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BAYESIAN SPECIFICATION ANALYSIS AND  
ESTIMATION OF SIMULTANEOUS EQUATION  
MODELS USING MONTE CARLO METHODS

ARNOLD ZELLNER

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MRG WORKING PAPER #M8804

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**ABSTRACT**

Bayesian methods for specification analysis or diagnostic checking of the simultaneous equation model are formulated and applied in analysis of two models. In this work, a direct Monte Carlo simulation approach is employed to compute exact posterior distributions of parameters measuring discrepancies from specifying assumptions, e.g., identifying restrictions exogeneity, etc. Also, a new approach for calculating the posterior distributions of a structural equation's parameters is developed and applied using Monte Carlo numerical methods. It is concluded that the methods developed will permit convenient computation of exact finite sample specification error and estimation results for simultaneous equation models.

Bayesian Specification Analysis and Estimation of  
Simultaneous Equation Models Using Monte Carlo Methods

by

Arnold Zellner, Luc Bauwens and Herman K. van Dijk\*

I. Introduction

There have been many studies relating to limited information estimation of the parameters of the simultaneous equation model (SEM) from both the Bayesian and non-Bayesian points of view--see, e.g. Zellner (1971, Ch. 9), Drèze (1976), Drèze and Richard (1983), Hausman (1983), Tsurumi (1985, 1987), and the references cited in these works. In non-Bayesian approaches, there is usually reliance on asymptotic approximations in making inferences.<sup>1</sup> Some previous Bayesian approaches also involve asymptotic approximations. A problem in previous exact Bayesian analyses is that posterior distributions of structural parameters are in most cases not analytically tractable<sup>2</sup> and thus must be integrated numerically to obtain their moments, marginal distributions, etc. As regards Monte Carlo numerical integration, usual posterior distributions of structural parameters do not have simple forms from which draws can be made easily. As a consequence, the success

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<sup>1</sup>A brief discussion of small sample results in non-Bayesian limited information estimation of the SEM is given by Anderson (1984, pp. 518 and 519). Tsurumi (1987) reports Monte Carlo experimental results.

<sup>2</sup>An exception is Drèze (1976) where the posterior density is in the poly-t family. Then one can, in some cases, compute moments of structural coefficients analytically. See also Tsurumi (1985, 1987).

of Monte Carlo integration procedures depends importantly on an investigator's ability to find distribution functions that are good approximations to posterior distributions and from which pseudo-random drawings can be made easily. Also, past Bayesian analyses of the SEM have not devoted much attention to diagnostic checking of models' assumptions, that is to specification error analysis.

In the present paper, we start from the reduced form of the SEM and make a distinction between "unrestricted reduced form analysis" (URFA) and "restricted reduced form analysis" (RRFA). In our URFA, we define indirect least squares, generalized indirect least squares, two-stage least squares and limited information maximum likelihood mappings or functions of unrestricted reduced form coefficients which do not require that overidentifying restrictions hold exactly and obtain complete posterior distributions of these mappings or functions by a direct Monte Carlo simulation approach. Also discrepancy vectors and discrepancy functions are introduced which measure the extent to which overidentifying restrictions are in error and we indicate how to obtain their posterior distributions by a direct simulation approach. One may also use Bayesian realized error analysis, Zellner (1975), to provide further diagnostic checks of the SEM.

In the case that exact identifying restrictions are imposed, we present a RRFA and discuss a method for computing posterior distributions of structural parameters which makes use of Monte Carlo integration in a relatively simple way, namely a direct simulation approach.

The plan of our paper is as follows. In Section 2 we consider simple, canonical models to illustrate our approach and go on to specify a

general system. Then various mappings of the URF coefficients are introduced and we indicate how to compute their posterior distributions, moments, etc. This is followed by an analysis of the RRF system to obtain posterior distributions of structural coefficients. Section 3 is devoted to further diagnostic checking procedures. In Section 4, our methods are applied in illustrative analyses of several well known models using actual data. Section 5 provides some concluding remarks. An efficient algorithm for generating pseudo-random drawings from a matrix-Student-t distribution is presented in the Appendix.

## 2. Model Specification, Interpretation and Analysis

In this section we first consider canonical models to illustrate features of our approach. Then we specify unrestricted reduced form (URF) systems and indicate how to compute posterior distributions for interesting functions or mappings of URF coefficients. These functions or mappings are related to discrepancy vectors which measure departures of the URF coefficients from satisfying usual overidentifying restrictions. Next, we impose identifying and normalizing restrictions, derive the posterior distribution of the parameters of a single structural equation using diffuse and informative prior distributions and discuss a Monte Carlo integration procedure for the computation of posterior moments and densities. Also, various conditional posterior distributions centered at OLS, 2SLS, LIML, and MELO point estimates and diagnostic checks of the validity of over-identifying restrictions are provided.

## 2.1 Canonical Models

The first canonical model is a "means model" for two endogenous variables, namely,

$$y_{1i} = \eta_i + v_{1i} \quad i = 1, 2, \dots, n \quad (2.1a)$$

$$y_{2i} = \xi_i + v_{2i} \quad (2.1b)$$

where  $\eta_i$  and  $\xi_i$  are means of  $y_{1i}$  and  $y_{2i}$ , respectively, and the zero-mean disturbance terms,  $v_{1i}$  and  $v_{2i}$  are assumed independently drawn from a bivariate normal distribution with  $2 \times 2$  positive definite symmetric (pds) covariance matrix. For example,  $\eta_i$  and  $\xi_i$  can be interpreted as the  $i$ 'th individual's "permanent" or "anticipated" consumption and income, respectively, whereas  $y_{1i}$  and  $y_{2i}$  are their measured counterparts. Interest may center on various functions of the  $\eta_i$ 's and  $\xi_i$ 's, for example  $\eta_i/\xi_i$ ,  $i = 1, 2, \dots, n$ , the "permanent consumption-income" ratios,  $\bar{\eta} = \sum_{i=1}^n \eta_i/n$ ,  $\bar{\xi} = \sum_{i=1}^n \xi_i/n$ ,  $\sigma_{\xi\xi} = \sum_{i=1}^n (\xi_i - \bar{\xi})^2/n$ ,  $\sigma_{\eta\eta} = \sum_{i=1}^n (\eta_i - \bar{\eta})^2/n$ ,  $\sigma_{\xi\eta} = \sum_{i=1}^n (\xi_i - \bar{\xi})(\eta_i - \bar{\eta})/n$ , higher order moments, skewness and kurtosis measures, etc. Further, weighted averages of the ratios  $\eta_i/\xi_i$ , e.g.,  $\bar{\gamma}_1 = \bar{\eta}/\bar{\xi} = \sum_{i=1}^n (\eta_i/\xi_i)\xi_i / \sum_{i=1}^n \xi_i$  or  $\bar{\gamma}_2 = \sum_{i=1}^n (\eta_i/\xi_i)\xi_i^2 / \sum_{i=1}^n \xi_i^2 = \sum_{i=1}^n \eta_i \xi_i / \sum_{i=1}^n \xi_i^2 = \xi' \eta / \xi' \xi$ , where  $\eta' = (\eta_1, \eta_2, \dots, \eta_n)$  and  $\xi' = (\xi_1, \xi_2, \dots, \xi_n)$  might be of interest. If we write

$$\underline{\eta} = \xi \gamma + \underline{\Delta}_1 \quad (2.2)$$

where  $\gamma$  is a scalar parameter and  $\underline{\Delta}_1$  is an  $n \times 1$  discrepancy vector, which measures the extent to which the  $\eta_i/\xi_i$  depart from a common value  $\gamma$ , then  $\bar{\gamma}_2 = \xi' \eta / \xi' \xi$  is the value of  $\gamma$  that minimizes  $\underline{\Delta}_1' \underline{\Delta}_1 = (\underline{\eta} - \xi \gamma)' (\underline{\eta} - \xi \gamma)$ , a discrepancy function. Also, the functions  $\bar{\sigma}_1^2 = (\underline{\eta} - \xi \bar{\gamma}_1)' (\underline{\eta} - \xi \bar{\gamma}_1) / n$  and  $\bar{\rho}_1^2 = 1 - \frac{\bar{\sigma}_1^2}{\eta' \eta}$  are of interest and have obvious regression interpretations.

Given a posterior distribution for the  $2n$  parameters,  $\eta$  and  $\xi$ , draws can be made from it and complete posterior distributions for  $\eta_i/\xi_i$ ,  $\bar{\eta}$ ,  $\bar{\xi}$ ,  $\sigma_{\xi\xi}$ ,  $\sigma_{\eta\eta}$ ,  $\sigma_{\xi\eta}$ ,  $\bar{\gamma}$ ,  $\bar{\sigma}_1^2$ ,  $\bar{\rho}_1^2$ , etc., can be obtained by a direct Monte Carlo approach, that is by repeated evaluations of these quantities using independent draws from the joint distribution. If it is the case that the distribution of  $\bar{\sigma}_1^2$  is centered far from zero, there is little support for the assumption  $\Delta_1 = 0$  or  $\eta = \gamma\xi$ . On the other hand, if  $\bar{\sigma}_1^2$ 's distribution is centered close to zero, this provides some support for the assumption  $\Delta_1 = 0$  and with this assumption, the model becomes a form of the usual "errors-in-the-variables" model. While we do not pursue the matter now, it is also possible to compute posterior odds relating to the hypotheses  $\Delta_1 = 0$  and  $\Delta_1 \neq 0$ .

If in addition to (2.1), we have proxies for  $\eta_i$  and  $\xi_i$ , namely,

$$\eta_i = \underline{x}_i^T \underline{\pi}_1 \quad (2.3a)$$

$$\xi_i = \underline{x}_i^T \underline{\pi}_2 \quad (2.3b)$$

where  $\underline{x}_i^T$  is a  $1 \times k$  vector of predetermined variables, a typical row of an  $n \times k$  matrix  $X$ , assumed of full column rank, and  $\underline{\pi}_1$  and  $\underline{\pi}_2$  are  $k \times 1$  coefficient vectors, the number of location parameters is reduced from  $2n$   $\xi_i$ 's and  $\eta_i$ 's to  $2k$   $\pi$ 's. Using (2.3), we can express (2.1) in matrix form as follows:

$$\underline{y}_1 = X \underline{\pi}_1 + \underline{v}_1 \quad (2.4a)$$

$$\underline{y}_2 = X \underline{\pi}_2 + \underline{v}_2 \quad (2.4b)$$

where  $\underline{y}_1$ ,  $\underline{y}_2$ ,  $\underline{v}_1$  and  $\underline{v}_2$  are  $n \times 1$  vectors with typical elements  $y_{1i}$ ,  $y_{2i}$ ,  $v_{1i}$  and  $v_{2i}$ , respectively.

In (2.4), we have two URF equations. Just as with (2.1), we may be interested in various functions or mappings of the URF coefficients, the analogues of those for  $\eta_i$  and  $\xi_i$  with  $x_i'\pi_1$  and  $x_i'\pi_2$  replacing  $\eta_i$  and  $\xi_i$ , respectively in their definitions. Also, we can introduce

$$X\pi_1 = X\pi_2\gamma + \Delta_2 \quad (2.5)$$

where  $\Delta_2$  is an  $n \times 1$  discrepancy vector. Then  $\bar{\gamma}_2 = \pi_2'X'X\pi_1/\pi_2'X'X\pi_2$  is the value of  $\gamma$  that minimizes  $\Delta_2'\Delta_2$ . Further,  $\bar{\sigma}_2^2 = (X\pi_1 - X\pi_2\bar{\gamma}_2)'(X\pi_1 - X\pi_2\bar{\gamma}_2)/n$  and  $\bar{\rho}_2^2 = 1 - n\bar{\sigma}_2^2/\pi_1'X'X\pi_1$  are regression-like mappings of the  $\pi$ 's which are of interest. Also, if we consider

$$\bar{\pi}_1 = \bar{\pi}_2\gamma + \Delta_3, \quad (2.6)$$

where  $\Delta_3$  is a  $k \times 1$  discrepancy vector, then the value of  $\gamma$ , say  $\bar{\gamma}_3$  minimizing  $\Delta_3'\Delta_3$  is just  $\bar{\gamma}_3 = \pi_2'\pi_1/\pi_2'\pi_2$  and  $\bar{\sigma}_3^2 = (\bar{\pi}_1 - \bar{\pi}_2\bar{\gamma}_3)'(\bar{\pi}_1 - \bar{\pi}_2\bar{\gamma}_3)/k$  and  $\bar{\rho}_3^2 = 1 - \bar{\sigma}_3^2/\pi_1'\pi_1$  are measures of the extent to which  $\Delta_3 = 0$  holds.

Given a joint posterior pdf for  $\pi_1$  and  $\pi_2$  from which draws can be made, a direct Monte Carlo simulation approach can be employed to obtain the posterior distributions of  $\bar{\gamma}_2$ ,  $\bar{\gamma}_3$ ,  $\bar{\sigma}_2^2$ ,  $\bar{\sigma}_3^2$ ,  $\bar{\rho}_2^2$ ,  $\bar{\rho}_3^2$ , etc., since these quantities are given functions or mappings of the unrestricted  $\pi$ 's.

If  $\Delta_2 = 0$  in (2.6) or  $\Delta_3 = 0$  in (2.7), we have the case of exact restrictions. Then (2.4) can be written as

$$Y_1 = X\pi_2\gamma + v_1 \quad (2.7a)$$

$$Y_2 = X\pi_2 + v_2 \quad (2.7b)$$

or

$$Y_1 = Y_2\gamma + u_1 \quad (2.7c)$$

$$Y_2 = X\pi_2 + u_2 \quad (2.7d)$$

where  $\underline{u}_1 = \underline{v}_1 - \underline{v}_2 \gamma$  and  $\underline{u}_2 = \underline{v}_2$ . (2.7a-b) is the restricted reduced form (RRF) equation system which can also be expressed in structural form as shown in (2.7c-d). On introducing a prior distribution for  $\gamma$ ,  $\pi_2$  and the reduced form disturbance covariance matrix, we can obtain a posterior distribution for these parameters. Note that in working with (2.7a-b), it is assumed that the overidentifying restrictions hold exactly, that is  $\Delta_2 = 0$  in (2.5) or  $\Delta_3 = 0$  in (2.6). The number of coefficients in (2.7) is  $k+1$ , usually a large reduction from the  $2k$  coefficients in (2.4) for  $k > 1$ . When  $k=1$ , the case of "just identification," the number of coefficients in the URF and RRF is the same. Also, relative to the  $2n$  location parameters in (2.1), the reduction is much larger. This reduction, however, is dependent not only on the identifying restrictions holding exactly but also on the appropriateness of the proxy expressions in (2.3). Diagnostic checking procedures relating to these assumptions will be described in a subsequent section.

We now turn to provide results for general cases including mappings of reduced form coefficients in the unrestricted case and posterior distributions for structural parameters in the restricted reduced form case after introducing some needed notation. Let  $Y_a = (y_1 : Y_1 : Y_0)$  denote an  $n \times m'$  matrix of observations on  $m'$  endogenous variables with URF,

$$(y_1 : Y_1 : Y_0) = X(\pi_1 : \pi_1 : \pi_0) + (\underline{v}_1 : v_1 : v_0) \quad (2.8)$$

where  $X$  is an  $n \times k$  matrix of observations on  $k$  predetermined variables of rank  $k$  and the rows of the disturbance matrix have been independently drawn from a zero-mean multivariate normal distribution with a pds covariance matrix. A structural equation, say the first, with normalization imposed can

be written as

$$(y_1 : y_1 : y_0) \begin{pmatrix} 1 \\ -y_1 \\ 0 \end{pmatrix} = (x_1 : x_0) \begin{pmatrix} \beta_1 \\ \tilde{\beta}_1 \\ 0 \end{pmatrix} + u_1 \quad (2.9a)$$

or

$$y_1 - y_1 y_1 = x_1 \beta_1 + u_1 \quad (2.9b)$$

where  $y_0$  and  $x_0$  are observations on endogenous and predetermined variables excluded from the first equation and  $X = (x_1 : x_0)$ . The  $m_1 \times 1$  vector  $y_1$  and the  $k_1 \times 1$  vector  $\beta_1$  are the structural coefficients and  $u_1$  is an  $n \times 1$  vector of structural disturbance terms.

To obtain the well known restrictions on the reduced form coefficients, we write (2.8) as

$$(y_1 : y_1 : y_0) = (x_1 : x_0) \begin{pmatrix} \pi_{11} : \pi_{11} : \pi_{01} \\ \pi_{10} : \pi_{10} : \pi_{00} \end{pmatrix} + (v_1 : v_1 : v_0) \quad (2.10)$$

and on multiplying both sides of (2.10) on the right by  $(1 : -y_1' : 0')'$ , the result is:

$$y_1 - y_1 y_1 = (x_1 : x_0) \begin{pmatrix} \pi_{11} - \pi_{11} y_1 \\ \pi_{10} - \pi_{10} y_1 \end{pmatrix} + v_1 - v_1 y_1 \quad (2.11)$$

For compatibility with (2.9b),  $u_1 = v_1 - v_1 y_1$  and

$$\pi_{11} - \pi_{11} y_1 = \beta_1 \quad (2.12a)$$

$$\pi_{10} - \pi_{10} y_1 = \tilde{\beta}_1 \quad (2.12b)$$

which are restrictions on the reduced form coefficients with  $y_1$  and  $\beta_1$  appearing in them, a generalization of (2.6) with  $\beta_3 = 0$ . In (2.12b)  $\pi_{10}$  is assumed to be of full column rank.

On substituting for  $(\pi'_{11} : \pi'_{10})'$  in (2.10) from (2.12), the RRF equations for  $y_1$  and  $Y_1$  are:

$$y_1 = X\pi'_{11}y_1 + X_1\pi'_{10} + v_1 \quad (2.13a)$$

$$Y_1 = X\pi'_{11} + V_1 \quad (2.13b)$$

where  $\pi'_1 = (\pi'_{11} : \pi'_{10})$ . It is seen that (2.13) is in the form of a multivariate nonlinear regression model, a generalization of (2.7). The system in (2.13) will serve as the starting point for an analysis of the RRF system, whereas

$$(y_1 : Y_1) = X(\pi_1 : \pi_1) + (v_1 : V_1). \quad (2.14a)$$

will serve as the starting point for the URF analysis of the data  $(y_1 : Y_1)$ .

## 2.2 Mappings of Unrestricted Reduced Form (URF) Coefficients

We shall obtain a posterior distribution for the parameters of (2.14a) and use it to obtain posterior distributions of interesting functions or mappings of the URF coefficients,  $(\pi_1 : \pi_1)$ . For convenience, we write  $Y = (y_1 : Y_1)$ ,  $\pi = (\pi_1 : \pi_1)$  and  $V = (v_1 : V_1)$  and thus (2.14a) becomes

$$\begin{matrix} Y \\ n \times m \end{matrix} = \begin{matrix} X \\ n \times k \end{matrix} \begin{matrix} \pi \\ k \times m \end{matrix} + \begin{matrix} V \\ n \times m \end{matrix} \quad (2.14b)$$

The  $n$  rows of  $V$  are assumed to be independently drawn from a multivariate normal distribution with zero mean vector and  $m \times m$  pds covariance matrix  $\Omega$ , i.e.  $MVN(0, \Omega)$ . If  $X$  includes lagged endogenous variables, we assume that initial or starting values are given. Then the likelihood function for (2.14b) is

$$\begin{aligned} \ell(\pi, \Omega | Y, X) &= |\Omega|^{-n/2} \exp\{-\frac{1}{2}\text{tr}(Y - X\pi)'(Y - X\pi)\Omega^{-1}\} \\ &\approx |\Omega|^{-n/2} \exp\{-\frac{1}{2}\text{tr}[S + (\pi - \hat{\pi})'X'X(\pi - \hat{\pi})]\Omega^{-1}\} \end{aligned} \quad (2.15)$$

where  $\propto$  denotes "is proportional to",

$$\hat{\Pi} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad (2.16a)$$

and

$$\mathbf{S} = (\mathbf{Y} - \mathbf{X}\hat{\Pi})'(\mathbf{Y} - \mathbf{X}\hat{\Pi}). \quad (2.16b)$$

It is seen that the likelihood function in (2.15) is in the same form as that for a multivariate regression model--see, e.g., Zellner (1971, Ch. 8) with  $\hat{\Pi}$  and  $\mathbf{S}$  sufficient statistics.

We shall employ the following standard diffuse prior distribution for  $\Pi$  and the distinct elements of  $\Omega$ ,<sup>3</sup>

$$p(\Pi, \Omega) \propto |\Omega|^{-(m+1+v_0)/2}, \quad (2.17)$$

where  $v_0 \geq 0$ , that is the elements of  $\Pi$  and  $\Omega$  are independent, with the former being uniformly distributed and the latter in the form of a degenerate, inverted Wishart distribution.

On multiplying (2.15) and (2.17) and using  $v_0 = 0$ , we obtain by Bayes's Theorem the joint posterior density of  $\Pi$  and  $\Omega$ , namely

$$p(\Pi, \Omega | D) \propto |\Omega|^{-(n+m+1)/2} \exp\{-\frac{1}{2}\text{tr}[\mathbf{S} + (\Pi - \hat{\Pi})' \mathbf{X}' \mathbf{X} (\Pi - \hat{\Pi})] \Omega^{-1}\} \quad (2.18)$$

where  $D$  denotes the given sample information  $(\mathbf{Y}, \mathbf{X})$  and prior information in (2.17). On integrating (2.18) with respect to  $\Omega$ , we obtain the well known marginal posterior density for  $\Pi$ ,

$$p(\Pi | D) \propto |\mathbf{S} + (\Pi - \hat{\Pi})' \mathbf{X}' \mathbf{X} (\Pi - \hat{\Pi})|^{-n/2} \quad (2.19)$$

which is in the form of a matrix Student-t density--see, e.g. Dickey (1967),

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<sup>3</sup>The value  $v_0 = k$  in the exponent of (2.17) has been suggested by Drèze (1976) while Zellner (1971) employs  $v_0 = 0$ .

Box and Tiao (1973), Drèze and Richard (1983), Geisser (1965), and Zellner (1971) for properties of this distribution. As explained below, it is possible to make independent draws from (2.19) and to use them to determine the posterior distributions of interesting functions or mappings of the elements of  $\Pi$ . Some of these mappings are given below.

We first consider the case of "just-identification" in which the matrix  $\Pi_{10}$  in (2.12b) is square and non-singular and the matrix  $(\pi_{10} : -\Pi_{10})$  is not of full column rank. Then (2.12b) has a unique solution for  $\gamma_1$ —see Graybill (1969, p. 140), and this solution can be substituted in (2.12a) to express  $\beta_1$  in terms of the RF coefficients. Explicitly, we have,

$$\beta_1 = \pi_{11} - \Pi_{11} \Pi_{10}^{-1} \pi_{10} \quad (2.20a)$$

$$\gamma_1 = \Pi_{10}^{-1} \pi_{10} \quad (2.20b)$$

which we call the Indirect Least Squares (ILS) mapping since if least squares estimates of the  $\Pi$ 's are inserted in (2.20), the result is the "indirect least squares" estimate of non-Bayesian econometrics. In the Bayesian approach, with the posterior distribution for  $\Pi$  in (2.19), the least squares quantity  $\hat{\Pi} = (X'X)^{-1}X'Y$  is the modal value and mean of (2.19) and the ILS estimate is the modal value of the posterior distribution of  $\beta_1$  and  $\gamma_1$  in this case of "exact identification" since (2.20) is a one-to-one transformation from the  $\Pi$ 's to  $\beta_1$  and  $\gamma_1$ . Further, as explained below, we can make independent draws from the matrix Student-t posterior distribution for  $\Pi$  in (2.19) and evaluate  $\beta_1$  and  $\gamma_1$  for each draw by use of (2.20) and thus obtain the complete posterior distributions for the elements of  $\beta_1$  and  $\gamma_1$ . Also, various measures associated with these distributions can be cal-

culated, for example medians, inter-quartile ranges, means (if they exist), etc., as will be illustrated in computed examples below.<sup>4</sup>

In the case of overidentification, the matrix  $\Pi_{10}$  in (2.12b) has dimension  $k_0 \times m_1$ , where  $k_0$  is the number of columns of  $X_0$  or the number of predetermined variables left out of the first structural equation in (2.9b) and  $m_1$  is the number of columns of  $Y_1$  or the number of endogenous variables included in (2.9b) less one. The rank condition for identification of the structural coefficients  $\gamma_1$  and  $\beta_1$  is that the rank of  $\Pi_0$  is  $m_1$ , which requires  $k_0 > m_1$ , the order condition in the overidentified case. In the over-identified case, we cannot go from the URF coefficients, the elements of  $\Pi$  in (2.14b) and (2.19) to the elements of  $\gamma_1$  and  $\beta_1$ . For example in (2.6) with  $\Delta_3 = 0$ ,  $\pi_1 = \pi_2 \gamma$  and given that  $\pi_1$  and  $\pi_2$  are a.s. linearly independent in the URF, we cannot solve for  $\gamma$  in terms of the elements of the vectors of URF coefficients,  $\pi_1$  and  $\pi_2$ . In fact, we can only find an approximate solution (Graybill (1969), p. 103ff.) as follows. Just as in (2.6), we shall append a discrepancy vector,  $\Delta_2$ , to (2.12). This yields

$$\pi_{11} - \Pi_{11}\gamma_1 = \beta_1 \quad (2.21a)$$

$$\pi_{10} - \Pi_{10}\gamma_1 = \Delta_2. \quad (2.21b)$$

We can now define discrepancy functions and obtain values of  $\gamma_1$  and  $\beta_1$  which minimize them. One example of a discrepancy function is  $\Delta_2'\Delta_2$  and the value of  $\gamma_1$  which minimizes this function, denoted by  $\gamma_1^*$  and the associated value of  $\beta_1$ ,  $\beta_1^*$  are

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<sup>4</sup>Drèze (1976, p. 1055) discusses conditions for existence of moments of structural coefficients.

$$\tilde{\beta}_1^* = \tilde{\pi}_{11} - \tilde{\pi}_{11}\tilde{y}_1^* \quad (2.22a)$$

$$\tilde{y}_1^* = (\tilde{\pi}'_{10}\tilde{\pi}_{10})^{-1}\tilde{\pi}'_{10}\tilde{\pi}_{10}. \quad (2.22b)$$

We shall call the mapping in (2.22) the Generalized Indirect Least Squares (GILS) mapping since when least squares estimates of the  $\pi$ 's are inserted in (2.22), the result is the GILS estimate--see Khazoom (1976). In our Bayesian approach, the posterior distribution of the elements of  $\tilde{\beta}_1^*$  and  $\tilde{y}_1^*$  can be computed by direct Monte Carlo simulation based on draws from the matrix Student-t posterior distribution for  $\tilde{\pi}$  in (2.19). Also posterior distributions for discrepancy functions can be computed, for example

$$\tilde{\Delta}'\tilde{\Delta}_2/k_0 = (\tilde{\pi}_{10} - \tilde{\pi}_{10}\tilde{y}_1^*)'(\tilde{\pi}_{10} - \tilde{\pi}_{10}\tilde{y}_1^*)/k_0 \quad (2.23a)$$

and

$$\tilde{\rho}_2^2 = 1 - \tilde{\Delta}'\tilde{\Delta}_2/\tilde{\pi}'_{10}\tilde{\pi}_{10}. \quad (2.23b)$$

Also, the posterior distributions of the elements of  $\tilde{\Delta}_2 = \tilde{\pi}_{10} - \tilde{\pi}_{10}\tilde{y}_1^*$  can be computed by direct Monte Carlo simulation. The posterior distributions of  $\tilde{\Delta}_2$ ,  $\tilde{\Delta}'\tilde{\Delta}_2/k_0$  and  $\tilde{\rho}_2^2$  will provide information regarding the validity of the exact restrictions in (2.12) in the frequently encountered overidentified case.

We next turn to a mapping that involves the matrix of predetermined variables by multiplying both sides of (2.12) on the left by  $X = (X_1 \ X_0)$  to obtain

$$\begin{aligned} X\tilde{\pi}_1 &= X\tilde{\pi}_{11}\tilde{y}_1 + X_1\tilde{\beta}_1 \\ &= \bar{Z}_1\tilde{\delta}_1 \end{aligned} \quad (2.24)$$

where  $\tilde{\pi}'_1 = (\tilde{\pi}'_{11} \ \tilde{\pi}'_{10})$ ,  $\tilde{\pi}'_1 = (\tilde{\pi}'_{11} \ \tilde{\pi}'_{10})$ ,  $\bar{Z}_1 = (X\tilde{\pi}_1 \ X_1)$  and  $\tilde{\delta}'_1 = (\tilde{y}_1' \ \tilde{\beta}_1')$ . To allow for possible errors in the exact restrictions in (2.24), we introduce a discrepancy vector,  $\tilde{\Delta}_3$  as follows,

$$X\tilde{\pi}_1 = \bar{Z}_1\tilde{\delta}_1 + \tilde{\Delta}_3. \quad (2.25)$$

Then, just as in the cases considered above, we can minimize the discrepancy function  $\tilde{\Delta}_3^{\prime} \tilde{\Delta}_3$  with respect to  $\tilde{\delta}_1$  to obtain,

$$\tilde{\delta}_1^* = (\tilde{Z}_1' \tilde{Z}_1)^{-1} \tilde{Z}_1' X_{\tilde{\pi}_1} \quad (2.26)$$

as the minimizing value which defines a mapping of the  $\pi$ 's which resembles that arising in 2SLS estimation.<sup>5</sup> Thus we call (2.26) the 2SLS Mapping.

Also from (2.25) and (2.26), we can define

$$\tilde{\Delta}_3 = X_{\tilde{\pi}_1} - \tilde{Z}_1 \tilde{\delta}_1^*, \quad (2.27a)$$

$$\tilde{\Delta}_3' \tilde{\Delta}_3 / n = (X_{\tilde{\pi}_1} - \tilde{Z}_1 \tilde{\delta}_1^*)' (X_{\tilde{\pi}_1} - \tilde{Z}_1 \tilde{\delta}_1^*) / n \quad (2.27b)$$

and

$$\tilde{\rho}_3^2 = 1 - \tilde{\Delta}_3' \tilde{\Delta}_3 / \tilde{\pi}_1' X' X_{\tilde{\pi}_1} \quad (2.27c)$$

Posterior distributions of  $\tilde{\delta}_1^*$ ,  $\tilde{\Delta}_3$ ,  $\tilde{\Delta}_3' \tilde{\Delta}_3$ ,  $\tilde{\rho}_3^2$  and other interesting functions of the URF coefficients can be calculated using a direct Monte Carlo simulation approach based on draws from the matrix Student-t distribution in (2.19).

Last, we define a LIML Mapping as follows. Write the URF system for  $Y = (y_1, Y_1)$  in (2.14) as

$$Y = X_1 \Pi_{1.} + X_0 \Pi_{0.} + V \quad (2.28)$$

where  $\Pi' = (\Pi_1', \Pi_0')$  and multiply both sides of (2.28) on the right by  $\gamma_a = (1 : \gamma_1')$  to obtain

$$\begin{aligned} Y \gamma_a &= X_1 \Pi_{1.} \gamma_a + X_0 \Pi_{0.} \gamma_a + V \gamma_a \\ &= X \Pi \gamma_a + V \gamma_a. \end{aligned} \quad (2.29)$$

Note that  $\Pi_{0.} \gamma_a = 0$  if the restrictions in (2.12) hold and thus we introduce a "variance ratio" discrepancy function,

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<sup>5</sup>An alternative procedure to compute  $\tilde{\delta}_1^*$  is presented in Section 3.

$$\phi = \frac{y_a' V' V r_r y_a}{y_a' V' V y_a} \quad (2.30)$$

where  $V = Y - X\pi$  and  $V_r = Y - X_1\pi_1$ . With  $\bar{\lambda}$  being the smallest root of  $|V_r' V_r - \lambda V' V| = 0$ , the value of  $y_a$  minimizing  $\phi$  in (2.30) is obtained by solving the following set of equations, given  $\pi$ ,  $X$  and  $Y$ ,

$$(V_r' V_r - \bar{\lambda} V' V) y_a = 0.$$

The solution is  $y_a^* = (1 : -y_1^*)'$  and we can then define  $\pi_1^* = \pi_{11} - \pi_{11} y_1^*$  from the restrictions in (2.12). Thus  $\pi_1^* = (y_1^* \ \pi_1^*)$  is the LIML Mapping which can be substituted in (2.30) to yield  $\phi^* = y_a^* V' V r_r y_a^* / y_a^* V' V y_a^*$ . The posterior distributions of  $\pi_1^*$ ,  $\phi^*$ ,  $\pi_{0.1}^*$ , etc. can be calculated by direct Monte Carlo simulation based on independent draws of  $\pi$  from its posterior distribution in (2.19).

We have discussed various mappings that are useful in connection with URF analysis which do not involve assuming that identifying restrictions hold exactly. One may extend the GILS mapping and the 2SLS mapping to the case of a full system of equations [see van Dijk (1985)]. We shall not pursue this extension herein. We turn now to the derivation of posterior distributions for structural parameters in a RRF framework.

### 2.3 Restricted Reduced Form Analysis (RRFA)

We now assume that the restrictions in (2.12) and in the line above (2.12), hold exactly and impose them to obtain the RRF system of the equations for  $y_1$  and  $Y_1$  as follows. Substitute the expression  $y_1 = u_1 + v_1 Y_1$  in (2.13a); use (2.13b) and (2.10), and reexpress (2.13) as

$$(y_1 \ Y_1) \begin{bmatrix} 1 & 0 \\ -y_1 & I \end{bmatrix} = (x_1 \ x_0) \begin{bmatrix} \pi_1 & \pi_{11} \\ 0 & \pi_{10} \end{bmatrix} + (u_1 \ v_1). \quad (2.13')$$

Assuming that the rows of  $(\underline{u}_1 \ \underline{v}_1)$  are independently drawn from a zero-mean normal distribution with PDS covariance matrix  $\Omega^*$  where

$$\Omega^* = \begin{bmatrix} \sigma_1^2 & \underline{w}_1' \\ \underline{w}_1 & \Omega_1 \end{bmatrix}, \quad (2.13'')$$

one can write the likelihood function

$$l(\underline{\delta}_1, \underline{\Pi}_1, \Omega^* | D) \propto |\Omega^*|^{-n/2} \exp\{-\frac{1}{2} \text{tr}[(\underline{u}_1 \ \underline{v}_1)' (\underline{u}_1 \ \underline{v}_1) \Omega^{*-1}]\} \quad (2.31)$$

where  $\underline{\delta}_1' = (\underline{y}_1' \ \underline{s}_1')$ ,  $D = (Y \ X)$  and  $(\underline{u}_1 \ \underline{v}_1)$  is restricted by the equations (2.13'). A well-known diffuse prior for the parameters of (2.31) is

$$p(\underline{\delta}_1, \underline{\Pi}_1, \Omega^*) \propto |\Omega^*|^{-(m_1 + 2 + v_0)/2} \quad (2.32)$$

where  $v_0$  ( $\geq 0$ ) can be chosen in accordance with invariance considerations. More informative priors are discussed below. Multiplying (2.31) and (2.32) gives the posterior pdf as

$$p(\underline{\delta}_1, \underline{\Pi}_1, \Omega^* | D) \propto |\Omega^*|^{-(n_* + m_1 + 2)/2} \times \exp\{-\frac{1}{2} \text{tr}[(\underline{u}_1 \ \underline{v}_1)' (\underline{u}_1 \ \underline{v}_1) \Omega^{*-1}]\} \quad (2.33)$$

where  $n_* = n + v_0$ . On integrating the posterior with respect to the elements of  $\Omega^*$ , one obtains the marginal pdf for  $\underline{\delta}_1$  and  $\underline{\Pi}_1$ , given as

$$p(\underline{\delta}_1, \underline{\Pi}_1 | D) \propto |(\underline{u}_1 \ \underline{v}_1)' (\underline{u}_1 \ \underline{v}_1)|^{-n_*/2}. \quad (2.34)$$

We now make use of

$$|(\underline{u}_1 \ \underline{v}_1)' (\underline{u}_1 \ \underline{v}_1)| = (\underline{u}_1' M_1 \underline{u}_1) |\underline{v}_1' \underline{v}_1|$$

where

$$M_1 = I - \underline{v}_1 (\underline{v}_1' \underline{v}_1)^{-1} \underline{v}_1'$$

and rewrite (2.34) as

$$p(\underline{\delta}_1, \underline{\Pi}_1 | D) \propto |(\underline{y}_1 - W_1 \underline{\delta}_1)' M_1 (\underline{y}_1 - W_1 \underline{\delta}_1)|^{-n_*/2} \times |(\underline{Y}_1 - X \underline{\Pi}_1)' (\underline{Y}_1 - X \underline{\Pi}_1)|^{-n_*/2} \quad (2.35)$$

where  $W_1 = (Y_1, X_1)$ . By making use of the definitions of the multivariate and matrix-variate Student-t density functions (see Zellner (1971), Appendix B) one can reexpress (2.35) as

$$p(\underline{\delta}_1, \underline{\pi}_1 | D) = p_1(\underline{\delta}_1 | \underline{\pi}_1, D)p_2(\underline{\pi}_1 | D) \quad (2.36)$$

where

$$p_1(\underline{\delta}_1 | \underline{\pi}_1, D) = c_1 \left| \frac{W_1' M_1 W_1}{s_1^2} \right|^{\frac{1}{2}} \{v_1 + (\underline{\delta}_1 - \tilde{\underline{\delta}}_1)' W_1' M_1 W_1 (\underline{\delta}_1 - \tilde{\underline{\delta}}_1) / s_1^2\}^{-\frac{1}{2}(v_1 + \underline{\kappa}_1)} \quad (2.36a)$$

and

$$p_2(\underline{\pi}_1 | D) = c_2 f(\underline{\pi}_1) \{c_3 |s_1 + (\underline{\pi}_1 - \underline{\pi}_1^*)' X' X (\underline{\pi}_1 - \underline{\pi}_1^*)|^{-n_*/2}\} \quad (2.36b)$$

with  $f(\underline{\pi}_1)$  given as

$$f(\underline{\pi}_1) = |W_1' M_1 W_1|^{-\frac{1}{2}(s_1^2)^{-v_1/2}}. \quad (2.36c)$$

The normalizing constants  $c_1$  and  $c_3$  are well known in terms of elementary functions [see the Appendix and Zellner (1971), Appendix B]. The parameters of (2.36a) are given as  $v_1 = n_* - \underline{\kappa}_1$ ,  $\underline{\kappa}_1 = m_1 + k_1$ , and

$$\tilde{\underline{\delta}}_1 = (W_1' M_1 W_1)^{-1} W_1' M_1 Y_1, \quad v_1 s_1^2 = (Y_1 - W_1 \tilde{\underline{\delta}}_1)' M_1 (Y_1 - W_1 \tilde{\underline{\delta}}_1) \quad (2.37a)$$

with  $|W_1' M_1 W_1| > 0$ ,  $v_1 s_1^2 > 0$ . The parameters of (2.36b) are

$$\underline{\pi}_1^* = (X' X)^{-1} X' Y_1, \quad s_1 = (Y_1 - X \underline{\pi}_1^*)' (Y_1 - X \underline{\pi}_1^*). \quad (2.37b)$$

Note that  $p(\underline{\delta}_1 | \underline{\pi}_1, D)$  in (2.36a), the conditional posterior density for  $\underline{\delta}_1$  given  $\underline{\pi}_1$  and  $D$ , is in the form of a  $\underline{\kappa}_1$ -variate Student-t pdf with  $v_1$  degrees of freedom with mean  $\tilde{\underline{\delta}}_1$  and covariance matrix  $(W_1' M_1 W_1)^{-1} v_1 s_1^2 / (v_1 - 2)$ , both of which depend on  $\underline{\pi}_1$ . On integration over the elements of  $\underline{\delta}_1$  in (2.36a), the marginal posterior density for  $\underline{\pi}_1$  is given in (2.36b) which is written as  $c_2 f(\underline{\pi}_1)$  times a normalized matrix-Student-t factor with  $c_2$  the normalizing constant that, to the best of our knowledge, is not known in terms of elementary functions.

To obtain the unconditional moments of the elements of  $\underline{\delta}_1$ , we make  $N$  draws  $\underline{\pi}_1^{(i)}$ ,  $i = 1, \dots, N$  from the matrix-Student-t factor in (2.36b) (see the algorithm described in the Appendix) and use well-known formulas to compute marginal moments from conditional moments. For example, to compute the unconditional mean of  $\underline{\delta}_1$ , we have

$$\begin{aligned} E(\underline{\delta}_1 | D) &= \int \tilde{\delta}_1 p_2(\underline{\pi}_1 | D) d\underline{\pi}_1 \\ &= \int \tilde{\delta}_1 f(\underline{\pi}_1) p_3(\underline{\pi}_1 | D) d\underline{\pi}_1 / \int f(\underline{\pi}_1) p_3(\underline{\pi}_1 | D) d\underline{\pi}_1 \end{aligned} \quad (2.38)$$

where  $p_3(\underline{\pi}_1 | D) = c_3 |S_1 + (\underline{\pi}_1 - \hat{\underline{\pi}}_1)' X' X (\underline{\pi}_1 - \hat{\underline{\pi}}_1)|^{-n_*/2}$ . To approximate the ratio of integrals in (2.38), we make  $N$  draws from  $p_3(\underline{\pi}_1 | D)$ , evaluate  $\tilde{\delta}_1 f(\underline{\pi}_1)$  and  $f(\underline{\pi}_1)$  for each draw and then compute

$$\sum_{i=1}^N \tilde{\delta}_1^{(i)} f(\underline{\pi}_1^{(i)}) / \sum_{i=1}^N f(\underline{\pi}_1^{(i)}) \quad (2.39)$$

where  $\tilde{\delta}_1^{(i)}$  is  $\tilde{\delta}_1$  evaluated at  $\underline{\pi}_1 = \underline{\pi}_1^{(i)}$ . The marginal covariance matrix of  $\underline{\delta}_1$  is defined as the sum of the expectation of the conditional variance and the variance of the conditional expectation, i.e.

$$\begin{aligned} V(\underline{\delta}_1 | D) &= \frac{v_1 s_1^2}{v_1 - 2} (W_1' M_1 W_1)^{-1} p_2(\underline{\pi}_1 | D) d\underline{\pi}_1 \\ &+ \int (\tilde{\delta}_1 - E(\tilde{\delta}_1)) (\tilde{\delta}_1 - E(\tilde{\delta}_1))' p_2(\underline{\pi}_1 | D) d\underline{\pi}_1. \end{aligned} \quad (2.40)$$

Each integral in the formula above can also be approximated by ratios of sums.

To compute the posterior density of an element of  $\underline{\delta}_1$ , say  $\delta_{1i}$ , we integrate (2.36a) analytically to obtain the conditional posterior pdf for  $\delta_{1i}$ ,  $p(\delta_{1i} | \underline{\pi}_1, D)$  which is in the form of a univariate Student-t pdf with  $v_1$  degrees of freedom. Then we consider

$$p(\delta_{1i} | D) = \int p(\delta_{1i} | \pi_1, D) p_2(\pi_1 | D) d\pi_1 \quad (2.41)$$

with  $p_2(\pi_1 | D)$  given in (2.36b). A Monte Carlo numerical integration procedure can be employed to evaluate the integral in (2.41). To approximate  $p(\delta_{1i} | D)$  at a given value of  $\delta_{1i}$ , say  $\delta_{1i}^*$ , compute simply

$$\sum_{i=1}^N p(\delta_{1i}^* | \pi_1^{(i)}, D) f(\pi_1^{(i)}) / \sum_{i=1}^N f(\pi_1^{(i)}).$$

In this way, complete marginal posterior pdfs for the elements of  $\underline{\delta}_1$  can be calculated. Also joint posterior pdfs for  $\delta_{1i}$  and  $\delta_{1j}$  can be calculated in a similar manner since, from (2.36b),  $p(\delta_{1i}, \delta_{1j} | \pi_1, D)$  has a bivariate Student-t form and  $\int p(\delta_{1i}, \delta_{1j} | \pi_1, D) p_2(\pi_1 | D) d\pi_1$  can be evaluated using Monte Carlo integration procedures. Finally, we note that (2.33) can be integrated analytically with respect to the elements of  $\underline{\delta}_1$ ,  $\underline{w}_1$  and  $\Omega_1$  to obtain  $p(\sigma_1^2 | \pi_1, D) p_2(\pi_1 | D)$  and numerical integration procedures can be utilized to obtain the marginal posterior pdf for  $\sigma_1^2$ ,  $p(\sigma_1^2 | D)$ .

Above, we have employed the diffuse prior assumptions in (2.32). As an alternative, we can use the following informative prior density

$$p(\underline{\delta}_1, \pi_1, \Omega^*) = p_1(\underline{\delta}_1, \pi_1 | \Omega^*) p_2(\Omega^*) \quad (2.42)$$

where  $p_1(\underline{\delta}_1, \pi_1 | \Omega^*)$  is a multivariate normal density with mean  $(\bar{\underline{\delta}}_1, \bar{\pi}_1)$  and covariance matrix  $\Omega^* \otimes C^{-1}$  and  $p_2(\Omega^*)$  is an inverted Wishart form. With this prior, operations similar to those presented above in the case of a diffuse prior are easily performed given values of  $\bar{\underline{\delta}}_1$ ,  $\bar{\pi}_1$ ,  $C$ , and other prior parameters. It is also possible to use an informative prior for  $\underline{\delta}_1$  given  $\Omega^*$  and diffuse priors for the other parameters.

Various conditional posterior densities associated with (2.35) are now considered. If we condition on  $X\pi_1 = Y_1 - K\bar{V}_1$ , where  $K > 0$  is a given

constant and  $\hat{v}_1 = Y_1 - X\hat{\beta}_1$ , we have  $X\hat{\beta}_1 = (1-K)Y_1 + KX\hat{\beta}_1$ , or  $V_1 = KV_1$ , where  $V_1 = Y_1 - X\hat{\beta}_1$ . Then on defining  $\hat{M}_1 = I - \hat{v}_1(\hat{v}_1'\hat{v}_1)^{-1}\hat{v}_1$  and  $\tilde{\delta}_1 = (\tilde{Y}_{1c}', \tilde{\beta}_{1c})$ , the conditional posterior mean vector, given by  $\tilde{\delta}_{1c} = (W_1'\hat{M}_1W_1)^{-1}W_1'\hat{M}_1Y_1$  is by direct evaluation,

$$\tilde{\delta}_{1c} = \begin{pmatrix} \tilde{Y}_{1c} \\ \tilde{\beta}_{1c} \end{pmatrix} = \begin{pmatrix} Y_1'Y_1 - KV_1'\hat{v}_1 & X_1'Y_1 \\ Y_1'X_1 & X_1'X_1 \end{pmatrix}^{-1} \begin{pmatrix} Y_1' - KV_1' \\ X_1' \end{pmatrix} Y_1. \quad (2.43)$$

With these conditioning assumptions,  $\tilde{\delta}_{1c}$ , the conditional posterior mean of  $\delta_1$ , is in the form of a K-class estimate. As is well known, for  $K=1$ ,  $\tilde{\delta}_{1c}$  is the 2SLS estimate, for  $K=\lambda$ , the smallest root of a determinantal equation encountered in maximum likelihood estimation,  $\tilde{\delta}_{1c}$  is the LIML estimate and for  $k = 1 - k/(v-2)$ , with  $v = n-k-m_1 > 2$ ,  $\tilde{\delta}_{1c}$  is the MELO estimate; see Zellner (1986). Note that if  $K=0$ ,  $\tilde{\delta}_{1c}$  is the OLS estimate. While the above conditional results are interesting, it is often the case that conditional means, etc. are not very good approximations to unconditional means, etc. in small or even moderate sized samples. This is illustrated in computed examples presented in Section 4.

We end this section with two remarks. First, the model (2.13) or (2.13') does not include a reduced form equation for  $Y_0$ , the endogenous variables excluded from the structural equation. This means that, in fact, our analysis in this section is conditional on the hypothesis that  $Y_0$  is independent of  $y_1$ , and  $Y_1$ . This hypothesis can be suppressed easily and the Bayesian analysis of the RRF can be adapted to the more general case. We note that one may interpret the model (2.13') as an incomplete simultaneous equation model [see Richard (1984)]. Second, we did not discuss conditions for the existence of the marginal posterior moments of  $\delta_1$ . Given

that our approach of computing posterior moments may be considered as an alternative to Drèze's (1976) approach, one may argue that Drèze's discussion of existence conditions [see also Drèze and Richard (1983)] is also applicable to our case. A more explicit discussion of conditions for existence of moments will be given in future work.

### 3. Some Bayesian Diagnostics for the Model Specification

In this section we extend the computational procedures of the previous section in order to compute posterior moments and densities of parameters (or functions of parameters) that give diagnostic checks of the specification of the model (2.13) or, equivalently, (2.13').

First, we discuss how to check the hypothesis of weak exogeneity (as defined by Engle et. al. (1983)) of the included endogenous variables  $y_1$  in equation (2.13').<sup>6</sup> In non-Bayesian econometrics this can be done by testing whether  $\eta_1 = 0$  in the expanded first equation of (2.13'), which is written as

$$y_1 = Y_1 y_1 + X_1 \beta_1 + \hat{V}_1 \eta_1 + \varepsilon_1 \quad (3.1)$$

where  $\hat{V}_1 = Y_1 - X_1 \hat{\beta}_1$  is the  $n \times m_1$  matrix of ordinary least squares residuals of the set of reduced form equations for  $Y_1$ . [For details, see, e.g., Hausman (1983), Holly (1982) and Engle (1984, Ch. 9.3)]. In our unrestricted reduced form (URF) approach one may proceed as follows:

- (i) Use independent random drawings  $\eta_1^{(1)}, \dots, \eta_1^{(i)}, \dots, \eta_1^{(N)}$ , that are generated from a matrix Student-t distribution with a density function

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<sup>6</sup>For earlier Bayesian results on testing for exogeneity, see Reynolds (1980, 1982).

proportional to (2.19) and compute the sequence  $v_1^{(1)}, \dots, v_1^{(i)}, \dots, v_1^{(N)}$  where  $v_1^{(i)} = Y_1 - X_1 \eta_1^{(i)}$ ,  $i = 1, \dots, N$ ;

- (ii) Run  $N$  ordinary least squares regressions on (3.1) with  $v_1^{(i)}$  instead of  $\hat{v}_1$ . This yields the sequence  $\hat{\eta}_1^{(1)}, \dots, \hat{\eta}_1^{(i)}, \dots, \hat{\eta}_1^{(N)}$  where  $\hat{\eta}_1^{(i)}$  is the well-known OLS expression;
- (iii) Compute the moments and densities of the elements of the vector  $\hat{\eta}_1$  by standard sampling theory formulas. If the posterior density of  $\hat{\eta}_1$  is located around zero, one has an indication that the variables  $Y_1$  in equation (2.13') are weakly exogenous in the sense that the stochastic component  $V_1$  of the variables  $Y_1$  does not contribute much to the equation (2.13'). The smaller the dispersion of  $\hat{\eta}_1$  around zero the greater one's confidence in this indication.

The sequence  $(\hat{y}_1^{(i)}, \hat{\beta}_1^{(i)}), i = 1, \dots, N$  that is obtained in the OLS-regression described in step (ii) above is equal to the sequence  $\{\hat{\delta}_1^{(i)}\}$ ,  $i = 1, \dots, N$  that is obtained by usng the 2SLS-mapping (2.26). This follows by direct verification. As a consequence, one expects that the sample mean  $\hat{\eta}_1$  from the sequence  $\{\hat{\eta}_1^{(i)}\}$ ,  $i = 1, \dots, N$  contains an approximation error with respect to  $\eta_1$  when the system (2.13') is strongly overidentified since  $\hat{v}_1 \neq v_1$  in general.

In order to deal with the overidentified case in an exact way, we consider again the RRF-system (2.13) and (2.13') and reformulate this model as follows. First, denote the  $i$ 'th row of  $(u_1 \ v_1)$  by  $(u_1 \ v_{1i}')$  and decompose the  $(1+m_1)$ -multivariate normal density of  $(u_1 \ v_{1i}')$  as a conditional normal density of  $u_1$  given a value of  $v_{1i}'$  and a marginal multivariate normal density of  $v_{1i}'$ . This yields  $(u_1 | v_{1i}') \sim N(v_{1i}' \eta_1, \sigma_1^2 - \omega_1' \Omega_1^{-1} \omega_1)$  with  $\eta_1 =$

$\Omega_1^{-1}\omega_1$  and  $v_{1i} \sim N(0, \Omega_1)$ . Next, perform the transformation of random variables from  $(u_1 | v_1)$  to  $(y_1 | Y_1)$  and from  $v_1$  to  $Y_1$ . This yields

$$(y_1 | Y_1) \sim N(Y_1 y_1 + X_1 \beta_1 + (Y_1 - X_1 \Omega_1) v_1, (\sigma_1^2 - \omega_1' \Omega_1^{-1} \omega_1) I) \quad (3.2)$$

$$Y_1 \sim N(X_1 \Omega_1, \Omega_1 \otimes I). \quad (3.3)$$

From (3.2) and (3.3) one can write the model

$$\begin{aligned} y_1 &= Y_1 y_1 + X_1 \beta_1 + v_1 \Omega_1 + \varepsilon \\ Y_1 &= X_1 \Omega_1 + v_1 \end{aligned} \quad (3.4)$$

where  $(\varepsilon_i, v_{1i}')$ ,  $i = 1, \dots, n$ , are independent random drawings from a multivariate normal distribution with mean zero and covariance matrix

$$\begin{bmatrix} \sigma_\varepsilon^2 & 0' \\ 0 & \Omega_1 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 - \omega_1' \Omega_1^{-1} \omega_1 & 0' \\ 0 & \Omega_1 \end{bmatrix}. \quad (3.4')$$

Note that  $\text{cov}(\varepsilon_i, v_{1i}) = 0$  which follows from direct verification. Therefore, testing whether  $\omega_1 = 0$  in the model given in (2.13') and (2.13'') is equivalent to testing whether  $\Omega_1 = 0$  in the model given in (3.4) and (3.4'). Further, note that if  $\Omega_1 = -Y_1$ , one can substitute  $X_1 \Omega_1 = Y_1 - v_1$  in the first equation of (3.4). As a consequence, there are only predetermined variables on the right hand side of equation (3.4).

The likelihood function of the parameters  $\delta_1' = (y_1', \beta_1')$ ,  $\Omega_1$ ,  $\sigma_\varepsilon^2$  and  $\Omega_1$  is obtained from (3.4) and (3.4') as

$$l(\delta_1, \Omega_1, \sigma_\varepsilon^2, \Omega_1 | D) \propto (\sigma_\varepsilon^2)^{-n/2} \exp\{-\varepsilon' \varepsilon / 2\sigma_\varepsilon^2\} |\Omega_1|^{-n/2} \times \exp\{-\frac{1}{2} \text{tr}[(v_1' v_1)^{-1} \Omega_1^{-1}]\} \quad (3.5)$$

where  $\varepsilon$  and  $v_1$  are given by equations in (3.4). As a next step we have to transform the prior density on  $(\delta_1, \Omega_1, \sigma_\varepsilon^2, \omega_1, \Omega_1)$  [see (2.32)] to a prior

density on the parameter set  $(\underline{\delta}_1, \underline{\pi}_1, \sigma_\epsilon^2, \underline{\eta}_1, \underline{\alpha}_1)$ . The relevant part is the transformation from  $(\sigma_\epsilon^2, \underline{\eta}_1, \underline{\alpha}_1)$  to  $(\sigma_\epsilon^2, \underline{\eta}_1, \underline{\alpha}_1)$  which gives as Jacobian  $|\underline{\alpha}_1|$ .

As a consequence the prior information specified in (2.32) is given in terms of  $(\underline{\delta}_1, \underline{\pi}_1, \sigma_\epsilon^2, \underline{\eta}_1, \underline{\alpha}_1)$  as

$$p(\underline{\delta}_1, \underline{\pi}_1, \sigma_\epsilon^2, \underline{\eta}_1, \underline{\alpha}_1) \propto (\sigma_\epsilon^2)^{-(m_1+v_0+2)/2} |\underline{\alpha}_1|^{-(m_1+v_0)/2} \quad (3.6)$$

The posterior density of the p-vector  $\underline{\theta}' = (\underline{\delta}_1', \underline{\eta}_1')$ , with  $p = \underline{\delta}_1 + \underline{\pi}_1$ , and  $\underline{\pi}_1$ ,  $\sigma_\epsilon^2$ ,  $\underline{\alpha}_1$  is given by

$$p(\underline{\theta}, \underline{\pi}_1, \sigma_\epsilon^2, \underline{\alpha}_1 | D) \propto (\sigma_\epsilon^2)^{-(n_*+m_1+2)/2} \exp\{-\underline{\varepsilon}' \underline{\varepsilon} / 2\sigma_\epsilon^2\} \times |\underline{\alpha}_1|^{-(n_*+m_1)/2} \exp\{-\frac{1}{2} \text{tr}[(\underline{V}_1' \underline{V}_1) \underline{\alpha}_1^{-1}]\}. \quad (3.7)$$

Integrating (3.7) with respect to  $\sigma_\epsilon^2$  and  $\underline{\alpha}_1$  yields the marginal posterior  $p(\underline{\theta}, \underline{\pi}_1 | D)$ ,

$$p(\underline{\theta}, \underline{\pi}_1 | D) \propto (\underline{\varepsilon}' \underline{\varepsilon})^{-(n_*+m_1)/2} |\underline{V}_1' \underline{V}_1|^{-(n_*-1)/2} \quad (3.8)$$

where  $\underline{\varepsilon}$  and  $\underline{V}_1$  are given in (3.4). The density (3.8) may be compared with the pdf given in (2.34) and (2.35). In a similar way as done below (2.35) in Subsection 2.3, we can rewrite (3.8) as

$$p(\underline{\theta}, \underline{\pi}_1 | D) = p_1(\underline{\theta} | \underline{\pi}_1, D) p_2(\underline{\pi}_1 | D) \quad (3.9)$$

where

$$p_1(\underline{\theta} | \underline{\pi}_1, D) = c_1 \left( \frac{\underline{W}' \underline{W}}{s_1^2} \right)^{\frac{1}{2}} \{v_1 + (\underline{\theta} - \tilde{\underline{\theta}})' \underline{W}' \underline{W} (\underline{\theta} - \tilde{\underline{\theta}}) / s_1^2\}^{-(v_1+p)/2} \quad (3.9a)$$

and

$$p_2(\underline{\pi}_1 | D) \propto c_2 h(\underline{\pi}_1) \{c_3 |s_1 + (\underline{\pi}_1 - \tilde{\underline{\pi}}_1)' \underline{X}' \underline{X} (\underline{\pi}_1 - \tilde{\underline{\pi}}_1)|^{-(n_*-1)/2} \quad (3.9b)$$

and

$$h(\underline{\pi}_1) = |\underline{W}' \underline{W}|^{-\frac{1}{2}} (s_1^2)^{-v_1/2}. \quad (3.9c)$$

From the definition of  $W = (w_1, v_1)$  it follows that  $|W'W| = |W_1' M_1 W_1| |v_1' v_1|$ , where  $M_1$  is given below (2.34). Therefore

$$h(\pi_1) = f(\pi_1) |v_1' v_1|^{-\frac{1}{2}} \quad (3.10)$$

with  $f(\pi_1)$  given in (2.36c). It follows that the posterior density of  $\pi_1$  given in (2.36b) is equivalent to the posterior density given in (3.9b). The parameters of the conditional multivariate Student-t density of the p-vector  $\theta$  are given as  $v_1 = n_* - \ell_1$ , and

$$\tilde{\theta} = (W'W)^{-1} W' y_1, \quad v_1 s_1^2 = (y_1 - W\tilde{\theta})' (y_1 - W\tilde{\theta}). \quad (3.11)$$

The conditional density  $p_1(\theta|\pi_1, D)$  in (3.9a) is in the form of a p-variate conditional Student-t pdf of  $\theta$  given  $\pi_1$  and  $D$  with  $v_1$  degrees of freedom, with mean  $\tilde{\theta}$  and covariance matrix  $(W'W)^{-1} v_1 s_1^2 / (v_1 - 2)$ , both of which depend on  $\pi_1$ . Similar remarks that were made with respect to (2.36a) apply to (3.9a) and are not repeated. We mention here only that if the marginal pdf of  $\pi_1$  is centered around zero, then one has an indication that  $Y_1$  is weakly exogenous in the sense discussed before.

We note that one may use diffuse or informative priors other than (3.6). For instance an alternative type of diffuse prior is given by

$$p(\theta, \pi_1, \sigma_\epsilon^2, \Omega_1) \propto (\sigma_\epsilon^2)^{-(m_1 + v_0 + 2)/2} |W'W|^{\frac{1}{2}} |\Omega_1|^{-(m_1 + v_0)/2}. \quad (3.12)$$

This prior is equal to (3.6) times  $|W'W|^{\frac{1}{2}}$ , which is the root of the determinant of the information matrix of  $\theta$  given  $\pi_1$ . As a result the factor  $|W'W|^{-\frac{1}{2}}$  will not appear in (3.9c). Further, we note that conditional moments associated with (3.9) can be formulated in a similar way as was done in Subsection 2.3. In particular, if we condition  $p_1(\theta|\pi_1, D)$  on  $\pi_1 = \hat{\pi}_1$

and integrate out  $\delta_1$ , the posterior density  $p_1(\eta_1 | \hat{t}_1, D)$  is an  $m_1$ -variate Student pdf with mean  $\hat{\eta}_1$ , the OLS estimate of  $\eta_1$  in (3.1). The non-Bayesian test procedure for the weak exogeneity of  $Y_1$  using (3.1) is to reject the null hypothesis if a  $(1-\alpha)\%$  confidence region centered at  $\eta_1 = 0$  does not contain the point  $\hat{\eta}_1$ . The Bayesian decision is to reject the null if a  $(1-\alpha)\%$  posterior probability region centered at  $\hat{\eta}_1$  does not contain the point  $\eta_1 = 0$ . An exact Bayesian decision procedure relies on the marginal posterior density  $p_1(\eta_1 | D)$  rather than on the conditional density  $p_1(\eta_1 | \hat{t}_1, D)$ . Some illustrative results on exogeneity testing are presented in Section 4.1.

Next, we discuss how we can check whether the overidentifying restrictions in (2.12a) and (2.12b) seem acceptable. It follows from the discussion, given in Subsection 2.2 (between equations (2.20) and (2.21)), that the degree of overidentification is equal to the number  $k_0$  of omitted predetermined variables in equation (2.9b) minus the number  $m_1$  of included endogenous variables on the right hand side of (2.9b). Thus, we may include some predetermined variables in (2.9b) that were, at first, excluded from this equation. If we add  $k_0 - m_1$  predetermined variables to the right hand side of (2.9b), then we have an exactly identified equation instead of an overidentified equation. As a consequence, one can make use of the URF approach and compute highest posterior density (HPD) regions around zero for the parameters of the  $k_0 - m_1$  included variables. This yields a check on the value of the overidentifying restrictions. If we add fewer than  $k_0 - m_1$  predetermined variables to (2.9b), then this equation is still overidentified and the RRF approach can be used to analyze the HPD regions around zero of the parameters of the included variables.

Several other diagnostic checks may be constructed, i.e. restricted reduced form moments may be compared with unrestricted reduced form moments. Diagnostic checks on autocorrelation and outliers may be constructed from posterior distributions of realized error terms. Further, one may compute posterior odds relating to exogeneity hypotheses. There are thus ample opportunities for much applied work using the methods discussed above.

#### 4. Applications of Methods

In this section we illustrate the methods of Sections 2 and 3 for the case of an exactly identified simultaneous equation model and for the case of an overidentified model. As an example of an exactly identified model we consider the Belgian beef market model (see Drèze and Richard (1983, Section 2.4)) which is given as

$$Q_t = \alpha_1 + \beta_1 P_t + \gamma_1 Y_t + u_{1t} \quad (4.1)$$

$$Q_t = \alpha_2 + \beta_2 P_t + \gamma_2 S_t + u_{2t} \quad (4.2)$$

where  $Q_t$  is the quantity of beef consumed per capita in period  $t$ ;  $P_t$  is the price-index;  $Y_t$  is real national income per capita and  $S_t$  is the cattle stock per capita (measured as the number of heads at the beginning of each period). The variables  $Q_t$  and  $P_t$  are endogenous and the variables  $Y_t$ ,  $S_t$  and the constant term are assumed exogenous. The data are annual observations for the period, 1950-1965. Given our uniform prior with  $v_0 = 0$  and given that the model is exactly identified, posterior first and higher order moments do not exist. In Figure 1 we present the marginal posterior density of  $\beta_1$  and give the computed quartiles of the posterior distribution. The density is concentrated around the mode but has a long tail to

the left. We note that the mode and the median are almost equal; however the first and fourth quartiles indicate that the density is skewed to the left. Further, we find evidence that the exogeneity of the price variable is rejected. The results reported are based on  $N = 100,000$  drawings in order to obtain an accurate figure. We emphasize, however, that the figure is already rather accurate with  $N = 10,000$  or  $N = 20,000$ .

### UNIVARIATE POSTERIOR OF BETA1 (BBM)

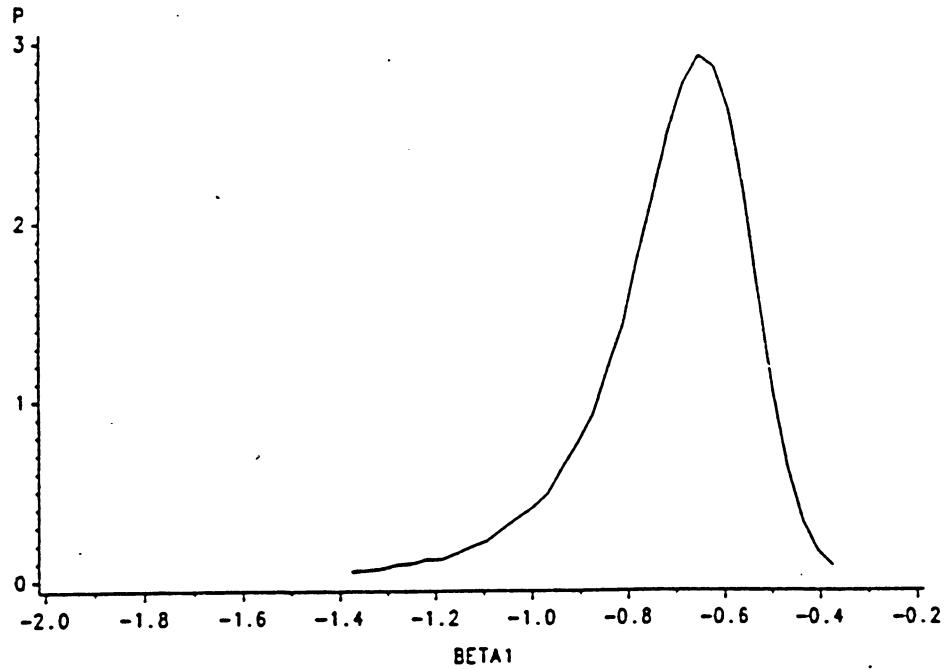


Fig. 1 Marginal posterior density of  $\beta_1$  in the Belgian Beef Market (BBM) model.

As an example of an overidentified simultaneous equations model we take Klein's Model I [see Klein (1950)] which is given as

$$C = \alpha_1 P + \alpha_2 P_{-1} + \alpha_3 W + \alpha_0 + u_1 \quad (4.3)$$

$$I = \beta_1 P + \beta_2 P_{-1} + \beta_3 K_{-1} + \beta_0 + u_2 \quad (4.4)$$

$$W_1 = \gamma_1 X + \gamma_2 X_{-1} + \gamma_3 t + \gamma_0 + u_3 \quad (4.5)$$

$$X = C + I + G \quad (4.6)$$

$$P = X - W_1 - T \quad (4.7)$$

$$K = K_{-1} + I \quad (4.8)$$

$$W = W_1 + W_2 \quad (4.9)$$

Consumption expenditure ( $C$ ) is structurally dependent on profits ( $P$ ), profits lagged one year ( $P_{-1}$ ) and on total wages ( $W$ ). Net investment expenditure ( $I$ ) depends on profits, lagged profits and on the capital stock at the beginning of the year ( $K_{-1}$ ). Finally, private wage income ( $W_1$ ) depends on net private product at market prices ( $X$ ), the same variable lagged ( $X_{-1}$ ) and on a trend term ( $t$ ). The model is closed by four identities, which provide links with three exogenous variables: the government wage bill ( $W_2$ ), government nonwage expenditure, including the net foreign balance, ( $G$ ) and business taxes ( $T$ ). The model has seven jointly dependent variables ( $C, I, W_1, X, P, W$ ) and eight predetermined variables ( $1, P_{-1}, X_{-1}, K_{-1}, G, T, W_2, t$ ). All variables (except 1 and  $t$ ) are measured in constant dollars. Posterior moments for Klein's Model I are reported in Tables 1-3 and univariate and bivariate marginal posterior densities of a structural parameter and an exogeneity parameter in the investment equation are given in Figures 2-3.

TABLE 1: INVESTMENT EQUATION OF KLEIN'S MODEL I: POSTERIOR EXPECTATIONS AND STANDARD DEVIATIONS<sup>a</sup>

	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_0$	$\pi_{1P}$
<u>Prior with <math>v_0 = 0</math>:</u>					
URFA with prior (2.17)					
GILS	0.59	(0.45)	0.48	(0.45)	-0.05
TSLS	0.35	(0.19)	0.45	(0.18)	-0.13
RRFA with prior (2.32) or (3.6)					
conditional on $\hat{\pi}_1$ <sup>b</sup>	0.15	(0.11)	0.62	(0.10)	-0.16
marginal	0.20	(0.17)	0.57	(0.17)	-0.15
<u>Prior with <math>v_0 = k</math>:</u>					
URFA with prior (2.17)					
GILS	0.54	(0.42)	0.52	(0.39)	-0.06
TSLS	0.27	(0.16)	0.51	(0.15)	-0.14
RRFA with prior (2.32) or (3.6)					
conditional on $\hat{\pi}_1$ <sup>b</sup>	0.15	(0.09)	0.62	(0.09)	-0.16
marginal	0.17	(0.15)	0.60	(0.15)	-0.16

<sup>a</sup>Quantities in parentheses are posterior standard deviations.

<sup>b</sup>The conditional posterior means are equal to the non-Bayesian two-stage least-squares estimates as explained in Section 2.3.

TABLE 2: WAGE EQUATION OF KLEIN'S MODEL I: POSTERIOR EXPECTATIONS AND STANDARD DEVIATIONS,<sup>a</sup> PRIOR WITH  $v_0 = 0$

	$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_0$	$\eta_{1X}$
<u>URFA with prior (2.17)</u>					
GILS	0.50	(0.15)	0.03	(0.13)	0.17
TSLS	0.44	(0.04)	0.15	(0.05)	0.13
<u>RRFA with prior (2.32) or (3.6)</u>					
conditional on $\hat{\pi}_1$	0.44	(0.04)	0.15	(0.05)	0.13
marginal	0.44	(0.07)	0.15	(0.07)	0.13

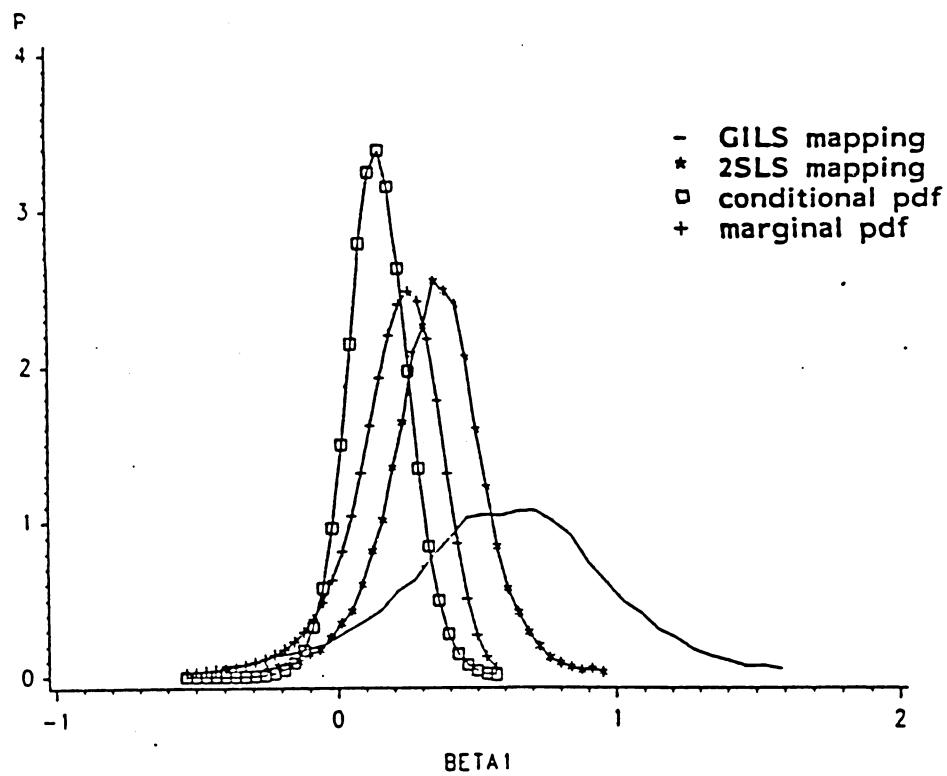
<sup>a</sup>Quantities in parentheses are posterior standard deviations.

TABLE 3: CONSUMPTION EQUATION OF KLEIN'S MODEL I: POSTERIOR EXPECTATIONS AND STANDARD DEVIATIONS<sup>a</sup>

	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha$	$\pi_{1P}$	$\pi_{1W}$
<u>prior with <math>v_0 = 0</math>:</u>						
URFA with prior (2.17)						
GILS	0.47	(0.34)	0.04	(0.24)	0.41	(0.25)
TSLS	0.10	(0.14)	0.15	(0.12)	0.81	(0.04)
RRFA with prior (2.32) or (3.6)						
conditional on $\hat{\pi}_1$	0.02	(0.10)	0.22	(0.09)	0.81	(0.03)
marginal	-0.08	(0.16)	0.29	(0.17)	0.83	(0.05)
<u>prior with <math>v_0 = k</math>:</u>						
URFA with prior (2.17)						
GILS	0.42	(0.32)	0.07	(0.21)	0.42	(0.23)
TSLS	0.07	(0.11)	0.18	(0.10)	0.81	(0.03)
RRFA with prior (2.32) or (3.6)						
conditional on $\hat{\pi}_1$	0.02	(0.08)	0.22	(0.07)	0.81	(0.03)
marginal	-0.08	(0.13)	0.33	(0.13)	0.80	(0.03)

<sup>a</sup>Quantities in parentheses are posterior standard deviations.

## UNIVARIATE POSTERIOR OF BETA1 (KLEIN 1)



## UNIVARIATE POSTERIOR OF ETA1 (KLEIN 1)

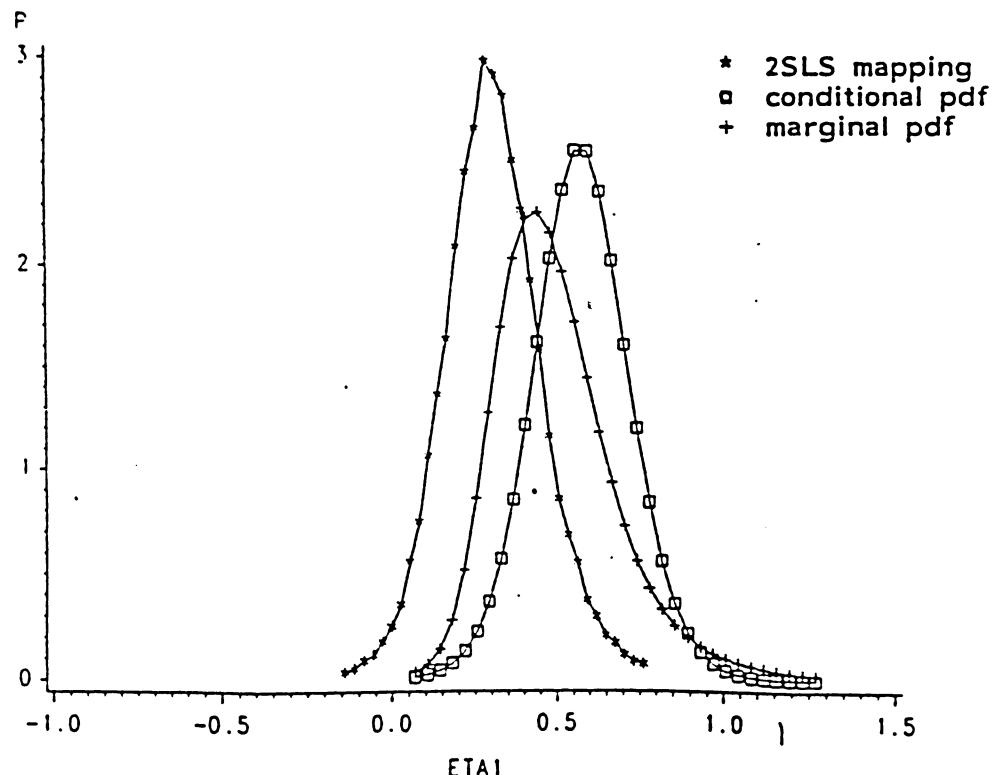
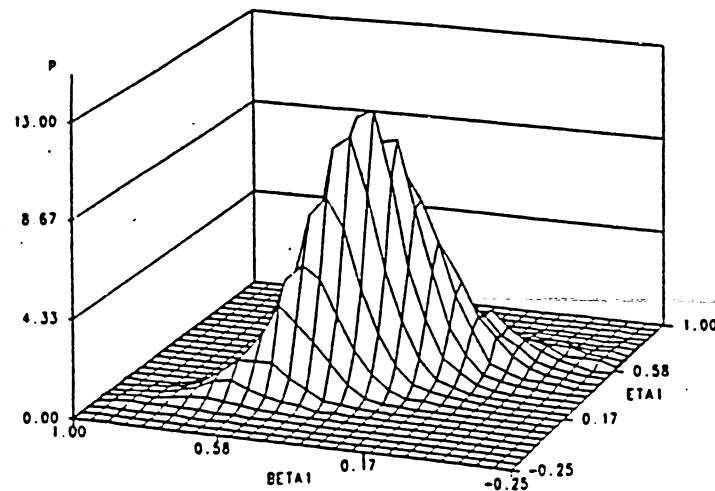
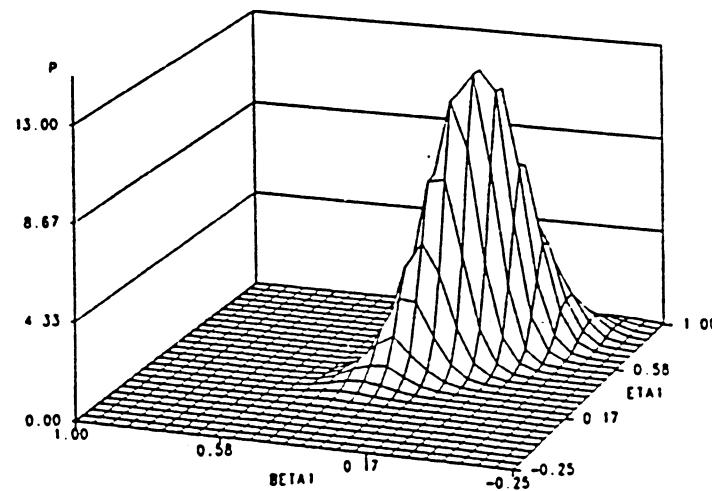


Fig. 2 Univariate marginal posterior densities of  $\beta_1$  and  $\eta_{1p}$  in the investment equation of Klein's Model 1.

## URF APPROACH: 2SLS MAPPING POSTERIOR



## RRF APPROACH: CONDITIONAL POSTERIOR



## RRF APPROACH: MARGINAL POSTERIOR

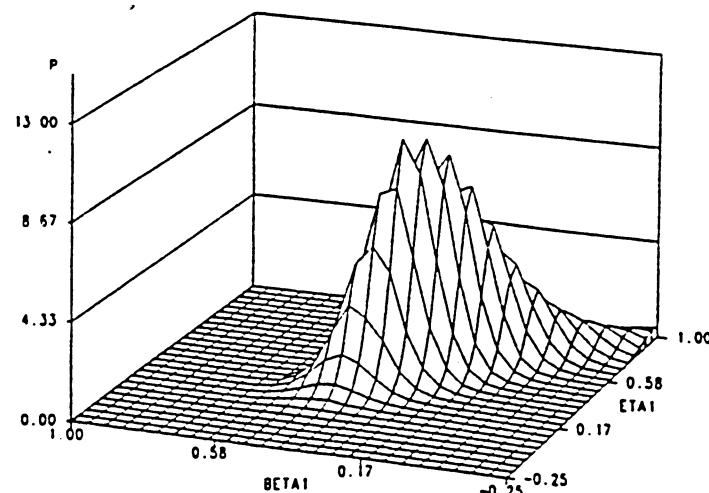


Fig. 3 Bivariate marginal posterior density functions for  $(\beta_1, \eta_{1p})$  in the investment equation of Klein's Model I.

It is seen from the results on the investment equation in Table 1 that the URF approach, in particular the GILS mapping, yields gross approximation errors for several parameters. The posterior means and standard deviations of the parameter of the included endogenous variable, of the constant term, and of the exogeneity parameter differ substantially from the results of the restricted reduced form approach. The results of the latter approach are based on  $N = 20,000$  drawings. We note that the marginal results differ also from the conditional results in the RRF approach but less than from the results given by the URF approach. The sensitivity with respect to the particular choices of  $v_0 = 0$  and  $v_0 = k$  is as expected. A larger value of  $v_0$  implies smaller variances due to lighter tails. It is of interest that the exogeneity of profits appears to be rejected while some preliminary results on overidentifying restrictions (not reported) suggest that these restrictions are not to be rejected. More details will be reported in future work. It is also of interest that conditional standard deviations are always smaller than the asymptotic TSLS standard deviations. The reason is that in the conditional approach the values of  $s_1^2$  is smaller than in the non-Bayesian approach. The results for the wage income equation given in Table 2 produced by different methods are similar. The hypothesis that net private product is exogenous is not rejected while from preliminary results, it appears that the overidentifying restrictions are rejected. The consumption function was the most complex case to analyze. The posterior means differ substantially for the different approaches. The posterior standard deviations for the exogeneity parameters for profits and wage income show a surprising result. The marginal standard deviations are

smaller than the conditional ones. It appears that the effect of the weight function  $f(\Pi_1)$  [see Subsection 2.3] is very nonlinear. This is a topic of current research. Exogeneity and preliminary results on over-identification, not reported here, suggest that both hypotheses are rejected. Figures 2 and 3 show the skewness of the marginal pdf's and differences between the results of the URF, the conditional RRF and the marginal RRF approaches.

### 5. Concluding Remarks

In this paper, we have shown how Monte Carlo numerical methods can be employed to compute exact posterior densities of the parameters of a structural equation using diffuse or informative prior distributions. In addition, operational procedures for Bayesian diagnostic checking or specification analysis were described. For example, discrepancy parameter vectors were introduced to represent departures from exact identifying restrictions and it was shown how to compute posterior densities for them and interesting functions of their elements which we refer to as discrepancy functions. In addition, a Bayesian procedure for evaluating exogeneity hypotheses was described. That diagnostic checking or specification analysis be performed is quite important and the fact that operational Bayesian procedures for diagnostic checking or specification analysis can be carried through without much difficulty is fortunate.

Applications of our methods were presented and yielded useful results. In particular, it was found in several instances that certain specifying assumptions, exogeneity hypotheses and identifying restrictions, were of doubtful validity. Also, it was found that exact marginal poste-

rior densities differed considerably from conditional posterior densities based on conditioning assumptions which are often employed in non-Bayesian procedures, for example in the 2SLS approach or other K-class estimation approaches. Thus we consider it very important to use appropriate marginal posterior densities for structural parameters rather than approximate conditional posterior densities. That the former can be computed using Monte Carlo techniques without much difficulty is indeed fortunate.

In future research, we plan to extend our consideration of diagnostic checking procedures to consider checks for autocorrelation of error terms, outliers and possible left out variables. Also, the single equation analysis in this paper will be extended to provide results for sets of structural equations and complete structural equation systems.

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**Appendix: The Generation of Pseudo-Random Drawings from a Matrix-Student Distribution**

Because the matrix-Student (Mt) distribution is related to the matrix-normal (MN) and to the inverted-Wishart (iW) distributions, we define these three families of distributions through their density functions and state a few properties that are useful to build an algorithm for generating a pseudo-random drawing from an Mt distribution.

**Definitions**

Let  $\Pi \in \mathbb{R}^{km}$  be a  $k \times m$  random matrix.

(i)  $\Pi$  has an MN distribution if its density function is

$$\begin{aligned} p(\Pi) &= f_{MN}^{k \times m}(\Pi | \bar{\Pi}, \Omega \otimes M^{-1}) \\ &= [(2\Pi)^{km} |\Omega|^k |M|^m]^{-\frac{1}{2}} \exp -\frac{1}{2} \text{tr}[\Omega^{-1}(\Pi - \bar{\Pi})' M(\Pi - \bar{\Pi})] \end{aligned} \quad (\text{A.1})$$

where  $\bar{\Pi} \in \mathbb{R}^{km}$  is a  $k \times m$  constant matrix,  $\Omega$  is an  $m \times m$  PDS constant matrix and  $M$  is a  $k \times k$  PDS constant matrix.

From here on, let  $\Omega$  be a random PDS matrix.

(ii)  $\Omega$  has an iW distribution if its density function is

$$\begin{aligned} p(\Omega) &= f_{iW}^m(\Omega | W, v) \\ &= [2^{\frac{1}{2}vm} \prod_{i=1}^m \Gamma(\frac{v+1-i}{2})]^{-1} |W|^{\frac{1}{2}v} |\Omega|^{-\frac{1}{2}(v+m+1)} \exp -\frac{1}{2} \text{tr} \Omega^{-1} W \end{aligned} \quad (\text{A.2})$$

where  $W$  is an  $m \times m$  constant matrix and the constant  $v > m-1$ .

(iii)  $\Pi$  has an Mt distribution if its density function is

$$\begin{aligned} p(\Pi) &= f_{Mt}^{k \times m}(\Pi | \bar{\Pi}, W, M, v) \\ &= [\prod_{i=1}^{\frac{1}{2}km} \Gamma(\frac{v+1-i}{2}) / \Gamma(\frac{v+k+1-i}{2})]^{-1} \\ &\quad \times |W|^{\frac{1}{2}v} |M|^{\frac{1}{2}m} |W + (\Pi - \bar{\Pi})' M(\Pi - \bar{\Pi})|^{-\frac{1}{2}(v+k)} \end{aligned} \quad (\text{A.3})$$

where  $\bar{\Pi}$ ,  $W$ ,  $M$  and  $v$  are defined as in (i) and (ii).

Some properties of these distributions

(1) If  $p(\boldsymbol{\Pi}|\boldsymbol{\Omega}) = f_{MN}^{k \times m}(\boldsymbol{\Pi}|\bar{\boldsymbol{\Pi}}, \boldsymbol{\Omega} \otimes \boldsymbol{M}^{-1})$  and  $p(\boldsymbol{\Omega}) = f_{iW}^m(\boldsymbol{\Omega}|\boldsymbol{W}, \boldsymbol{v})$ , then  $p(\boldsymbol{\Pi})$  is given by formula (A.3). This property states that an  $Mt$  distribution is a marginal distribution from an  $MN-iW$  one.

(2) Let  $\boldsymbol{\Pi}$  have the density (A.1).

(i) If  $\mathbf{A}$  is an  $r \times k$  matrix of rank  $r \leq k$ , and  $\mathbf{B}$  is an  $m \times s$  matrix of rank  $s \leq m$ , then

$$p(\mathbf{A}\boldsymbol{\Pi}\mathbf{B}) = f_{MN}^{r \times s}(\mathbf{A}\boldsymbol{\Pi}\mathbf{B} | \mathbf{A}\bar{\boldsymbol{\Pi}}\mathbf{B}, \mathbf{B}'\boldsymbol{\Omega}\mathbf{B} \otimes \mathbf{A}\mathbf{M}^{-1}\mathbf{A}'). \quad (A.4)$$

(ii) In particular, if  $\mathbf{B}'\boldsymbol{\Omega}\mathbf{B} = \mathbf{I}_m$  and  $\mathbf{A}\mathbf{M}^{-1}\mathbf{A}' = \mathbf{I}_k$ ,  $\mathbf{Z} := \mathbf{A}(\boldsymbol{\Pi} - \bar{\boldsymbol{\Pi}})\mathbf{B}$  is a matrix of independent standard normal variables.

(3) Let  $\boldsymbol{\Omega}$  have the density (A.2).

(i) If  $\mathbf{C}$  is an  $m \times s$  matrix of rank  $s \leq m$ , then

$$p(\mathbf{C}'\boldsymbol{\Omega}\mathbf{C}) = f_{iW}^s(\mathbf{C}'\boldsymbol{\Omega}\mathbf{C} | \mathbf{C}'\mathbf{W}\mathbf{C}, v-m+s). \quad (A.5)$$

(ii) Partition  $\boldsymbol{\Omega}$  into  $\boldsymbol{\Omega}_{11}(m_1 \times m_1, PDS)$ ,  $\boldsymbol{\Omega}_{22}(m_2 \times m_2, PDS)$ ,  $\boldsymbol{\Omega}_{21}(m_2 \times m_1)$ ,  $\boldsymbol{\Omega}_{12} = \boldsymbol{\Omega}'_{21}$  and let  $\boldsymbol{\Omega}_{22 \times 1} = \boldsymbol{\Omega}_{22} - \boldsymbol{\Omega}_{21}\boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12}$ . Then  $\boldsymbol{\Omega}$  and  $(\boldsymbol{\Omega}_{11}, \boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12}, \boldsymbol{\Omega}_{22 \times 1})$  are in one-to-one correspondence and

$$p(\boldsymbol{\Omega}_{11}, \boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12}, \boldsymbol{\Omega}_{22 \times 1}) = p(\boldsymbol{\Omega}_{11})p(\boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12} | \boldsymbol{\Omega}_{22 \times 1})p(\boldsymbol{\Omega}_{22 \times 1}) \quad (A.6)$$

with

$$p(\boldsymbol{\Omega}_{11}) = f_{iW}^{m_1}(\boldsymbol{\Omega}_{11} | \mathbf{W}_{11}, v-m_2) \quad (A.7)$$

$$p(\boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12}) = f_{MN}^{m_1 \times m_2}(\boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12} | \mathbf{W}_{11}^{-1}\mathbf{W}_{12}, \boldsymbol{\Omega}_{22 \times 1} \otimes \mathbf{W}_{11}^{-1}) \quad (A.8)$$

$$p(\boldsymbol{\Omega}_{22 \times 1}) = f_{iW}^{m_2}(\boldsymbol{\Omega}_{22 \times 1} | \mathbf{W}_{22 \times 1}, v) \quad (A.9)$$

where  $\mathbf{W}_{11}$ ,  $\mathbf{W}_{22}$  and  $\mathbf{W}_{22 \times 1}$  are defined from  $\mathbf{W}$  as  $\boldsymbol{\Omega}_{11}$ ,  $\boldsymbol{\Omega}_{22}$  and  $\boldsymbol{\Omega}_{22 \times 1}$  are defined from  $\boldsymbol{\Omega}$ .

(iii) In particular, if  $C'WC = I_m$  in (A.5),  $\Psi = C'\Omega C$  is in one-to-one correspondence with  $\frac{1}{2}m(m+1)$  independent random variables:  $\frac{1}{2}m(m-1)$  standard normal variables, plus  $m$  variables  $\lambda_i$ , each of them having an inverted-gamma density defined as

$$f_{iW}^1(\lambda_i | 1, v-i+1), \quad \text{for } i = 1, 2, \dots, m.$$

This follows from the property 3-ii applied to  $\Psi$   $m$  times: one starts e.g. with  $m_2=1$  and  $m_1=m-1$  and notices that 3-ii can be applied again to  $p(\Psi_{11})$  which is an iW density with parameters  $I_{m_1}$  and  $v-1$ .

Other properties of these distributions can be found in Zellner (1971, Appendix B.4 and B.5), Drèze and Richard (1983, Appendix A) and Bauwens (1984, Appendix A.I and A.II), who give separate algorithms for the generation of random numbers from MN and iW distribution. These algorithms can be combined to draw from an Mt distribution, with density given by (A.3), by drawing firstly an iW  $\Omega$  matrix with density (A.2), and by drawing subsequently an MN matrix, with density (A.1) where  $\Omega$  is the iW matrix obtained at the iW step.

#### Mt Algorithm

To obtain a drawing  $\Pi$  from the Mt distribution defined by (A.3):

1- Compute the lower triangular (LT) matrices  $Q'$  and  $P$  such that  $W = Q'Q$  and  $M^{-1} = PP'$ .

2- iW-Step:

- (i) generate  $\frac{1}{2}m(m-1)$  standard normal drawings and  $m$  inverted-gamma drawings  $\lambda_i$ , with  $p(\lambda_i) = f_{iW}^1(\lambda_i | 1, v-i+1)$ ;
- (ii) compute the  $m \times m$  LT matrix  $\Phi$  such that  $\Phi\Phi' = \Omega$  is a drawing from the iW distribution of  $\Omega$  defined by (A.2) (but one does not need to compute  $\Phi\Phi'$ ).

Let  $\phi = (\phi_{ij})$ : then  $\phi_{ij} = 0$  for  $i < j$ . The lower triangle of  $\phi$  can be filled by the following steps:

- (1)  $i \leftarrow 0$ ;  $1 \leftarrow \frac{1}{2}m(m+1) + 1$ ; let  $\phi$  be a vector of  $i-1$  elements that will finally contain the column expansion of the LT of  $\phi$ , i.e.,  $\phi = (\phi_{11}, \phi_{21}, \dots, \phi_{m1}, \phi_{22}, \phi_{32}, \dots, \phi_{m-1, m-1}, \phi_{mm}, \phi_{mm})$
- (2)  $i \leftarrow i + 1$ ; if  $i > m$ , stop
- (3)  $1 \leftarrow 1 - i$   $\phi(1) = \sqrt{\lambda_i} (\lambda_i \text{ obtained at 2-i})$
- (4) if  $i = 1$ , go to (2); or else go to (5)
- (5) pick  $i-1$  standard normal drawings obtained at step 2-(i) and assign them in a vector  $u$ . Compute  $y = \sqrt{\lambda_i} \cdot \phi_{i-1} \cdot u$  where  $y$  is a vector of  $i-1$  elements and  $\phi_{i-1}$  denotes the LT matrix whose column expansion of the lower triangle is stored in the last  $1/2i(i-1)$  elements of the vector  $\phi$  (but  $\phi_1$  is the scalar  $\phi_{mm} = \sqrt{\lambda_{11}}$ ). Finally,  $\phi(1+k) \leftarrow y(k)$ ,  $k = 1, 2, \dots, i-1$  and go to (2).

3- MN step:

- (i) generate  $km$  standard normal drawings  $z_{ij}$  ( $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, m$ ). Let  $Z = (z_{ij})$ ;
- (ii) compute  $\Pi = \bar{\Pi} + PZ\phi'Q'$  where  $\phi$  is the LT matrix obtained at step 2-ii.

To draw standard normal variables, one can use the polar algorithm --see e.g. Knuth (1971). To draw inverted gamma variables, one can use the GRUB algorithm of Kinderman and Monahan (1980) that is efficient since the computer time required to obtain one inverted gamma drawing is almost perfectly independent of the value of  $v$  (as is not the case if one generates gamma drawings as sums of  $v$  independent squared normal drawings). To get

one drawing  $\mathbb{I}$ , one needs  $\frac{1}{2}m(m-1) + mk$  univariate standard normal drawings, plus the  $m$  inverted gamma drawings; all these drawings must be independent.

The proposed Mt algorithm has the advantage that the marginal cost of a drawing (steps 2 and 3) is not affected by the value of the degrees of freedom parameter  $v$ .

Provided  $v$  is an integer, one could replace the implementation of the  $iW$  step by (i) drawing a Wishart matrix  $\Omega^{-1}$  as  $\sum_{j=1}^v Z_j Z_j'$  where the  $m \times 1$  independent vectors  $Z_j$  have a multivariate normal density with zero expected value and covariance matrix given by  $W$ , (ii) inverting  $\Omega^{-1}$  and (iii) computing the LT matrix  $\Phi$  such that  $\Omega = \Phi \Phi'$ . This implementation requires  $vm$  standard normal drawings at the  $iW$  step, instead of  $\frac{1}{2}m(m-1)$  of these plus the  $m$  inverted gamma drawings. So for very small values of  $v$  and  $m$ , this implementation may be more efficient. Notice however that a Cholesky decomposition of  $\Omega$ , giving  $\Phi$ , has to be performed, whereas  $\Phi$  is obtained directly in the implementation we use.

Another method to generate from the Mt distribution (A.3) that is expected to be less efficient, is to use the property that

$$p(\mathbb{I}) = p(\mathbb{I}_1 | \mathbb{I}_2 \mathbb{I}_3 \dots \mathbb{I}_m) p(\mathbb{I}_2 | \mathbb{I}_3 \dots \mathbb{I}_m) \dots p(\mathbb{I}_m) \quad (\text{A.10})$$

where  $\mathbb{I}_i$  ( $i = 1, 2, \dots, m$ ) is the  $i$ 'th column of  $\mathbb{I}$ , and each of the densities on the right of (A.10) is a multivariate Student density [see Zellner (1971, p. 397), or Drèze and Richard (1983, p. 589)]. Formula (A.10) suggests a sequential drawing procedure.

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