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FLEXIBLE LEAST SQUARES FOR  
APPROXIMATELY LINEAR SYSTEMS\*

R. KALABA and L. TEFATSION

MRG WORKING PAPER #M8926

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## ABSTRACT

The problem of filtering and smoothing for a system described by approximately linear dynamic and measurement relations has been studied for many decades. Yet the potential problem of misspecified dynamics, which makes the usual probabilistic assumptions involving normality and independence questionable at best, has not received the attention it merits. This paper proposes a probability-free multicriteria "flexible least squares" filter which meets this misspecification problem head on. A Fortran program implementation is provided for this filter, and references to simulation and empirical results are given. Although there are close connections with the standard Kalman filter, there are also important conceptual and computational distinctions. The Kalman filter, relying on probability assumptions for model discrepancy terms, provides a unique estimate for the state sequence. In contrast, the flexible least squares filter provides a family of state sequence estimates, each of which is vector-minimally incompatible with the prior dynamical and measurement specifications.

## I. INTRODUCTION

Following World War II, probabilistic methods attained a dominant position in filtering and smoothing theory [1]. Early studies focused on linear system identification problems arising in radar and communications for which the theoretical specifications were essentially correct, and for which model discrepancy terms were reasonably modelled as random quantities with known distributions. For such problems, probabilistic methods could credibly be used to construct scalar measures for theory and data incompatibility in the form of likelihood or posterior distribution functions.

More recently, however, the social and biological sciences have presented filtering and smoothing problems of critical importance for which the processes of interest are highly nonlinear and poorly understood. In attempting to apply standard filtering and smoothing techniques to such a problem, a data analyst typically has to replace the unknown nonlinear process relations with an approximate system of linear relations. The resulting model discrepancy terms then incorporate model specification errors from various conceptually distinct sources—e.g., imperfectly specified measurements versus imperfectly specified state dynamics; hence it is questionable whether these discrepancy terms are either jointly or separately governed by meaningful probability relations. More generally, it is difficult to provide any credible way to scale and weigh the discrepancy terms relative to one another.

In decision theory, incommensurability of this type is typically handled by multicriteria optimization techniques [2]. However, such techniques have not yet been exploited systematically in state estimation theory. Rather, currently available filtering and smoothing techniques *require* the data analyst to provide probability assessments for all discrepancy terms. In consequence, social and biological scientists attempting to apply these techniques are often forced to resort to conventional probability specifications such as normality and independence which may have little public credibility.

This paper proposes a probability-free multicriteria filter for the estimation of ap-

proximately linear dynamical systems. Briefly stated, this "flexible least squares" (FLS) filter solves the following multicriteria optimization problem: Characterize the set of all state sequence estimates which achieve vector-minimal incompatibility between imperfectly specified linear theoretical relations and process observations.

The FLS filtering and smoothing problem for approximately linear dynamical systems is set out in Section II. The FLS recurrence relations for the solution of this problem are derived in Section III. Section IV considers the relationship between FLS and Kalman filtering. Concluding remarks are given in Section V. A Fortran program *GFLS* which implements the FLS recurrence relations for this application is provided in an appendix.

## II. THE BASIC PROBLEM

Consider a system whose state at time  $t$ ,  $t = 1, 2, \dots$ , is an  $n$ -dimensional vector  $x_t$ . It is believed that the state transition equations for the system take the approximately linear form

$$x_{t+1} \approx F(t)x_t + a(t), \quad t = 1, 2, \dots, \quad (1)$$

where  $F(t)$  is a known  $n \times n$  square matrix, and  $a(t)$  is a known  $n$ -dimensional column vector. At each time  $t$ , an  $m$ -dimensional vector  $y_t$  of observations is obtained. The measurement relations are assumed to take the approximately linear form

$$y_t \approx H(t)x_t + b(t), \quad t = 1, 2, \dots, \quad (2)$$

where  $H(t)$  is a known  $m \times n$  rectangular matrix and  $b(t)$  is a known  $m$ -dimensional column vector.

Each possible sequence of estimates  $\hat{x}_1, \hat{x}_2, \dots$  for the state vectors entails two conceptually distinct types of model specification errors: namely, measurement errors consisting of the discrepancies  $[y_t - H(t)\hat{x}_t - b(t)]$  between the actual and the estimated observation at each time  $t$ ; and dynamic errors consisting of the discrepancies  $[\hat{x}_{t+1} - F(t)\hat{x}_t - a(t)]$  which arise due to misspecification of the state transition equations. The basic filtering



and smoothing problem then involves *multicriteria* optimization. Given a sequence of observation vectors  $y_1, y_2, \dots, y_T$  up to time  $T$  with  $T \geq 1$ , determine the state sequence estimates  $\hat{X}_T = (\hat{x}_1, \dots, \hat{x}_T)$  which in some sense make *both* types of specification error as small as possible.

Suppose a dynamic cost  $c_D(\hat{X}_T, T)$  and a measurement cost  $c_M(\hat{X}_T, T)$  are separately assessed for the two disparate types of model specification errors entailed by the choice of a state sequence estimate  $\hat{X}_T$ . On the basis of both tractability and general intuitive appeal, these costs are taken to be sums of squared discrepancy terms.

More precisely, for any given state sequence estimate  $\hat{X}_T$ , the dynamic cost associated with  $\hat{X}_T$  is taken to be

$$c_D(\hat{X}_T, T) = \sum_{t=1}^{T-1} \left[ \hat{x}_{t+1} - (F(t)\hat{x}_t + a(t)) \right]' D(t) \left[ \hat{x}_{t+1} - (F(t)\hat{x}_t + a(t)) \right] \quad (3)$$

and the measurement cost associated with  $\hat{X}_T$  is taken to be

$$c_M(\hat{X}_T, T) = \sum_{t=1}^T \left[ y_t - (H(t)\hat{x}_t + b(t)) \right]' M(t) \left[ y_t - (H(t)\hat{x}_t + b(t)) \right]. \quad (4)$$

Here  $D(t)$  and  $M(t)$  are square, symmetric, positive definite scaling matrices of orders  $n$  and  $m$ , respectively. Having non-zero off-diagonal terms in these matrices would presume knowledge about the relative signs of the discrepancy terms, a presumption which is not very reasonable when discrepancy terms result from model misspecification. Nevertheless, these matrices are left in general form because it does not impede the analytical treatment presented below.

If the prior beliefs (1) and (2) concerning the dynamic and measurement relations are absolutely true, then the actual state sequence  $X_T = (x_1, \dots, x_T)$  would result in zero values for both  $c_D$  and  $c_M$ . In any real-world application, we would of course expect to see positive dynamic and measurement costs associated with each potential state sequence estimate  $\hat{X}_T$ . Nevertheless, not all of these state sequence estimates are equally interesting. Specifically, we would not be interested in a state sequence estimate  $\hat{X}_T$  if it were cost-



subordinated by another estimate  $\hat{X}_T^*$  in the sense that  $\hat{X}_T^*$  yielded a lower value for one type of cost without increasing the value of the other.

We therefore focus attention on the set of state sequence estimates which are not cost-subordinated by any other state sequence estimate. Such estimates are referred to as *flexible least squares* (FLS) estimates. Each FLS estimate shows how the state vector could have evolved over time in a manner minimally incompatible with the prior dynamic and measurement specifications (1) and (2). Without additional model criteria to augment (1) and (2), restricting attention to any proper subset of the FLS estimates is a purely arbitrary decision. Consequently, the FLS approach envisions the generation and consideration of all of the FLS estimates in order to determine commonalities and divergencies displayed by these potential state trajectories.

The collection  $C^F(T)$  of cost vectors  $(c_D, c_M)$  associated with the FLS estimates is referred to as the *cost-efficient frontier*. Given the cost specifications (3) and (4), the frontier is a downward sloping strictly convex curve in the  $c_D$ - $c_M$  plane. (See Figure 1.)

— Insert Figure 1 About Here —

Once the FLS estimates and the cost-efficient frontier are determined, three different levels of analysis can be used to investigate the incompatibility of the theoretical relations (1) and (2) with the observation vectors  $y_1, \dots, y_T$ . First, the frontier can be examined to determine the efficient trade-offs between the dynamic and measurement costs  $c_D$  and  $c_M$ . For example, one can determine the minimum measurement cost which would have to be paid in order to achieve zero dynamic cost, i.e., an exact fit of the state transition equations (1). Second, descriptive summary statistics (e.g., average values and standard deviations) can be constructed for the trajectories traced out by the FLS estimates along the frontier. Finally, the trajectories traced out by the FLS estimates can be directly examined from left to right along the frontier to assess the effects of decreasing the implicit penalty imposed for dynamic versus measurement cost.

Ref. [3] applies this three-stage FLS analysis to a time-varying linear regression prob-

lem, a special case of (1) and (2) with scalar observations ( $m = 1$ ), no forcing terms, and state transition matrices  $F(t)$  set identically equal to the identity matrix. For this application the components of the  $1 \times n$  vectors  $H(t)$  are interpreted as explanatory variables for the scalar observations  $y_t$ , the state vectors  $x_t$  are interpreted as coefficient vectors for the "linear regression" relations (2), and the state transition equations (1) with  $F(t) \equiv I$  are interpreted as smoothness relations governing the evolution of the coefficient vectors over time.

Ref. [4] undertakes an empirical FLS study of coefficient stability for a well-known log-linear regression model of U.S. money demand over the volatile period 1959-1985. Interesting insights are obtained concerning shifts in the coefficients at economically reasonable points in time. In Ref. [5], the FLS approach is used to develop a new measure of productivity change; the coefficients characterizing the production process are allowed to evolve slowly over time. The new measure compared favorably with more traditional measures when tested for U.S. agricultural data.

How are the cost-efficient frontier and the FLS estimates actually generated? The next section suggests what might be done.

### III. THE FLEXIBLE LEAST SQUARES FILTER

In view of the strict convexity of the cost-efficient frontier, each point on this frontier solves a problem of the form "minimize  $c_M$  subject to  $c_D = \text{constant}$ ." Consequently, each FLS state sequence estimate  $\hat{X}_T = (\hat{x}_1, \dots, \hat{x}_T)$  can be generated as the solution to a problem of the form

$$\min_{X_T} \left[ \mu c_D(X_T, T) + c_M(X_T, T) \right], \quad (5)$$

where  $\mu$  is a suitably chosen Lagrange multiplier lying between 0 and  $+\infty$ . Hereafter the bracketed expression in (5) will be referred to as the *incompatibility cost* associated with  $X_T$ , conditional on  $\mu$  and  $T$ . The multiplier  $\mu$ , multiplied by  $-1$ , gives the slope of the cost-efficient frontier at the solution point for (5); thus  $\mu$  parameterizes the trade-offs

attainable between dynamic and measurement cost along the cost-efficient frontier.

The FLS approach envisions the generation of the entire cost-efficient frontier, together with the corresponding FLS state sequence estimates. Numerical experiments (e.g., [3]) have shown that the cost-efficient frontier can be adequately sketched out by solving the minimization problem (5) over a rough grid of  $\mu$ -points increasing by powers of ten.

How is this minimization to be done? The solution of (5) appears to be a formidable problem. Since each state vector  $x_t$  is  $n$ -dimensional, the first-order necessary conditions for the solution of (5) constitute a linear two-point boundary value problem in  $nT$  scalar unknowns. Fortunately, as will now be shown, problem (5) can be reduced to its proper dimensionality,  $n$ , through the use of a dynamic programming technique.

### III.1 The Basic FLS Filter

Let  $\mu > 0$  be given. A recursive procedure will now be developed for the exact sequential solution of the incompatibility cost minimization problem (5) as the duration  $T$  of the process increases and additional observation vectors are obtained.

Suppose that the time is  $T \geq 2$ . Observation vectors have previously been obtained for times  $1, \dots, T-1$ , and a new observation vector  $y_T$  has just become available. Any choice of an estimate  $x_T$  for the current time- $T$  state vector incurs two costs. First, a measurement cost is incurred if there is a discrepancy between the actual observation vector  $y_T$  and the estimated observation vector  $[H(t)x_T + b(T)]$ . Second, consideration must also be given to the minimum achievable incompatibility cost over the earlier part of the process, conditional on the state estimate for time  $T$  being  $x_T$ . The time-separability of the cost functions (3) and (4) implies that this latter cost depends only on  $x_T$  and the observation vectors through time  $T-1$ .

Let a function be introduced to represent the minimum incompatibility cost which can be achieved through time  $T-1$ , conditional on any given time- $T$  state vector  $x_T$ :

$\phi(x_T; \mu, T-1)$  = the minimum incompatibility cost attainable through choice of  $x_1, x_2, \dots, x_{T-1}$ , conditional on the state vector at time  $T$  being  $x_T$ . (6)

The FLS estimate for the time- $T$  state vector, conditional on  $\mu$  and the observation vectors obtained through time  $T$ , is then found by solving the minimization problem

$$\min_{x_T} \left\{ \left[ y_T - (H(T)x_T + b(T)) \right]' M(T) \left[ y_T - (H(T)x_T + b(T)) \right] + \phi(x_T; \mu, T-1) \right\}. \quad (7)$$

Let this FLS estimate be denoted by

$$x_T^{FLS}(\mu, T) = \arg \min_{x_T} \{ \dots \}. \quad (8)$$

At time  $T$  it is necessary to prepare for the appearance of an observation vector at time  $T+1$ . To do this, one needs to know the cost function  $\phi(x_{T+1}; \mu, T)$ . This cost function is given by

$$\begin{aligned} \phi(x_{T+1}; \mu, T) = \min_{x_T} \left\{ \mu \left[ x_{T+1} - (F(T)x_T + a(T)) \right]' D(T) \left[ x_{T+1} - (F(T)x_T + a(T)) \right] \right. \\ \left. + \left[ y_T - (H(T)x_T + b(T)) \right]' M(T) \left[ y_T - (H(T)x_T + b(T)) \right] \right. \\ \left. + \phi(x_T; \mu, T-1) \right\}. \end{aligned} \quad (9)$$

The recursive relationship (9) can be given a dynamic programming interpretation. Conditional on any possible state vector  $x_{T+1}$  for time  $T+1$ , the choice of a state estimate  $x_T$  for time  $T$  incurs three types of cost. First, there is a dynamic cost associated with the estimated state transition from time  $T$  to time  $T+1$ . Second, there is a measurement cost associated with the discrepancy between the estimated and the actual time- $T$  observation vector. And third, there is a minimum achievable incompatibility cost based on everything that is known about the process through time  $T-1$ , conditional on the time- $T$  state vector being  $x_T$ . Selecting  $x_T$  to minimize the sum of these three costs yields



the minimum achievable incompatibility cost based on everything that is known about the process through time  $T$ , conditional on the time- $(T + 1)$  state vector being  $x_{T+1}$ .

Using (9), the cost functions  $\phi(x_2; \mu, 1), \phi(x_3; \mu, 2), \dots$  can be determined one after the other. At time  $T$ , assume that the function  $\phi(x_T; \mu, T - 1)$  is known. An observation vector  $y_T$  then becomes available, and the function  $\phi(x_{T+1}; \mu, T)$  can be determined. To start matters off, it is assumed that an initial cost function  $\phi(x_1; \mu, 0)$  is given. For the particular cost specifications (3) and (4), this initial cost is identically zero. More generally, however, the initial cost could summarize whatever beliefs one has concerning the cost of estimating that the system is in state  $x_1$  at time  $T = 1$  before an observation vector at time  $T = 1$  has been received.

The connection between the minimization problems (5) and (7) is straightforward. Using relationship (9) with  $\phi(x_1; \mu, 0) \equiv 0$ , the cost function  $\phi(x_T; \mu, T - 1)$  can be expanded in the form

$$\phi(x_T; \mu, T - 1) = \min_{x_1, x_2, \dots, x_{T-1}} \left\{ \mu \sum_{t=1}^{T-1} \left[ x_{t+1} - F(t)x_t - a(t) \right]' D(t) \left[ x_{t+1} - F(t)x_t - a(t) \right] + \sum_{t=1}^{T-1} \left[ y_t - H(t)x_t - b(t) \right]' M(t) \left[ y_t - H(t)x_t - b(t) \right] \right\}. \quad (10)$$

Recalling definitions (3) and (4) for  $c_D$  and  $c_M$ , it is then immediately seen that the minimization problem (7) is an alternative representation for the incompatibility cost minimization problem (5).

The recurrence relation (9) is a special case of a multicriteria filter shown elsewhere [6] to generalize various well-known filters such as those of Kalman [7], Viterbi [8], Larson-Peschon [9], and Swerling [10]. It illustrates how one might formulate and update a cost-of-estimation function for a dynamic process when discrepancy terms are not given a probabilistic interpretation. The recurrence relation (9) thus replaces the use of Bayes' rule, which would be employed if discrepancy terms were interpreted as random quantities having known probability distributions and satisfying various independence restrictions.

This point will be elaborated in Section IV, below.

### III.2 A More Concrete Representation for the FLS Filter

It will now be shown how the basic recurrence relation (9) can be more concretely represented in terms of recurrence relations for an  $n \times n$  matrix  $Q_T(\mu)$ , an  $n \times 1$  vector  $p_T(\mu)$ , and a scalar  $r_T(\mu)$ .

From general considerations in linear-quadratic control theory, it is known that if the cost function appearing in the righthand side expression in Eq. (9) is given by

$$\phi(x_T; \mu, T-1) = x_T' Q_{T-1}(\mu) x_T - 2p_{T-1}(\mu)' x_T + r_{T-1}(\mu), \quad (11)$$

where  $Q_{T-1}(\mu)$  is a real  $n \times n$  symmetric matrix, then the cost function appearing on the lefthand side has the form

$$\phi(x_{T+1}; \mu, T) = x_{T+1}' Q_T(\mu) x_{T+1} - 2p_T(\mu)' x_{T+1} + r_T(\mu). \quad (12)$$

We shall show this below in detail.

First, suppose the initial cost function takes the quadratic form

$$\phi(x_1; \mu, 0) = x_1' Q_0(\mu) x_1 - 2p_0(\mu)' x_1 + r_0(\mu), \quad (13)$$

where the  $n \times n$  matrix  $Q_0(\mu)$  is symmetric and positive semidefinite. As earlier noted, this function summarizes our knowledge of the cost of estimating that the system is in state  $x_1$  at time  $T = 1$  before an observation vector at time  $T = 1$  has been received. For the particular cost specifications (3) and (4), the coefficient terms  $Q_0(\mu)$ ,  $p_0(\mu)$ , and  $r_0(\mu)$  are all zero.

Let us now determine the recurrence relations connecting  $Q_T(\mu)$ ,  $p_T(\mu)$ , and  $r_T(\mu)$  with  $Q_{T-1}(\mu)$ ,  $p_{T-1}(\mu)$ , and  $r_{T-1}(\mu)$  for an arbitrary time  $T \geq 1$ , where the  $n \times n$  matrix  $Q_{T-1}(\mu)$  is symmetric and positive semidefinite. Consider Eq. (9) for any given  $x_{T+1}$ . The large curly bracketed term in (9) breaks down into quadratic, linear, and constant

parts with respect to  $x_T$ , as follows:

$$\begin{aligned} \{ \dots \} = & x_T' [\mu F(T)' D(T) F(T) + H(T)' M(T) H(T) + Q_{T-1}(\mu)] x_T \\ & + \left( 2\mu [x_{T+1} - a(T)]' D(T) [-F(T)] + 2[y_T - b(T)]' M(T) [-H(T)] - 2p_{T-1}(\mu)' \right) x_T \\ & + \mu [x_{T+1} - a(T)]' D(T) [x_{T+1} - a(T)] + [y_T - b(T)]' M(T) [y_T - b(T)] + r_{T-1}(\mu). \end{aligned} \quad (14)$$

To do the minimization called for in Eq. (9), the derivative with respect to  $x_T$  of the right-hand side of Eq. (14) is set equal to the null vector, which yields

$$\begin{aligned} 0 = & \left[ \mu F(T)' D(T) F(T) + H(T)' M(T) H(T) + Q_{T-1}(\mu) \right] x_T \\ & - \left( \mu [x_{T+1} - a(T)]' D(T) F(T) + [y_T - b(T)]' M(T) H(T) + p_{T-1}(\mu)' \right)'. \end{aligned} \quad (15)$$

Assuming the bracketed term in (15) is invertible [e.g., assuming the positive semidefinite matrix  $Q_{T-1}(\mu)$  is positive definite, or that either  $F(T)$  or  $H(T)$  has full rank], the optimizing vector  $x_T$  is given by

$$\begin{aligned} x_T = & \left[ \mu F(T)' D(T) F(T) + H(T)' M(T) H(T) + Q_{T-1}(\mu) \right]^{-1} \\ & \times \left( \mu F(T)' D(T) [x_{T+1} - a(T)] + H(T)' M(T) [y_T - b(T)] + p_{T-1}(\mu) \right). \end{aligned} \quad (16)$$

To simplify the notation, let us now introduce the symmetric matrix  $V_T(\mu)$  as

$$V_T(\mu) = \left[ \mu F(T)' D(T) F(T) + H(T)' M(T) H(T) + Q_{T-1}(\mu) \right]^{-1}. \quad (17)$$

Then we may write the optimizing vector  $x_T$  in the form

$$x_T = s_T(\mu) + G_T(\mu) x_{T+1}, \quad (18)$$

where

$$s_T(\mu) = V_T(\mu) \left( H(T)' M(T) [y_T - b(T)] + p_{T-1}(\mu) - \mu F(T)' D(T) a(T) \right) \quad (19)$$

and

$$G_T(\mu) = V_T(\mu) \mu F(T)' D(T). \quad (20)$$

Now we are ready to find  $\phi(x_{T+1}; \mu, T)$ . Substituting Eq. (18) into Eq. (9), the quadratic terms in  $x_{T+1}$  have the matrix  $Q_T(\mu)$  given by

$$\begin{aligned} & \mu \left[ I - F(T)G_T(\mu) \right]' D(T) \left[ I - F(T)G_T(\mu) \right] \\ & + \left( H(T)G_T(\mu) \right)' M(T)H(T)G_T(\mu) + G_T(\mu)' Q_{T-1}(\mu)G_T(\mu) \\ & = G_T(\mu)' V_T(\mu)^{-1} G_T(\mu) + 2\mu D(T) \left[ -F(T) \right] G_T(\mu) + \mu D(T). \end{aligned} \quad (21)$$

But

$$G_T(\mu)' = \mu D(T)F(T)V_T(\mu), \quad (22)$$

so that

$$G_T(\mu)' V_T(\mu)^{-1} = \mu D(T)F(T). \quad (23)$$

It follows that

$$\begin{aligned} Q_T(\mu) &= \mu D(T)F(T)G_T(\mu) - 2\mu D(T)F(T)G_T(\mu) + \mu D(T) \\ &= \mu D(T) \left[ I - F(T)G_T(\mu) \right]. \end{aligned} \quad (24)$$

By standard matrix manipulations (see, e.g., [11, p. 7]), it can be shown that  $Q_T(\mu)$  in (24) is positive semidefinite given the positive semidefiniteness of  $Q_{T-1}(\mu)$  and the positive definiteness of the weight matrices  $D(T)$  and  $M(T)$  as assumed in Section II.

Next we shall determine the vector  $p_T(\mu)$ . Consider, again, the substitution of Eq. (18) into Eq. (9). The linear terms in  $x_{T+1}$  have the coefficient vector  $-2p_T(\mu)$  given by

$$\begin{aligned} & 2G_T(\mu)' V_T(\mu)^{-1} s_T(\mu) + 2\mu D(T) \left[ -F(T) \right] s_T(\mu) \\ & + G_T(\mu)' \left\{ 2\mu F(T)' D(T)a(T) + 2 \left[ -H(T) \right]' M(T) \left[ y_T - b(T) \right] - 2p_{T-1}(\mu) \right\} \\ & + 2\mu D(T) \left[ -a(T) \right]. \end{aligned} \quad (25)$$

It follows, after some simplification, that

$$p_T(\mu) = G_T(\mu)' \left[ H(T)' M(T) [y_T - b(T)] + p_{T-1}(\mu) \right] + Q_T(\mu)' a(T). \quad (26)$$

In a similar manner, we find for  $r_T(\mu)$  that

$$\begin{aligned} r_T(\mu) &= r_{T-1}(\mu) + \left[ y_T - b(T) \right]' M(T) \left[ y_T - b(T) \right] + \mu a(T)' D(T) a(T) \\ &\quad - s_T(\mu)' [V_T(\mu)']^{-1} s_T(\mu). \end{aligned} \quad (27)$$



The relations (24), (26), and (27) constitute the desired recurrence relations for  $Q_T(\mu)$ ,  $p_T(\mu)$ , and  $r_T(\mu)$ .

Finally, using these recurrence relations, the FLS filter estimate (8) for the state vector at time  $T \geq 1$  can also be given a more concrete representation. Let

$$U_T(\mu) = H(T)'M(T)H(T) + Q_{T-1}(\mu), \quad (28)$$

and let

$$z_T(\mu) = H(T)'M(T)[y_T - b(T)] + p_{T-1}(\mu). \quad (29)$$

Then

$$x_T^{FLS}(\mu, T) = [U_T(\mu)]^{-1} z_T(\mu). \quad (30)$$

### III.3 FLS Smoothed State Estimates

Consider the problem of obtaining the FLS smoothed estimate for the state vector  $x_T$  at time  $T$  as the length of the process increases from  $T$  to  $T + 1$  and an additional observation vector  $y_{T+1}$  is obtained.

In preparation for time  $T + 1$ , the quadratic, linear, and constant terms  $Q_T(\mu)$ ,  $p_T(\mu)$ , and  $r_T(\mu)$  characterizing the cost function in Eq. (12) have been calculated and stored. As a byproduct of this calculation, the unique cost-minimizing  $x_T$  as a function of  $x_{T+1}$  has been determined in accordance with Eq. (18) to be  $x_T = s_T(\mu) + G_T(\mu)x_{T+1}$ . Using Eq. (30) updated to time  $T + 1$ , the FLS filter estimate for the state vector at time  $T + 1$  is given by

$$x_{T+1}^{FLS}(\mu, T + 1) = [U_{T+1}(\mu)]^{-1} z_{T+1}(\mu). \quad (31)$$

The FLS smoothed estimate for the time- $T$  state vector  $x_T$ , based on the observation vectors  $y_1, \dots, y_{T+1}$  for times 1 through  $T + 1$ , is then given by

$$x_T^{FLS}(\mu, T + 1) = s_T(\mu) + G_T(\mu)x_{T+1}^{FLS}(\mu, T + 1). \quad (32)$$

More generally, given any fixed time  $t$ ,  $0 \leq t \leq T$ , the FLS smoothed estimate  $x_t^{FLS}(\mu, T+1)$  for the state vector  $x_t$  at time  $t$ , based on the observation vectors  $y_1, \dots, y_{T+1}$  for times 1 through  $T+1$ , is found by solving the system of equations

$$\begin{aligned} x_t &= s_t(\mu) + G_t(\mu)x_{t+1} \\ &\vdots \\ x_T &= s_T(\mu) + G_T(\mu)x_{T+1} \end{aligned} \tag{33a}$$

in reverse order, starting with the initial condition

$$x_{T+1} = x_{T+1}^{FLS}(\mu, T+1). \tag{33b}$$

Relations (30) and (33) for generating the FLS filtered and smoothed state estimates result naturally from the dynamic programming procedure used to update incompatibility cost. Alternative formulas for generating these state estimates could be obtained from (30) and (31) using appropriate matrix manipulations (see [11]). Based on past numerical experience, however, we elected to adhere closely to the dynamic programming formulation.

A Fortran program *GFLS* for generating the FLS filtered and smoothed state estimates by means of the relations (30) and (33) is provided in an appendix to this paper. In simulation experiments conducted to date with *GFLS* on an IBM Model 3090, the generated FLS estimates have satisfied the first-order necessary conditions for the cost-minimization problem (5) up to the maximum degree of accuracy (fourteen to sixteen digits) permitted by the double-precision word length employed. Our empirically based belief, then, is that the suggested procedure for determining the FLS filtered and smoothed state estimates is numerically stable and highly accurate.

#### IV. RELATIONSHIP WITH KALMAN FILTERING

FLS and Kalman filtering address conceptually distinct problems. FLS treats a multi-criteria model specification problem which does not require probability assumptions either for its motivation or for its solution: the characterization of the set of all state sequence

estimates which achieve vector-minimal incompatibility between imperfectly specified theoretical relations and process observations. Kalman filtering is a point estimation technique which determines the most probable state sequence for a stochastic model assumed to be correctly and completely specified. Nevertheless, when applied to approximately linear systems, the two approaches satisfy duality relations which generalize the well-known duality [7, p. 42] between the noise-free regulator problem and maximum a posteriori probability estimation.

Conceptual differences between FLS and Kalman filtering are examined in Section IV.1. In Section IV.2 the Kalman filter recurrence equations are derived by means of simple cost-function arguments which mimic the steps outlined in Section III.2 for the derivation of the FLS recurrence relations. Probabilistic arguments (e.g., Bayes' Rule or iterated expectations) are not required. Conversely, in Section IV.3 it is seen that the FLS recurrence relations for generating any particular state sequence estimate along the cost-efficient frontier reduce to information filter equations, the "inverse" of Kalman filter equations, if the model discrepancy terms are assumed to satisfy various independence and normality restrictions. Implications of these duality relations are discussed in Section IV.4.

#### IV.1 Conceptual Differences Between FLS and Kalman Filtering

Previous sections of this paper investigate how filtering and smoothing might be undertaken for the approximately linear system (1) and (2) when the dynamic and measurement discrepancy terms  $w_t \equiv [x_{t+1} - F(t)x_t - a(t)]$  and  $v_t \equiv [y_t - H(t)x_t - b(t)]$  are incommensurable model specification errors. A multicriteria FLS solution is proposed for this problem. As seen in Section III, this multicriteria solution can be implemented by means of a *family* of Riccati-type recurrence relations. The Riccati-equation form of these recurrence relations is not surprising; it has been known for decades [12] that linear-quadratic minimization leads to recurrence relations of this type. What is new is the probability-free motivation provided for why one should be interested in this entire family of recurrence relations.

Suppose, instead, that the following probability relations, commonly assumed in Kalman filtering studies, are introduced for the discrepancy terms  $w_t$  and  $v_t$  and for the initial state vector  $x_1$ :

- [PDF for  $w_t$ ] =  $N(0, S(t))$ ;
- [PDF for  $v_t$ ] =  $N(0, R(t))$ ;
- $(w_t)$  and  $(v_t)$  are mutually and serially independent processes; (34)
- [PDF for  $x_1$ ] =  $N(x_1^*, \Sigma_1)$ ;
- $x_1$  is distributed independently of  $v_t$  and  $w_t$  for each  $t$ .

Under assumptions (34), the discrepancy terms  $w_t$  and  $v_t$  are interpreted as white noise random vectors with known Gaussian probability density functions (PDF's) governing both their individual and joint behavior. In particular,  $w_t$  and  $v_t$  are now supposed to be perfectly commensurable quantities which can be scaled and weighed relative to one another. The FLS interpretation for  $w_t$  and  $v_t$  as conceptually distinct apple-and-orange model specification errors incorporating everything unknown about the dynamic and measurement aspects of the process is thus dramatically altered.

Combining the measurement relations (2) with the probability relations (34) permits the derivation of a probability density function  $P(Y_T | X_T)$  for the observation sequence  $Y_T = (y_1, \dots, y_T)$  conditional on the state sequence  $X_T = (x_1, \dots, x_T)$ . Combining the dynamic relations (1) with the probability relations (34) permits the derivation of a "prior" probability density function  $P(X_T)$  for  $X_T$ . The multiplication of these two derived probability density functions yields the joint probability density function for  $X_T$  and  $Y_T$ ,

$$P(Y_T | X_T) \cdot P(X_T) = P(X_T, Y_T). \quad (35)$$

The joint probability density function (35) elegantly combines the two distinct sources of theory and data incompatibility—measurement and dynamic—into a single *scalar* measure of incompatibility for any considered state sequence  $X_T$ .

Given the probability relations (34), the usual Kalman filter objective is to determine the maximum a posteriori (MAP) state sequence, i.e., the state sequence which maximizes



the posterior probability density function  $P(X_T | Y_T)$ . Since the observation sequence  $Y_T$  is assumed to be given, this objective is equivalent to determining the state sequence which maximizes the product of  $P(X_T | Y_T)$  and  $P(Y_T)$ . By the agreed upon rules of probability theory,

$$P(X_T | Y_T) \cdot P(Y_T) = P(Y_T | X_T) \cdot P(X_T), \quad (36)$$

where, as earlier explained, the right-hand expression in (36) can be evaluated using (1), (2), and the probability relations (34). Determining the MAP state sequence is thus equivalent to determining the state sequence which minimizes the scalar "incompatibility cost function"

$$c(X_T, T) = -\log[P(Y_T | X_T) \cdot P(X_T)]. \quad (37)$$

What has been achieved by the introduction of the probability relations (34)? Without relations such as (34), the dynamic and measurement discrepancy terms cannot be scaled and weighed relative to one another. The filtering and smoothing problem is thus intrinsically a multicriteria optimization problem: Conditional on the given observations, determine the state sequence estimates which are in some sense minimally incompatible with each of the imperfectly specified theoretical relations (1) and (2). Given the probability relations (34), however, the discrepancy terms are transformed into perfectly commensurable "disturbance terms" impinging on correctly specified theoretical relations in accordance with known probability distributions. In this case, MAP estimation seems an eminently reasonable way to proceed. *The multicriteria optimization problem is thus transformed into the scalar optimization problem of determining the most probable state sequence for a stochastic model assumed to be correctly and completely specified.*

Making use of Bayes' rule, Larson and Peschon [9] develop a recurrence relation for the sequential updating of the posterior density function  $P(X_T | Y_T)$  as the duration  $T$  of the process increases and additional observation vectors are obtained. This recurrence relation is used to determine recursively the MAP state sequence for each time  $T$ . The Larson-Peschon filter is derived under assumptions (34) without the requirement that the PDF's

be Gaussian; nonlinearity of the dynamic and measurement relations is also permitted. Larson and Peschon show that their filter reduces to the Kalman filter when Gaussian distributions and linear dynamic and measurement relations are assumed.

For example, suppose for simplicity that the forcing terms  $a(t)$  and  $b(t)$  in the dynamic and measurement relations (1) and (2) are identically zero. For this case, Larson and Peschon obtain the relations

$$\begin{aligned}\Sigma^{-1}(T+1 | T+1) &= H(T+1)' R(T+1)^{-1} H(T+1) + \left[ F(T) \Sigma(T | T) F(T)' + S(T) \right]^{-1}; \\ x(T+1 | T+1) &= F(T) x(T | T) \\ &\quad + \Sigma(T+1 | T+1) H(T+1)' R(T+1)^{-1} \left[ y_{T+1} - H(T+1) F(T) x(T | T) \right].\end{aligned}\tag{38}$$

In equations (38),  $x(T+1 | T+1)$  is the MAP estimate for the state vector at time  $T+1$ , conditional on the observation vectors obtained through time  $T+1$ ; and  $\Sigma(T+1 | T+1)$  is the error covariance matrix for  $x(T+1 | T+1)$ . By use of appropriate matrix inversion formulas, the relations (38) can be transformed into a pair of recurrence relations either for the error covariance matrix  $\Sigma(T | T)$  and the state estimate  $x(T | T)$ —the standard Kalman filter equations (see [7] and [13, pp. 105-120])—or for the inverse “information matrix”  $\Sigma^{-1}(T | T)$  and the modified state estimate  $\Sigma^{-1}(T | T)x(T | T)$ , yielding the “information filter equations” (see [13, pp. 139-142]).

#### IV.2 Cost Derivation of the Kalman Filter Recurrence Relations

It will now be shown that the recursive relations (38) can alternatively be derived by means of simple intuitive cost considerations, without reliance on probabilistic arguments.

As in Section IV.1, suppose for simplicity that the forcing terms  $a(t)$  and  $b(t)$  in (1) and (2) are identically zero. For any time  $T > 1$ , let  $X_T$  denote the  $T$ -length state

trajectory  $(x_1, \dots, x_T)$ ; and let the time- $T$  incompatibility cost function be specified by

$$c(X_T, T) = \left\{ \sum_{t=1}^{T-1} [x_{t+1} - F(t)x_t]' S(t)^{-1} [x_{t+1} - F(t)x_t] + \sum_{t=1}^T [y_t - H(t)x_t]' R(t)^{-1} [y_t - H(t)x_t] + [x_1 - x_1^*]' \Sigma_1^{-1} [x_1 - x_1^*] \right\}. \quad (39)$$

Also, let the time-1 incompatibility cost function be specified by

$$c(X_1, 1) = [x_1 - x_1^*]' \Sigma_1^{-1} [x_1 - x_1^*]. \quad (40)$$

Given the probability relations (34), the time- $T$  incompatibility cost function (39) coincides with the previously defined incompatibility cost function (37) apart from a nonessential constant term. Finally, for any time  $T \geq 1$ , let  $C^F(x_T, T)$  denote the minimum cost (39) attainable at time  $T$ , conditional on the time- $T$  state vector being  $x_T$ .

By definition, the state-conditioned cost function  $C^F(x_1, 1)$  for time 1 coincides with the time-1 cost function  $c(X_1, 1)$ ; hence it has the quadratic form

$$C^F(x_1, 1) = [x_1 - x(1 | 1)]' \Sigma^{-1}(1 | 1) [x_1 - x(1 | 1)], \quad (41a)$$

where

$$\Sigma^{-1}(1 | 1) \equiv \Sigma_1^{-1}; \quad (41b)$$

$$x(1 | 1) \equiv x_1^*. \quad (41c)$$

Note that  $x(1 | 1)$  is the state vector  $x_1$  which minimizes the state-conditioned cost function  $C^F(x_1, 1)$ .

Suppose the state-conditioned cost function  $C^F(x_T, T)$  for some time  $T \geq 1$  has the quadratic form

$$C^F(x_T, T) = [x_T - x(T | T)]' \Sigma^{-1}(T | T) [x_T - x(T | T)] + k_T, \quad (42)$$

where  $k_T$  is independent of  $x_T$ . As shown in [6, Section 4.3], the state-conditioned cost function for time  $T + 1$  satisfies the recurrence relation

$$C^F(x_{T+1}, T+1) = \min_{x_T} \left\{ \Delta c(x_T, x_{T+1}, T+1) + C^F(x_T, T) \right\}, \quad (43a)$$

where

$$\begin{aligned} \Delta c(x_T, x_{T+1}, T+1) \equiv & \left[ x_{T+1} - F(T)x_T \right]' S(T)^{-1} \left[ x_{T+1} - F(T)x_T \right] \\ & + \left[ y_T - H(T)x_T \right]' R(T)^{-1} \left[ y_T - H(T)x_T \right] \end{aligned} \quad (43b)$$

denotes the total change in cost associated with the transition from  $T$  to  $T+1$ . Substituting (42) into (43), it follows by straightforward calculations (analogous to those in Section III.2) that the state-conditioned cost function for time  $T + 1$  has the quadratic form

$$C^F(x_{T+1}, T+1) = \left[ x_{T+1} - x(T+1 | T+1) \right]' \Sigma^{-1}(T+1 | T+1) \left[ x_{T+1} - x(T+1 | T+1) \right] + k_{T+1}, \quad (44)$$

where  $\Sigma(T+1 | T+1)$  and  $x(T+1 | T+1)$  satisfy the recursive relations (38). As is clear from (44),  $x(T+1 | T+1)$  is the state vector  $x_{T+1}$  which minimizes the state-conditioned cost function  $C^F(x_{T+1}, T+1)$ .

The terms  $\Sigma(T+1 | T+1)$  and  $x(T+1 | T+1)$  appearing in the cost expression (44) thus coincide with the error covariance matrix and state estimate generated by the Kalman filter recurrence relations derived from (38). Note, also, that the quadratic and linear coefficient terms  $\Sigma^{-1}(T+1 | T+1)$  and  $\Sigma^{-1}(T+1 | T+1)x(T+1 | T+1)$  for the cost expression (44), considered as a function of  $x_{T+1}$ , coincide with the information matrix and modified state estimate generated by the information filter equations. It is not surprising, then, that the cost arguments used to derive the recursive relations (38) for these terms are entirely analogous to the cost arguments used in Section III.2 to determine recursive relations for the quadratic and linear coefficient terms  $Q_T(\mu)$  and  $p_T(\mu)$  for the cost expression  $\phi(x_{T+1}; \mu, T)$ .



In summary, the Kalman and information filter recurrence relations can be derived for approximately linear systems using simple cost arguments, without recourse to probabilistic arguments such as Bayes' rule or iterated expectations. All that is needed is that the basic cost function used to measure theory and data incompatibility be a quadratic function exhibiting time-separability.

### IV.3 The FLS Recurrence Relations as Information Filter Equations

Conversely, the FLS recurrence relations associated with any given point  $\mu$  on the cost-efficient frontier reduce to a variant of the information filter equations if the theoretical relations (1) and (2) are augmented by probability relations of the form (34).

Specifically, suppose the dynamic weight matrix  $\mu D(t)$  is taken to be the inverse of the covariance matrix  $S(t)$  for  $w_t$ , and the measurement weight matrix  $M(t)$  is taken to be the inverse of the covariance matrix  $R(t)$  for  $v_t$ , for each time  $t$ ; and suppose also that the initial cost matrix  $Q_0(\mu)$  is taken to be the inverse of the covariance matrix  $\Sigma_1$  for the initial state vector  $x_1$ . In this case the matrix  $U_T(\mu)$  in (28) corresponds to the inverse of the "measurement-update" error covariance matrix  $\Sigma(T | T)$  and the vector  $z_T(\mu)$  in (29) corresponds to the modified state estimate  $\Sigma^{-1}(T | T)x(T | T)$ . Moreover, the matrix  $Q_T(\mu)$  corresponds to the inverse of the "time-update" error covariance matrix  $\Sigma(T + 1 | T)$ , defined [13, Chapter 3] to be the error covariance matrix for the MAP estimate of  $x_{T+1}$  based on observations through time  $T$ .

### IV.4 Duality Implications

If the probability relations (34) are justified for a given filtering and smoothing application, they should of course be incorporated in the estimation procedure. However, for many important applications—particularly in the social sciences—obtaining agreement among researchers regarding probability relations such as (34) can be difficult.

For example, the process observations may be the outcome of a nonreplicable experiment, so that no objective test of these relations can be carried out. Also, the theoretical

relations may represent tentatively held conjectures concerning a poorly understood process; or they may be a linearized set of relations obtained for an analytically intractable nonlinear process, as in many aerospace filtering and smoothing problems. In these cases it is doubtful whether the discrepancy terms are governed by *any* meaningful probability relations. Independence restrictions, in particular, are questionable and troublesome.

For these reasons, the FLS procedure, with its minimal assumptions concerning discrepancy terms, appears to offer a useful complement to existing filtering and smoothing techniques. Moreover, the FLS duality relations discussed in previous sections may shed some light on the robustness properties of the Kalman filter.

It is now conventional to interpret any quadratic criterion function representing sums of squared dynamic and measurement errors—e.g., the Kalman filter criterion function (39)—as a log-likelihood expression arising from some underlying stochastic model in which model discrepancy terms are interpreted as independent and normally distributed random variables. Yet it is also known that Kalman filtering works remarkably well in some contexts in which these strong stochastic assumptions are not even remotely satisfied. A partial explanation for this robustness is that the Kalman filter criterion function can be given an alternative interpretation: namely, as a cost function embodying the criterion that model discrepancy terms be *small*.

“Smallness” should not be confused with “randomness.” Postulating that  $x_{t+1}$  is close to  $[F(t)x_t + a(t)]$  does not mean that the discrepancy term  $[x_{t+1} - F(t)x_t - a(t)]$  is necessarily a random vector. As numerous experiments with FLS have shown (see, e.g., [3]), the postulate of small dynamic and measurement discrepancy terms is a powerful assumption which allows state trajectories to be tracked and recovered with surprising qualitative accuracy at each point along the cost-efficient frontier.

## V. CONCLUDING REMARKS

The main purpose of this paper is to present a probability-free multicriteria approach to the problem of filtering and smoothing when prior beliefs concerning dynamics and

measurements take an approximately linear form. In particular, model discrepancy terms are treated as model specification errors which may not have any meaningful probabilistic description. Applications are envisioned in various fields, particularly in the social and biological sciences, where obtaining agreement among researchers regarding probability relations for discrepancy terms is difficult.

The essence of the proposed FLS procedure is the cost-efficient frontier. This frontier, a curve in a two-dimensional cost plane, provides an explicit and systematic way to determine the efficient trade-offs between the separate costs incurred for dynamic and measurement specification errors.

The estimated state sequences whose associated cost vectors attain the cost-efficient frontier, referred to as FLS estimates, show how the state vector could have evolved over time in a manner minimally incompatible with the prior dynamic and measurement specifications. Each FLS estimate has the property that it is not possible simultaneously to reduce both the dynamic and the measurement cost by choice of an alternative state sequence estimate. The similarities displayed by the FLS estimates suggest working hypotheses regarding the evolution of the actual state vector. The divergencies displayed by these estimates reflect the residual uncertainty inherent in the problem specifications regarding the exact nature of this evolution. Without additional prior information, restricting attention to any proper subset of the FLS estimates is an arbitrary decision.

A Fortran program *GFLS* for implementing the FLS filtering and smoothing procedure for approximately linear systems is provided in the appendix. This program has been used in both simulation and empirical studies of time-varying linear regression ([3-5]).

Nonlinear systems are studied from the multicriteria FLS point of view in [6].

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This appendix provides a Fortran program *GFLS* which implements the sequential FLS solution of the bicriteria filtering and smoothing problem posed in Section II. The program has received extensive testing. In addition, the program incorporates a check of the sequential FLS solution based upon using the standard first-order conditions for the solution of the incompatibility cost minimization problem (5).

The variable names used in the *GFLS* program adhere strictly to those used in the body of the paper. Moreover, numerous comment statements are interspersed throughout the program which are geared to the equation numbers used in the paper.

User inputs are required in a subroutine INPUT. This subroutine initializes the penalty weight  $\mu$ , the total number of observation vectors  $TCAP$ , the state vector dimension  $n$ , the observation vector dimension  $m$ , and the initial cost function coefficient terms  $Q_0(\mu)$ ,  $p_0(\mu)$ , and  $r_0(\mu)$ . The program is currently dimensioned for  $TCAP \leq 110$ ,  $n \leq 15$ , and  $m \leq 15$ .

Subroutine INPUT also requires the user to set two flags. The first flag, IFLAGR, is set equal to 1 if the user wishes to generate evaluations for the constant terms  $r_T(\mu)$  in the cost functions (12), and is set equal to 0 otherwise. The second flag, IFLAGS, is set equal to 1 if the user wishes to generate smoothed state estimates in addition to filtered state estimates, and is otherwise set equal to 0. If the user sets IFLAGS = 1, the program automatically carries out a test of the first-order conditions for the incompatibility cost minimization problem (5).

User inputs are also required in a subroutine MODEL. For each current time  $T$ , subroutine MODEL generates the  $n \times n$  state transition matrix  $F(T)$ , the  $n \times 1$  dynamic forcing term  $a(T)$ , the  $m \times n$  measurement matrix  $H(T)$ , the  $m \times 1$  measurement forcing term  $b(T)$ , the  $n \times n$  dynamic weight matrix  $D(T)$ , the  $m \times m$  measurement weight matrix  $M(T)$ , and the  $m \times 1$  observation vector  $y_T$ . For simulation studies, the observation vector  $y_T$  is generated in accordance with the relation  $y_T = H(T)x_T + b(T) + v_T$ , where  $x_T$  is an

$n \times 1$  user-specified state vector and  $v_T$  is an  $m \times 1$  user-specified discrepancy term. The user-specified state vector  $x_T$  is stored in an array TRUEX for later comparison with the numerically generated FLS smoothed estimate for  $x_T$ .

The *GFLS* program contains subroutines for all needed matrix operations. Currently, these subroutines are dimensioned for  $15 \times 15$  matrices. To keep the number of subroutines to a minimum, vector and scalar operations are carried out with these matrix subroutines by considering some vectors to lie in the first column of a  $15 \times 15$  matrix, and some scalars to be the upper-left component of a  $15 \times 15$  matrix.

```

// EXEC FORT1CLG,REGION=512K
// *
/*JOBPARM COPIES=4
C
C
C      GFLS: FLEXIBLE LEAST SQUARES FOR APPROXIMATELY LINEAR SYSTEMS
C      R. KALABA AND L. TESFATSION
C
C      FILENAME: GFLS.CNTL
C      LAST UPDATED: 23 OCTOBER 1989
C
C      IMPLICIT REAL*8(A-H,O-Z)
C      INTEGER T,TCAP,TCAP1
C      REAL*8 M
C
C      THIS PROGRAM IS CURRENTLY DIMENSIONED FOR A MAXIMUM OF TCAP=110
C      OBSERVATION VECTORS Y OF MAXIMUM DIMENSION MOBS = 15 WITH STATE
C      VECTORS X OF MAXIMUM DIMENSION N = 15.
C
C      DIMENSION QO(15,15),PO(15,15),RO(15,15),QZERO(15,15)
C      DIMENSION PZERO(15,15),RZERO(15,15)
C      DIMENSION F(15,15),A(15,15),H(15,15),B(15,15),D(15,15)
C      DIMENSION M(15,15),Y(15,15),TRUEX(15,110),YY(15,110)
C      DIMENSION HT(15,15),U(15,15),C(15,15),W(15,15),V(15,15)
C      DIMENSION E(15,15),Z(15,15),G(15,15),QNEW(15,15),PNEW(15,15)
C      DIMENSION S(15,15),RNEW(15,15),XTCAP(15,15),X(15,110)
C      DIMENSION AA(15,15),BB(15,15),CC(15,15),DD(15,15),EE(15,15)
C      DIMENSION FF(15,15),HH(15,15),OO(15,15),PP(15,15),QQ(15,15)
C      DIMENSION RR(15,15),TT(15,15)
C
C      ADDITIONAL ARRAYS IF SMOOTHED ESTIMATES ARE TO BE CALCULATED
C      FOR INTERMEDIATE X VALUES (I.E., IF IFLAGS IS SET AT 1)
C
C      DIMENSION GG(15,15,110),SS(15,110)
C
C      THE FOLLOWING SUBROUTINE INITIALIZES THE PENALTY WEIGHT AMU,
C      THE NUMBER OF OBSERVATIONS TCAP, THE STATE VECTOR DIMENSION
C      N, THE OBSERVATION VECTOR DIMENSION MOBS, AND THE INITIAL COST
C      FUNCTION CHARACTERISTICS QZERO, PZERO, AND RZERO. IT ALSO SETS
C      THE VALUE FOR A FLAG "IFLAGR" TO DETERMINE IF RNEW IS TO BE
C      CALCULATED (1) OR NOT (0) AND A FLAG "IFLAGS" TO DETERMINE IF
C      SMOOTHED ESTIMATES FOR INTERMEDIATE X VALUES ARE TO BE CALCULATED
C      (1) OR NOT (0).
C
C      CALL INPUT(AMU,TCAP,N,MOBS,QZERO,PZERO,RZERO,IFLAGR,IFLAGS)
C      CALL SHIFT(N,N,QZERO,QO)
C      CALL SHIFT(N,1,PZERO,PO)
C      CALL SHIFT(1,1,RZERO,RO)
C
C      ENTERING THE MAIN DO LOOP FOR GENERATING Q,P,AND R FOR
C      SUCCESSIVE TIMES T = 1,TCAP USING EQS.(24),(26), AND (27).
C

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DO 50 T=1,TCAP	00000530
CALL MODEL(T,F,A,H,B,D,M,Y,TRUEX)	00000540
DO 5 I=1,MOBS	00000550
YY(I,T) = Y(I,1)	00000560
5 CONTINUE	00000570
C	00000580
C GETTING U=HT*M*H + QO IN EQ.(28)	00000590
C	00000600
CALL MUL(MOBS,MOBS,N,M,H,AA)	00000610
CALL TRANS(MOBS,N,H,HT)	00000620
CALL MUL(N,MOBS,N,HT,AA,BB)	00000630
CALL ADD(N,N,BB,QO,U)	00000640
C	00000650
C GETTING C=FT*D	00000660
C	00000670
CALL TRANS(N,N,F,AA)	00000680
CALL MUL(N,N,N,AA,D,C)	00000690
C	00000700
C GETTING W=AMU*C*F+U	00000710
C	00000720
CALL MUL(N,N,N,C,F,AA)	00000730
CALL MULCON(N,N,AMU,AA,BB)	00000740
CALL ADD(N,N,BB,U,W)	00000750
C	00000760
C GETTING V=WINV IN EQ.(17)	00000770
C	00000780
CALL INV(N,W,V)	00000790
C	00000800
C GETTING E = (Y-B)	00000810
C	00000820
CALL SUB(MOBS,1,Y,B,E)	00000830
C	00000840
C GETTING Z = HT*M*E + PO IN EQ.(29)	00000850
C	00000860
CALL MUL(MOBS,MOBS,1,M,E,AA)	00000870
CALL MUL(N,MOBS,1,HT,AA,BB)	00000880
CALL ADD(N,1,BB,PO,Z)	00000890
C	00000900
C GETTING G = AMU*V*C IN EQ.(20)	00000910
C	00000920
CALL MUL(N,N,N,V,C,AA)	00000930
CALL MULCON(N,N,AMU,AA,G)	00000940
IF(IFLAGS.EQ.0) GO TO 110	00000950
C	00000960
C STORE G FOR CALCULATION OF SMOOTHED ESTIMATES	00000970
C	00000980
DO 10 I=1,N	00000990
DO 20 J=1,N	00001000
GG(I,J,T)=G(I,J)	00001010
20 CONTINUE	00001020
10 CONTINUE	00001030
110 CONTINUE	00001040
C	00001050

C	GETTING QNEW = AMU*D*(1-F*G) IN EQ.(24)	00001060
C		00001070
	CALL MUL(N,N,N,F,G,AA)	00001080
	CALL IDEN(N,BB)	00001090
	CALL SUB(N,N,BB,AA,CC)	00001100
	CALL MUL(N,N,N,D,CC,DD)	00001110
	CALL MULCON(N,N,AMU,DD,QNEW)	00001120
C		00001130
C	GETTING PNEW = GT*Z+QNEW*A IN EQ.(26)	00001140
C		00001150
	CALL TRANS(N,N,G,AA)	00001160
	CALL MUL(N,N,1,AA,Z,BB)	00001170
	CALL TRANS(N,N,QNEW,CC)	00001180
	CALL MUL(N,N,1,CC,A,DD)	00001190
	CALL ADD(N,1,BB,DD,PNEW)	00001200
C		00001210
C	GETTING S = V*(Z - AMU*C*A) IN EQ.(19)	00001220
C		00001230
	CALL MUL(N,N,1,C,A,BB)	00001240
	CALL MULCON(N,1,AMU,BB,CC)	00001250
	CALL SUB(N,1,Z,CC,DD)	00001260
	CALL MUL(N,N,1,V,DD,S)	00001270
	IF(IFLAGS.EQ.0) GO TO 210	00001280
C		00001290
C	STORE S FOR CALCULATION OF SMOOTHED ESTIMATES	00001300
C		00001310
	DO 30 I=1,N	00001320
	SS(I,T)=S(I,1)	00001330
30	CONTINUE	00001340
210	CONTINUE	00001350
	IF(IFLAGR.EQ.0) GO TO 310	00001360
C		00001370
C	GETTING RNEW = RO + ET*M*E + AMU*AT*D*A - ST*W*S IN EQ.(27)	00001380
C		00001390
	CALL MUL(MOBS,MOBS,1,M,E,AA)	00001400
	CALL TRANS(MOBS,1,E,BB)	00001410
	CALL MUL(1,MOBS,1,BB,AA,CC)	00001420
	CALL ADD(1,1,RO,CC,DD)	00001430
	CALL MUL(N,N,1,D,A,EE)	00001440
	CALL TRANS(N,1,A,FF)	00001450
	CALL MUL(1,N,1,FF,EE,HH)	00001460
	CALL MULCON(1,1,AMU,HH,OO)	00001470
	CALL ADD(1,1,DD,OO,PP)	00001480
	CALL MUL(N,N,1,W,S,QQ)	00001490
	CALL TRANS(N,1,S,RR)	00001500
	CALL MUL(1,N,1,RR,QQ,TT)	00001510
	CALL SUB(1,1,PP,TT,RNEW)	00001520
310	CONTINUE	00001530
	IF(T.EQ.TCAP) GO TO 50	00001540
C		00001550
C	UPDATING QO,PO, AND RO	00001560
C		00001570
	CALL SHIFT(N,N,QNEW,QO)	00001580

	CALL SHIFT(N,1,PNEW,PO)	00001590
	IF(IFLAGR.EQ.0) GO TO 50	00001600
	CALL SHIFT(1,1,RNEW,RO)	00001610
50	CONTINUE	00001620
C		00001630
C	GETTING THE FLS FILTER ESTIMATE FOR XTCAP = UINV*Z IN EQ.(30)	00001640
C		00001650
	CALL INV(N,U,AA)	00001660
	CALL MUL(N,N,1,AA,Z,XTCAP)	00001670
	DO 65 I=1,N	00001680
	X(I,TCAP)=XTCAP(I,1)	00001690
65	CONTINUE	00001700
	IF (IFLAGS.EQ.1) GOTO 410	00001710
C		00001720
C	PRINTING OUT THE FLS FILTER ESTIMATE FOR XTCAP	00001730
C		00001740
	CALL OUTPUT(TCAP,N,X,TRUEX)	00001750
	IF(IFLAGS.EQ.0) GOTO 510	00001760
410	CONTINUE	00001770
C		00001780
C	GETTING SMOOTHED ESTIMATES FOR X1,..., XTCAP-1 IN EQS.(33A)	00001790
C		00001800
	TCAP1=TCAP-1	00001810
	DO 70 T=1,TCAP1	00001820
	L=TCAP-T	00001830
	DO 80 I=1,N	00001840
	X(I,L)=SS(I,L)	00001850
	DO 90 J=1,N	00001860
	X(I,L)=X(I,L)+GG(I,J,L)*X(J,L+1)	00001870
90	CONTINUE	00001880
80	CONTINUE	00001890
70	CONTINUE	00001900
C		00001910
C	PRINTING OUT THE FLS ESTIMATES FOR X1,...,XTCAP	00001920
C		00001930
	DO 150 T=1,TCAP	00001940
	CALL OUTPUT(T,N,X,TRUEX)	00001950
150	CONTINUE	00001960
C	VALIDATION TEST: HOW WELL DO THE FLS ESTIMATES SATISFY THE	00001970
C	FIRST-ORDER CONDITIONS FOR THE COST MINIMIZATION PROBLEM (5)	00001980
	CALL FOCTST(X,YY)	00001990
510	CONTINUE	00002000
	STOP	00002010
	END	00002020
C		00002030
C	MATRIX SUBROUTINES FOR ADDITION, MULTIPLICATION, TRANSPOSITION,	00002040
C	SUBTRACTION, INVERSION, MULTIPLICATION BY A SCALAR, SHIFT, AND	00002050
C	FORMATION OF AN IDENTITY MATRIX	00002060
C		00002070
C	OBTAINING THE SUM C=A+B OF TWO NROW X MCOL MATRICES A AND B	00002080
C		00002090
	SUBROUTINE ADD(NROW,MCOL,A,B,C)	00002100
	IMPLICIT REAL*8(A-H,O-Z)	00002110

	DIMENSION A(15,15),B(15,15),C(15,15)	00002120
	DO 10 I=1,NROW	00002130
	DO 20 J=1,MCOL	00002140
	C(I,J)=A(I,J)+B(I,J)	00002150
20	CONTINUE	00002160
10	CONTINUE	00002170
	RETURN	00002180
	END	00002190
C		00002200
C	OBTAINING THE PRODUCT C=A*B OF AN NROW X L MATRIX A AND AN	00002210
C	L X MCOL MATRIX B	00002220
C		00002230
	SUBROUTINE MUL(NROW,L,MCOL,A,B,C)	00002240
	IMPLICIT REAL*8(A-H,O-Z)	00002250
	DIMENSION A(15,15),B(15,15),C(15,15)	00002260
	DO 10 I=1,NROW	00002270
	DO 20 J=1,MCOL	00002280
	SUM=0.0D+00	00002290
	DO 30 K=1,L	00002300
	SUM=SUM+A(I,K)*B(K,J)	00002310
30	CONTINUE	00002320
	C(I,J)=SUM	00002330
20	CONTINUE	00002340
10	CONTINUE	00002350
	RETURN	00002360
	END	00002370
C		00002380
C	OBTAINING THE TRANSPOSE B OF AN NROW X MCOL MATRIX A	00002390
C		00002400
	SUBROUTINE TRANS(NROW,MCOL,A,B)	00002410
	IMPLICIT REAL*8(A-H,O-Z)	00002420
	DIMENSION A(15,15),B(15,15)	00002430
	DO 10 I=1,NROW	00002440
	DO 20 J=1,MCOL	00002450
	B(J,I)=A(I,J)	00002460
20	CONTINUE	00002470
10	CONTINUE	00002480
	RETURN	00002490
	END	00002500
C		00002510
C	OBTAINING THE DIFFERENCE C=A-B BETWEEN NROW X MCOL MATRICES	00002520
C	A AND B	00002530
C		00002540
	SUBROUTINE SUB(NROW,MCOL,A,B,C)	00002550
	IMPLICIT REAL*8(A-H,O-Z)	00002560
	DIMENSION A(15,15),B(15,15),C(15,15)	00002570
	DO 10 I=1,NROW	00002580
	DO 20 J=1,MCOL	00002590
	C(I,J)=A(I,J)-B(I,J)	00002600
20	CONTINUE	00002610
10	CONTINUE	00002620
	RETURN	00002630
	END	00002640

C		00002650
C	OBTAINING THE INVERSE C OF A K X K MATRIX A	00002660
C		00002670
	SUBROUTINE INV(K,A,C)	00002680
	IMPLICIT REAL*8(A-H,O-Z)	00002690
	DIMENSION A(15,15),B(15,30),C(15,15)	00002700
	DO 5 J=1,K	00002710
	DO 6 I=1,K	00002720
	B(I,J)=A(I,J)	00002730
6	CONTINUE	00002740
5	CONTINUE	00002750
	K2=K*2	00002760
	DO 7 J=1,K	00002770
	DO 8 I=1,K	00002780
	B(I,K+J)=0.0D+00	00002790
	IF(I.EQ.J) B(I,K+J)=1.0D+00	00002800
8	CONTINUE	00002810
7	CONTINUE	00002820
C		00002830
C	THE PIVOT OPERATION STARTS HERE	00002840
C		00002850
	DO 9 L=1,K	00002860
	PIVOT = B(L,L)	00002870
	DO 13 J=L,K2	00002880
	B(L,J)=B(L,J)/PIVOT	00002890
13	CONTINUE	00002900
C		00002910
C	TO IMPROVE THE ROWS	00002920
C		00002930
	DO 14 I=1,K	00002940
	IF(I.EQ.L) GO TO 14	00002950
	AIL=B(I,L)	00002960
	DO 15 J=L,K2	00002970
	B(I,J)=B(I,J)-AIL*B(L,J)	00002980
15	CONTINUE	00002990
14	CONTINUE	00003000
9	CONTINUE	00003010
	DO 45 I=1,K	00003020
	DO 46 J=1,K	00003030
	C(I,J)=B(I,K+J)	00003040
46	CONTINUE	00003050
45	CONTINUE	00003060
	RETURN	00003070
	END	00003080
C		00003090
C	OBTAINING THE PRODUCT C*A OF A SCALAR C AND AN NROW X MCOL	00003100
C	MATRIX A	00003110
C		00003120
	SUBROUTINE MULCON(NROW,MCOL,C,A,CA)	00003130
	IMPLICIT REAL*8(A-H,O-Z)	00003140
	DIMENSION A(15,15),CA(15,15)	00003150
	DO 10 I=1,NROW	00003160
	DO 20 J=1,MCOL	00003170

	CA(I,J)=C*A(I,J)	00003180
20	CONTINUE	00003190
10	CONTINUE	00003200
	RETURN	00003210
	END	00003220
C		00003230
C	PUTTING AN NROW X MCOL MATRIX A INTO AN NROW X MCOL MATRIX B	00003240
C		00003250
	SUBROUTINE SHIFT(NROW,MCOL,A,B)	00003260
	IMPLICIT REAL*8(A-H,O-Z)	00003270
	DIMENSION A(15,15),B(15,15)	00003280
	DO 10 I=1,NROW	00003290
	DO 20 J=1,MCOL	00003300
	B(I,J)=A(I,J)	00003310
20	CONTINUE	00003320
10	CONTINUE	00003330
	RETURN	00003340
	END	00003350
C		00003360
C	FORMING THE N X N IDENTITY MATRIX E	00003370
C		00003380
	SUBROUTINE IDEN(N,E)	00003390
	IMPLICIT REAL*8(A-H,O-Z)	00003400
	DIMENSION E(15,15)	00003410
	ZERO=0.0D+00	00003420
	ONE=1.0D+00	00003430
	DO 10 I=1,N	00003440
	DO 20 J=1,N	00003450
	E(I,J)=ZERO	00003460
20	CONTINUE	00003470
10	CONTINUE	00003480
	DO 30 L=1,N	00003490
	E(L,L)=ONE	00003500
30	CONTINUE	00003510
	RETURN	00003520
	END	00003530
C		00003540
	SUBROUTINE INPUT(AMU,TCAP,N,MOBS,QZERO,PZERO,RZERO,IFLAGR,IFLAGS)	00003550
	IMPLICIT REAL*8(A-H,O-Z)	00003560
	INTEGER TCAP	00003570
	DIMENSION QZERO(15,15),PZERO(15,15),RZERO(15,15)	00003580
	AMU = 1.0D+00	00003590
	TCAP = 30	00003600
	N = 2	00003610
	MOBS = 1	00003620
	DO 10 J = 1,N	00003630
	DO 20 I = 1,N	00003640
	QZERO(I,J) = 0.0D+00	00003650
	PZERO(I,J) = 0.0D+00	00003660
	RZERO(I,J) = 0.0D+00	00003670
20	CONTINUE	00003680
10	CONTINUE	00003690
	IFLAGR=1	00003700

	IFLAGS=1	00003710
	RETURN	00003720
	END	00003730
C		00003740
	SUBROUTINE MODEL(T,F,A,H,B,D,M,Y,TRUEX)	00003750
	IMPLICIT REAL*8(A-H,O-Z)	00003760
	REAL*8 M	00003770
	REAL*4 GNORM	00003780
	INTEGER T,TCAP	00003790
	DIMENSION F(15,15),A(15,15),H(15,15),B(15,15),D(15,15),M(15,15)	00003800
	DIMENSION Y(15,15),TRUEX(15,110),ZERO(15,15)	00003810
	DIMENSION QZERO(15,15),PZERO(15,15),RZERO(15,15)	00003820
C		00003830
C	TIME-VARYING LINEAR REGRESSION STUDY WITH A SHIFT IN THE COEFF.	00003840
C	VECTOR AT MIDPOINT OBSERVATION TIME T=15 (SEE SECTION 2).	00003850
	CALL INPUT(AMU,TCAP,N,MOBS,QZERO,PZERO,RZERO,IFLAGR,IFLAGS)	00003860
	SIGMA = 0.00D+00	00003870
	DO 10 I=1,15	00003880
	DO 20 J=1,15	00003890
	ZERO(I,J) = 0.0D+00	00003900
20	CONTINUE	00003910
10	CONTINUE	00003920
	CALL IDEN(N,F)	00003930
	CALL SHIFT(N,1,ZERO,A)	00003940
	H(1,1)=1.0D+00	00003950
	H(1,2)=1.0D+00	00003960
	AT=DFLOAT(T)	00003970
	IF(T.EQ.1) GO TO 200	00003980
	H(1,1)=DSIN(10.0D+00+(AT))+0.01D+00	00003990
	H(1,2)=DCOS(10.0D+00+(AT))	00004000
200	CONTINUE	00004010
	CALL SHIFT(MOBS,1,ZERO,B)	00004020
	CALL IDEN(N,D)	00004030
	CALL IDEN(MOBS,M)	00004040
	IF (T.GT.15) GOTO 150	00004050
	TRUEX(1,T) = 2.0D+00	00004060
	TRUEX(2,T) = 3.0D+00	00004070
	GOTO 175	00004080
150	TRUEX(1,T) = 4.0D+00	00004090
	TRUEX(2,T) = 5.0D+00	00004100
175	CONTINUE	00004110
	UU = DBLE(GNORM(0))	00004120
	Y(1,1)=H(1,1)*TRUEX(1,T) + H(1,2)*TRUEX(2,T) + SIGMA*UU	00004130
	RETURN	00004140
	END	00004150
C		00004160
	SUBROUTINE OUTPUT(T,N,X,TRUEX)	00004170
	IMPLICIT REAL*8(A-H,O-Z)	00004180
	INTEGER T	00004190
	DIMENSION X(15,110),TRUEX(15,110)	00004200
	L = T	00004210
	WRITE(6,100) L,(X(I,L),I=1,N)	00004220
100	FORMAT(1H0,'TIME EQUALS',I3/1X,'FLS ESTIMATES',7X,2D25.10)	00004230

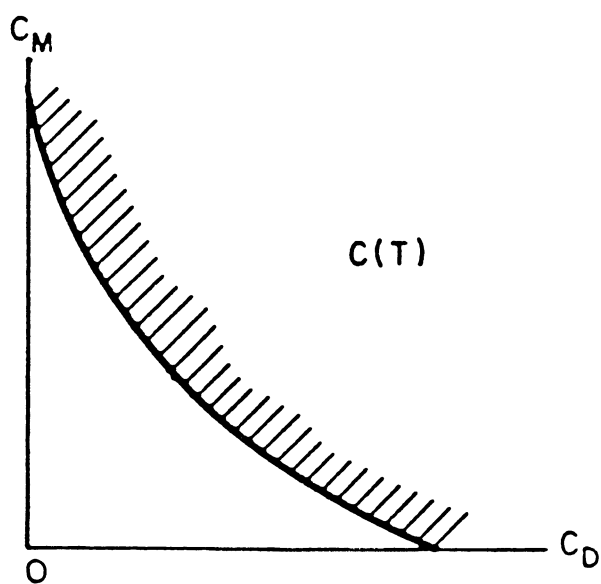


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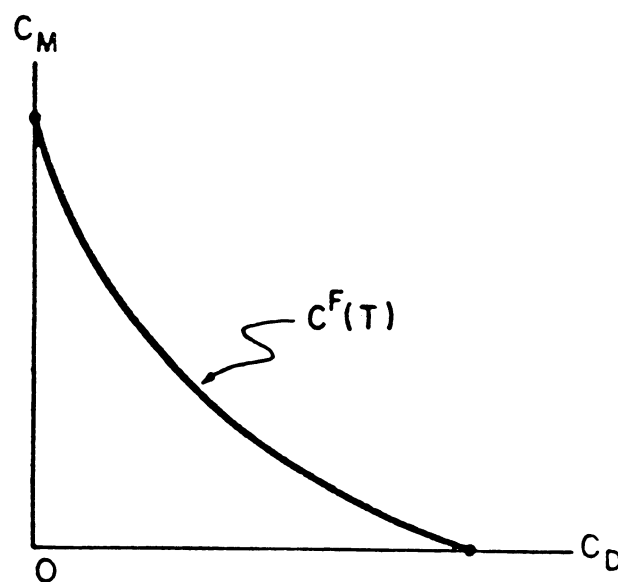
200 WRITE(6,200) (TRUEX(I,L),I=1,N) 00004240
    FORMAT(1X,'TRUE X VALUES',7X,2D25.10) 00004250
    RETURN 00004260
    END 00004270
C 00004280
C VALIDATION TEST: HOW WELL DO THE FLS ESTIMATES SATISFY THE 00004290
C FIRST-ORDER CONDITIONS FOR THE COST MINIMIZATION PROBLEM (5) 00004300
C 00004310
    SUBROUTINE FOCTST(X,YY) 00004320
    IMPLICIT REAL*8(A-H,O-Z) 00004330
    INTEGER T,TP1,TCAP,TCAP1 00004340
    REAL*8 M,MH 00004350
    DIMENSION QZERO(15,15),PZERO(15,15),RZERO(15,15) 00004360
    DIMENSION XT(15,15),X(15,110),XTT(15,15),E(15,15) 00004370
    DIMENSION PZEROT(15,15),EE(15,15),CO(15,15),YT(15,15),YY(15,110) 00004380
    DIMENSION F(15,15),A(15,15),H(15,15),B(15,15),D(15,15),M(15,15) 00004390
    DIMENSION Y(15,15),TRUEX(15,110) 00004400
    DIMENSION MH(15,15),EM(15,15),EMT(15,15),W(15,15),XTP1(15,15) 00004410
    DIMENSION ED(15,15),EDT(15,15),U(15,15),V(15,15),FOCD(15,15) 00004420
    C = -1.0D+00 00004430
C FORM THE STATE VECTOR FOR TIME T = 1 00004440
    CALL INPUT(AMU,TCAP,N,MOBS,QZERO,PZERO,RZERO,IFLAGR,IFLAGS) 00004450
    DO 100 I=1,N 00004460
        XT(I,1) = X(I,1) 00004470
    100 CONTINUE 00004480
C FORM THE INITIAL INCREMENTAL COST CO = -(X1'QO - PO') 00004490
    CALL TRANS(N,1,XT,XTT) 00004500
    CALL MUL(1,N,N,XTT,QZERO,E) 00004510
    CALL TRANS(N,1,PZERO,PZEROT) 00004520
    CALL SUB(1,N,E,PZEROT,EE) 00004530
    CALL MULCON(1,N,C,EE,CO) 00004540
C DO LOOP FOR THE SEQUENTIAL CHECK OF THE FOC FOR T=1,TCAP 00004550
    DO 200 T=1,TCAP 00004560
C FORM THE TIME-T STATE VECTOR XT 00004570
        DO 300 I=1,N 00004580
            XT(I,T) = X(I,T) 00004590
        300 CONTINUE 00004600
C FORM THE TIME-T OBSERVATION VECTOR YT 00004610
        DO 400 J=1,MOBS 00004620
            YT(J,T) = YY(J,T) 00004630
        400 CONTINUE 00004640
        CALL MODEL(T,F,A,H,B,D,M,Y,TRUEX) 00004650
C FORM W = (YT - H(T)XT - B(T)).'M(T)H(T) 00004660
        CALL MUL(MOBS,MOBS,N,M,H,MH) 00004670
        CALL RME(N,MOBS,YT,XT,H,B,EM) 00004680
        CALL TRANS(MOBS,1,EM,EMT) 00004690
        CALL MUL(1,MOBS,N,EMT,MH,W) 00004700
        IF(T.EQ.TCAP) GOTO 600 00004710
C FORM THE TIME-T+1 STATE VECTOR XTP1 00004720
        TP1 = T + 1 00004730
        DO 500 I=1,N 00004740
            XTP1(I,1) = X(I,TP1) 00004750
        500 CONTINUE 00004760

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C	FORM U = AMU*(XTP1 - F(T)XT - A(T))*D(T)	00004770
	CALL RDE(N,XTP1,XT,F,A,ED)	00004780
	CALL TRANS(N,1,ED,EDT)	00004790
	CALL MUL(1,N,N,EDT,D,E)	00004800
	CALL MULCON(1,N,AMU,E,U)	00004810
C	FORM V = U*F	00004820
	CALL MUL(1,N,N,U,F,V)	00004830
	GOTO 800	00004840
600	CONTINUE	00004850
	DO 700 I=1,N	00004860
	V(1,I) = 0.0D+00	00004870
700	CONTINUE	00004880
800	CONTINUE	00004890
C	DETERMINE THE FOC DISCREPANCIES FOR TIME T	00004900
C	GIVEN BY FOCD = CO + V + W	00004910
	CALL ADD(1,N,CO,V,E)	00004920
	CALL ADD(1,N,E,W,FOCD)	00004930
C	PRINT OUT THE FOC DISCREPANCIES FOCD FOR TIME T	00004940
	WRITE (6,36) T	00004950
36	FORMAT(1H0,'FOC DISCREPANCIES FOR TIME',I3)	00004960
	WRITE (6,37) (FOCD(1,I),I=1,N)	00004970
37	FORMAT(1X,13D10.2)	00004980
C	UPDATE THE INITIAL INCREMENTAL COST CO	00004990
	CALL MULCON(1,N,C,U,CO)	00005000
200	CONTINUE	00005010
	RETURN	00005020
	END	00005030
C		00005040
C	SUBROUTINE FOR EVALUATING THE MEASUREMENT SPECIFICATION ERROR	00005050
C	EM = (YT - H(T)XT - B(T)) FOR TIME T	00005060
C		00005070
	SUBROUTINE RME(N,MOBS,YT,XT,H,B,EM)	00005080
	IMPLICIT REAL*8(A-H,O-Z)	00005090
	DIMENSION YT(15,15),XT(15,15),H(15,15),B(15,15),EM(15,15)	00005100
	DIMENSION HX(15,15),HXPB(15,15)	00005110
	CALL MUL(MOBS,N,1,H,XT,HX)	00005120
	CALL ADD(MOBS,1,HX,B,HXPB)	00005130
	CALL SUB(MOBS,1,YT,HXPB,EM)	00005140
	RETURN	00005150
	END	00005160
C		00005170
C	SUBROUTINE FOR EVALUATING THE DYNAMIC SPECIFICATION ERROR	00005180
C	ED = (XTP1 - F(T)XT - A(T)) FOR TIME T	00005190
C		00005200
	SUBROUTINE RDE(N,XTP1,XT,F,A,ED)	00005210
	IMPLICIT REAL*8(A-H,O-Z)	00005220
	DIMENSION XTP1(15,15),XT(15,15),F(15,15),A(15,15),ED(15,15)	00005230
	DIMENSION FXT(15,15),FXTPA(15,15)	00005240
	CALL MUL(N,N,1,F,XT,FXT)	00005250
	CALL ADD(N,1,FXT,A,FXTPA)	00005260
	CALL SUB(N,1,XTP1,FXTPA,ED)	00005270
	RETURN	00005280
	END	00005290



(a) Cost Possibility Set  $C(T)$



(b) Cost-Efficient Frontier  $C^F(T)$

FIGURE 1

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