$$

$$
= \begin{bmatrix} E(\theta_1) \\ E(\theta_2) \\ E(\theta_3) \\ E(\theta_4) \\ E(\theta_5) \\ E(\theta_6) \\ E(\theta_7) \\ E(\theta_8) \\ E(\theta_9) \end{bmatrix}.
$$

After parameterizing the A matrix, the system's endogenous variables can be solved for any exogenous shock (x) as

$$
(123)\qquad y = A^{-1}x\,.
$$

For purposes of comparison, we parameterize the model by setting the retail demand elasticity equal to -1.0, and the factor supply elasticities equal to 1.0. In addition, we set the elasticities of substitution equal to zero ($\sigma_{12} = \sigma_{21} = 0.0$), and assume that the factor share of live cattle (K_1) equals 0.90 and the factor share of all other processing inputs (K_2) equals 0.10. Although the terms σ_{11} and σ_{22} have no economic meaning as elasticities of substitution, they must be included in the model if the system of equations is to be homogenous of degree zero in all prices (Silberberg). Hence, these values are calculated as $\sigma_{11} = -\frac{K_2 * \sigma_{12}}{K_1} = -\frac{0.30*0.0}{0.90} = 0.0$, and $\sigma_{22} = -\frac{K_1 * \sigma_{21}}{K_2} = -\frac{0.90*0.0}{0.10} = 0.0$.

Thus, the A matrix in equation (122) is parameterized as

$$(124)\quad \begin{bmatrix} 1 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.90 & -0.10 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0.0 & 0.0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0.0 & 0.0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1.0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Within any economic system, the scalar multiplication of all input and output prices should have no effect upon the quantities of inputs used or output produced within the system such that:

$$(125)\quad Pf_x - w = 0 \equiv f_x = \left(\frac{1}{P}\right) w = \left(\frac{1}{tP}\right) tw = t^o \left(\frac{1}{P}\right) w$$

for any scalar t. This implies that equal percentage increases in factor prices coupled with an identical percentage increase in producer output price should have no impact on equilibrium quantities.

To test this for a doubling of factor prices (i.e., a 100% increase), allow $E(\theta_8) = 1.00$ and $E(\theta_9) = 1.00$. The represents a tax on each factor so that the demand price of each factor will be greater than its supply price. Hence, the input price wedges are entered as positive numbers. Furthermore, to test for homogeneity of degree zero in all prices, a consumer output price subsidy of 100% is simultaneously modeled by allowing $E(\theta_7) = -1.00$. That is, the output supply price that producers receive will be 100%

larger than the output demand price that consumers must pay as a result of the subsidy. Thus, the $\boldsymbol{x}$ vector becomes

(126)
$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.00 \\ 1.00 \\ 1.00 \end{bmatrix}.$$

Using the matrices indicated in equations (124) and (126), the solution to equation (123) is

(127)
$$\boldsymbol{y} = \begin{bmatrix} E(q) \\ E(p^D) \\ E(p^S) \\ E(x_1) \\ E(x_2) \\ E(w_1^D) \\ E(w_2^D) \\ E(w_1^S) \\ E(w_2^S) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00 \\ 0 \\ 0 \\ 1.00 \\ 1.00 \\ 0 \\ 0 \end{bmatrix}.$$

The results show that a doubling of factor input prices and output price causes a 100% increase in input factor demand prices (a doubling) and a 100% increase in the output producer price. However, no changes in factor use or output production occurs. The illustration of this effect is presented in figures 1-3. Figure 1 illustrates the initial demand ($D^{x_1^0}$) and supply ($S^{x_1^0}$) of factor 1 and initial equilibrium point of $w_1^{d_0}$ and x_1^0. At this point, there is no difference between the demand and supply prices of factor 1. Hence, $w_1^{d_0} = w_1^{s_0}$ prior to the implementation of a tax on factor 1. However, if a tax were used to double the factor price of input 1 (w_1), the effect can be visualized as an upward shift in the supply of factor 1 from $S^{x_1^0}$ to $S^{x_1^1}$. This increases the factor demand price for input 1 and reduces the factor supply price.

Figure 2 shows the effect of a simultaneous subsidy on the price of the output as a shift of the demand curve upward and to the right from D^0 to D^1. Figure 3 illustrates that a 100% tax on factor input 2 (doubling its price) would reduce the amount of the factor used. But, the increase in "demand" caused by the subsidy increases the demand for factor 1 (from $D^{x_1^0}$ to $D^{x_1^1}$) and factor 2 (from $D^{x_2^0}$ to $D^{x_2^1}$). Consequently, the amount of each factor used does not change so that $x_1^0 = x_1^1$ and $x_2^0 = x_2^1$. However, because the demand prices for both factors are now higher, the output supply function declines from S^0 to S^1. The ultimate effect in the output market is that the price producers receive for their production is P_S^1 while the price that consumers pay for that production is P_D^1. However, no change in output occurs.

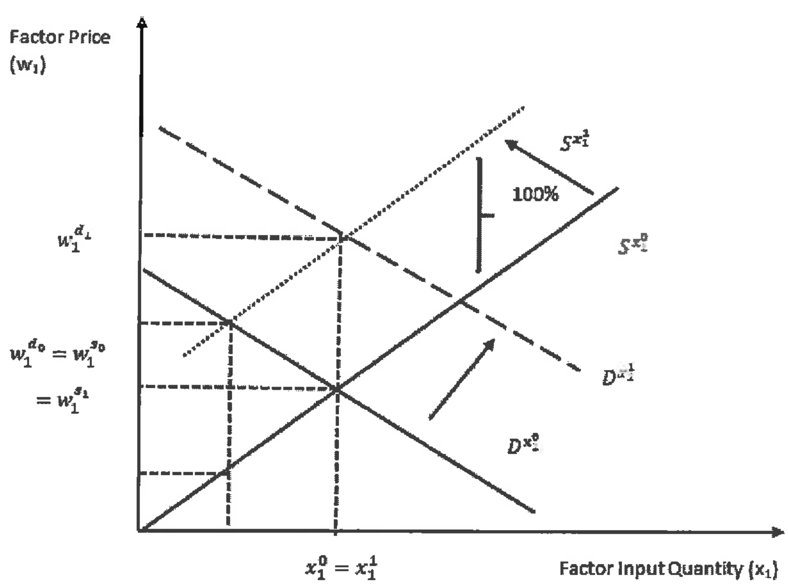

Figure 1. Factor 1 Homogeneity of Degree Zero in Prices

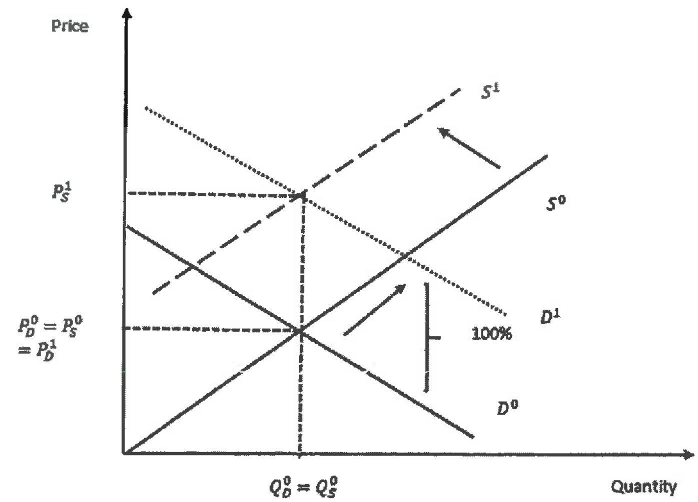

Figure 2. Output Market and Homogeneity of Degree Zero in Prices

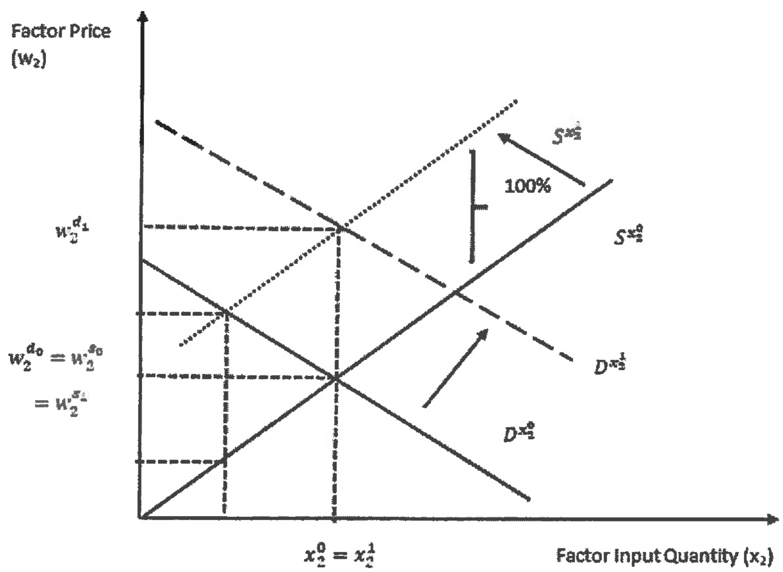

Figure 3. Factor 2 Homogeneity of Degree Zero in Prices

A 10% Shock to Factor 1

Given that the model in equations (103)-(111) is homogeneous of degree zero in all prices, we now compare the results for a 10% increase in the cost of producing live cattle to the result produced from the above *ad hoc* EDM. Recall that equilibrium equations

(109)-(111) were added to the original EDM for the purpose of testing for homogeneity of degree 0 in all prices. Those equations could be used to estimate the effects of various legislative policies such as taxes on inputs or an intervention in the output market. However, the equations are not needed to estimate the effects of a 10% increase in the cost of factor 1. For ease of illustration, we consider the cost increase using only equations (103)-(108). After removing the superscripts and applying the same parameters as those in equation (124), the matrix form of the EDM is given by

$$(128) \qquad \begin{bmatrix} 1 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -0.90 & -0.10 \\ -1 & 0 & 1 & 0 & 0.0 & 0.0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} E(q) \\ E(p) \\ E(x_1) \\ E(x_2) \\ E(w_1) \\ E(w_2) \end{bmatrix} = \begin{bmatrix} E(\theta_1) \\ E(\theta_2) \\ E(\theta_3) \\ E(\theta_4) \\ E(\theta_5) \\ E(\theta_6) \end{bmatrix}.$$

A 10% increase in the cost of factor 1 (live cattle) is modeled as $E(\theta_5) = -0.10$ in equation (107). Thus, the $\boldsymbol{x}$ vector becomes

$$(129) \qquad \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.10 \\ 0 \end{bmatrix}.$$

Using the parameterization in equation (124) and the shock presented in equation (129), equation (123) results in

$$(130) \qquad \boldsymbol{y} = \begin{bmatrix} E(q) \\ E(p) \\ E(x_1) \\ E(x_2) \\ E(w_1) \\ E(w_2) \end{bmatrix} = \begin{bmatrix} -0.045 \\ 0.045 \\ -0.045 \\ -0.045 \\ 0.055 \\ -0.045 \end{bmatrix}.$$

The equilibrium quantity of output at the retail level ($E(q)$) decreases 4.5%, and retail prices ($E(p)$) increase by 4.5%. The two values are identical in absolute value because of the assumed supply and demand elasticities of 1.0 and -1.0, respectively. The earlier model which was not homogeneous of degree zero in all prices estimated that the retail level quantity would decline by 3.3% and that the price would increase by 3.3%.

In addition, it might seem that a 10% reduction in the supply of a factor should cause a 5% increase in output price and a 5% decrease in output quantity given the assumed unitary elasticities. However, equation (104) illustrates that the change in price will be less than the expected 5% value as long as both inputs are used in some amount. This occurs even if input substitution (as in our case) is not allowed. Parenthetically, the "expected" 5% increase in price and decrease in quantity would occur if the supply of *both* factors is reduced by 10%.

Equation (130) also indicates that the decrease in the supply of factor 1 would decrease the quantity demanded of factor 1 by 4.5%. The earlier model predicted a decline of 6.7%. The price of factor 1 is estimated to increase by 5.5%, while the earlier model predicted a 3.3% increase. Once again, it may appear that the price of factor 1 should increase by the same percentage (4.5%) as the reduction in the quantity demanded of the factor given the assumed unitary elasticities. However, equation (108) indicates that the difference between $E(x_1)$ and $1.0 * E(w_1)$ must equal -10%. Given that $E(x_1)$ equals -4.5%, the value of $E(w_1)$ must equal 5.5%.

The results also indicate that the demand for factor 2 decreases by 4.5%. This is identical to the reduction in the quantity demanded for factor 1 because of the assumption of no input substitution between the factors and the reduction in output of 4.5%. In addition, the price of factor 2 decreases by 4.5% because of the decreased demand for the factor. Again, the change in factor demand is identical to the reduction in factor price because of the assumed unitary elasticities.

Summary

Equilibrium Displacement Models (EDMs) are used by applied economists for policy analyses in both vertically- and horizontally-related markets. Because complex interactions exist among many markets, EDMs provide a comprehensive approach to modeling changes in market equilibria. They are also often used to quantify impacts of a variety of economic and regulatory exogenous shocks and resulting changes in producer and consumer surplus.

Various approaches have been used to change a primal system of equations into a dual model that consists of linear approximations to unknown demand and supply functional forms. However, many EDMs do not meet the primary economic principle of homogeneity of degree zero in input and output prices. Hence, results from such models cannot be accurate.

We present a mathematical derivation of EDMs based on the assumption that production technologies are homogeneous of degree 1 in input quantities. The approach converts primal economic functions into their dual counterparts and results in empirical EDMs that are homogeneous of degree zero in all input and output prices.

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