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PROCUREMENT AUCTIONS*

SUDIPTO DASGUPTA

DANIEL F. SPULBER

MRG WORKING PAPER #M8729

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August, 1987

ABSTRACT

This paper examines the procurement problem of a buyer who wishes to procure an object from a given number of firms under conditions of asymmetric information about firms' cost. Optimal fixed and variable quantity procurements are analyzed and implementation issues discussed. It is shown that the optimal mechanism can be implemented by auction in which a payment schedule is announced by the buyer, and the firm bidding the largest output wins the contract.

*The support of the National Science Foundation under Grant No. SES-86-08115 is gratefully acknowledged.

1. INTRODUCTION

Government procurement is generally modelled by economists as the acquisition of a fixed quantity of a product with well defined characteristics. Most procurement, however, involves the design and development of new technology, variable production levels or the provision of services.[1] To address this problem, government acquisition through sealed bids, or "formal advertising" as it is known, usually involves some modification of the bids and the method of selecting winners to account for potential design changes and future production costs. The present paper presents an alternative to the standard fixed quantity procurement auction which allows the buyer to *vary the quantity* of the good purchased (or quality of service) based on bids by competing sellers. This model addresses more complex procurement techniques in which firms submit estimates of life-cycle costs or production designs in addition to price estimates.[2]

Cost estimates and implementation of production plans are an important feature of "second sourcing", a procurement procedure in which the buyer selects at least two producers. We extend this variable quantity auction to the case of *multiple winners* and show that the equal marginal cost rule no longer holds under asymmetric information.

The standard procurement auction model studies the purchase of a particular indivisible object.[3] Marginal and average production costs are constant as is the buyer's marginal value of the good. It is reasonable to suppose that sellers' marginal and average costs may exhibit scale effects and that the buyer may have diminishing marginal utility. In this general setting, the fixed quantity auction is only a second best procurement strategy for the buyer. We begin by showing, for general buyer benefits and seller costs, that the buyer may *optimally* select the fixed quantity to be purchased by auction. The optimal quantity is obtained by setting the buyer's marginal value of the good equal to expected marginal production costs adjusted on the basis of information rents obtained by firms and the number of potential bidders. The fixed quantity rule ignores the information conveyed by the winning bid. Accordingly, even if the buyer optimally adjusts the quantity to be purchased before the auction, we show that the buyer's net gain always can be improved by a variable-quantity schedule which makes future production a function of the winning firm's costs.

An *optimal* procurement mechanism is then derived for which the quantity schedule depends on the announced cost parameter. It is shown that the optimal procurement policy may be implemented by a procedure in which the buyer announces a *total* compensation schedule as a function of total outputs. Firms bid output levels and the winner is the largest output bidder. While the fixed-quantity procurement auction can be implemented as a standard sealed bid auction in which the firm bidding the lowest supply price is awarded the contract, the variable-quantity procurement cannot in general be implemented by choosing the firm that bids the lowest per unit price. We give sufficient conditions for implementability on the basis of per unit bid prices.

Our work is closely related to Riordan and Sappington (1986, 1987) who consider variable-quantity procurement in a two stage model.[4] Firms observe an informative signal about prospective production costs. These signals are reported to the buyer and constitute their bids. A single firm is awarded the contract. The actual production costs is subsequently observed by the firm. Final payments and quantity delivered depend on the reported signal and the reported observed cost. Their optimal quantity schedule is similar to ours. Our analysis differs primarily in our analysis of fixed *vs.* variable output in auctions and the issue of *implementation*. Rob (1986) considers the choice of an optimal fixed quantity although his analysis does not examine the connection with variable quantity procurement.

The paper is organized as follows. Selection of the optimal output in a standard fixed quantity auction is examined in section 2. The possibility of improving buyer benefits by a variable-quantity schedule is established in section 3. The optimal schedule is derived and characterized in section 4 and an implementation procedure is obtained in section 5. The multiple winners problem is solved in section 6. Conclusions are given in section 7.

2. Demand in the Fixed Quantity Auction

2a. The Auction Framework.

The purpose of this section is to show that in a fixed-quantity procurement, the buyer can optimally choose the quantity to be procured. The fixed quantity case is important to analyse not only because of its prevalence in practice, but also because it sheds some light on the nature of auctions. In particular, it is useful to contrast the procurement auction model with standard models of monopsony. We follow a two-stage procedure. First, the optimal auction is examined for a fixed output. Then, the optimal demand from the point of view of the buyer is considered.

Consider the standard auction model (see for instance Riley and Samuelson, 1981; Maskin and Riley, 1984). Assume that there is a single buyer facing n potential suppliers or firms. The buyer has a utility function of the form $V(Q)$, with $V'(Q) > 0$, $V''(Q) < 0$, where Q is the quantity of the output procured. The firms have cost functions of the form

$$C^i(Q) = \int_0^Q c(\tilde{Q}, \theta^i) d\tilde{Q} + K \tag{1}$$

where θ^i represents the firm's private information about production costs and $K > 0$ is the firm's fixed costs. We assume increasing marginal costs, $c_Q \geq 0$ so that average costs are u-shaped. Marginal costs are increasing and convex in the cost parameter, c_{θ^i}, $c_{\theta^i\theta^i} \geq 0$. Also, we may parameterize costs such that $c_{Q\theta^i} \geq 0$. These assumptions are satisfied for example by an affine cost function $C(Q, \theta) = K + \theta Q$. The cumulative distribution of the cost parameter is defined on $[\underline{\theta}, \bar{\theta}]$ and is given by $F(\cdot)$ which is common knowledge. So, firm cost parameters are independently and identically distributed. Let F be continuously differentiable, $dF(\theta)/d\theta \equiv F(\theta)$, $F(\underline{\theta}) = 0$, $F(\bar{\theta}) = 1$. The hazard rate $H(\theta) = F'(\theta)/F(\theta)$ is assumed to be decreasing in θ. This is satisfied by common distribution functions such as the uniform.

We confine our attention to the case in which a single firm is chosen by the buyer to produce the entire output. It is well known[4] that under our assumptions, an optimal buying strategy involves use of either of the auctions considered by Vickery (1961). Consider first the standard procurement auction in which the buyer invites sealed bids on a given output Q. The buyer announces a maximum reserve price b^0, such that there is one firm at least that is indifferent between selling and not selling at this price, *i.e.*

$$\pi(b^0, \theta^*) = 0 \qquad \text{for some} \qquad \theta^* \in [\underline{\theta}, \bar{\theta}] \tag{2}$$

where $\pi(b,\theta) = Qb - C(Q,\theta)$ denotes profits of a firm with cost parameter θ, which wins with bid b. The following theorem due to Maskin and Riley (1984a) will be useful.

THEOREM 1: *(Maskin and Riley). There exists an equilibrium in which each firm whose θ exceeds θ^* does not bid, while those with $\theta < \theta^*$ bid according to the unique symmetric bid function $b(\theta)$. Moreover, this $b(\theta)$ is increasing and differentiable.*

To apply the Maskin and Riley result in our framework we must verify that $(\partial/\partial\theta)[\pi(b,\theta)/\pi_1(b,\theta)] < 0$. This follows from $c_\theta > 0$. Given the existence of the bid function, one can use methods similar to Riley and Samuelson (1981) to derive explicit expressions for the bid function, expected profits of the firms, and the buyer's expected net gain. This is shown in appendix A1. The bid function is given by

$$b(\theta^i, Q) = \frac{\int_{\theta^i}^{\theta^*} C_\theta(Q,\theta)\big(1-F(\theta)\big)^{n-1} d\theta}{Q\big(1-F(\theta^i)\big)^{n-1}} + \frac{C(Q,\theta^i)}{Q} \tag{3}$$

Expected profit of a firm with cost parameter θ^i is given by

$$E\pi(\theta^i, Q) = \int_{\theta^i}^{\theta^*} C_\theta(Q,\theta)\big(1-F(\theta)\big)^{n-1} d\theta \tag{4}$$

while the expected net gain of the buyer is given by

$$\begin{aligned} g(Q,\theta^*) =& V(Q)\left[1-\big(1-F(\theta^*)\big)^n\right] \\ & - n\int_{\underline{\theta}}^{\theta^*} \left\{C(Q,\theta) + C_\theta(Q,\theta)\frac{F(\theta)}{F'(\theta)}\right\}\big(1-F(\theta)\big)^{n-1} F'(\theta) d\theta \end{aligned} \tag{5}$$

Eq. (3) shows that all firms that submit bids except the marginal firm have positive expected profits. Also, from (3) we have $b^0 = C(Q,\theta^*)/Q$, which determines b^0, given θ^* and Q. Eq. (3) also shows that the per unit bids of firms depend on average, and not marginal, costs. In particular, it is possible for marginal costs to exceed the per unit bid.

Differentiating (3) with respect to Q, we get

$$b_Q(Q,\theta^i) = \frac{C_\theta(Q,\theta^*)\big(1-F(\theta^*)\big)^{n-1}}{Q\big(1-F)\theta^i)\big)^{n-1}}\frac{d\theta^*}{dQ} + \frac{1}{Q}\Bigg[\left(C_Q(Q,\theta^i) - \frac{C(Q,\theta^i)}{Q}\right) + \frac{\int_{\theta_i}^{\theta^*}\left[C_{Q\theta}(Q,\theta) - \frac{C_\theta(Q,\theta)}{Q}\right]\big(1-F(\theta)\big)^{n-1}d\theta}{\big(1-F(\theta^i)\big)^{n-1}}\Bigg] \tag{6}$$

Thus, a change in the level of output has marginal effect on the per unit bid, represented by the first term in (2′), and an inframarginal effect, represented by the second term in (2′). Assuming θ^* is chosen optimally given Q, the sign of the marginal effect would depend on whether the effect of procuring a larger output is to allow more or fewer firms to bid. It can be shown that there exists $\bar{Q}$ such that $\frac{d\theta^*}{dQ} \lesseqgtr 0$ as $Q \gtreqless \bar{Q}$. The inframarginal effect is also of ambiguous sign, and depends on the relation between average and marginal costs.

Similarly, differentiating (4) with respect to Q, one gets

$$\frac{\partial}{\partial Q}E\pi(Q,\theta^i) = C_\theta(Q,\theta^*)\big(1-F(\theta^*)\big)^{n-1}\frac{d\theta^*}{dQ} + \int_{\theta^i}^{\theta^*} C_{q\theta}(Q,\theta)\big(1-F(\theta)\big)^{n-1}d\theta$$

Once again, there are marginal and inframarginal effects. For $Q < \bar{Q}$, increase in output will always increase expected profits of firms.

2b. Buyer Demand.

The buyer's demand for output is now characterized by extending the fixed quantity auction model. The standard model assumes that buyer demand is a constant quantity. However, it is in the buyer's interest to choose this quantity optimally given the buyer's value of the good and expected seller bids.

Maximize expected buyer gain in eq. (5) with respect to Q and θ^* to obtain the optimal fixed-quantity procurement. The first order conditions give:

$$V(Q) = C(Q,\theta^*) + C_\theta(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)} \tag{7}$$

and

$$V'(Q) = \frac{n\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)\left\{C_Q(Q,\theta) + C_Q\theta(Q,\theta)\frac{F(\theta)}{F'(\theta)}\right\}d\theta}{1-\left(1-F(\theta^*)\right)^n} \tag{8}$$

Noting that $1-\left(1-F(\theta^*)\right)^n = n\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)d\theta$, we may rewrite (8) as follows

$$V'(Q) = \frac{\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)\left\{C_Q(Q,\theta) + C_{Q\theta}(Q,\theta)\frac{F(\theta)}{F'(\theta)}\right\}d\theta}{\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)d\theta} \tag{9}$$

We need to ensure that the optimal levels of Q and θ^* are indeed interior, so that the first order conditions (7) and (9) are meaningful. Appendix A2 gives sufficient conditions for interior optimum.

Eq. (9) indicates that choosing the fixed quantity optimally entails setting marginal value of the output to the buyer equal to average "adjusted" marginal cost, where the latter refers to the term $C_Q(Q,\theta) + C_{Q\theta}(Q,\theta)\frac{F(\theta)}{F'(\theta)}$. The term $C_{Q\theta}(\cdot)\frac{F(\theta)}{F'(\theta)}$ represents an adjustment to marginal cost that has to be made on account of asymmetry of information and the need to prevent low cost firms from overbidding and masquerading as high-cost firms. Greater insight can be obtained regarding (9) once we discuss variable-quantity procurement. The connection between fixed and variable quantity procurement and the second-best nature of the former can then be understood more fully. Let us also note that since $V'(Q) \neq C_Q(Q,\theta_m)$ is general, where θ_m denotes that cost of the winning firm, the outcome is not in general *ex post* efficient.

Condition (7) shows that the optimal, fixed-quantity auction also involves setting the cut-off value of θ^* at a level where an inefficient outcome is not ruled out. It could be the case that the lowest-cost firm is between θ^* and $\tilde{\theta}$, where $\tilde{\theta}$ satisfies $V(Q) - C(Q,\tilde{\theta}) = 0$. In this case no output is procured, even though the cost of production is lower than the value of the output to the buyer. The reason why it is optimal for the buyer to allow this possibility of inefficiency is that it causes firms to bid lower than they otherwise would. This is evident from (3), which shows that for given level of Q, higher θ^* implies higher $b(\theta^i)$. Thus, in setting θ^* optimally, the buyer is trading off the loss from an inefficient outcome against the gains from lower bids by the firms.

We now state some results that follow in a straightforward manner from the first-order conditions and the assumptions regarding cost, utility and distribution functions made earlier.

PROPOSITION 1: *The marginal firm is dependent on the number of bidding firms.*

This result is different from the standard auction result, where the number of bidders has no effect on the marginal bidder. Here, the number of bidders affects the optimal level of Q (see (9)), and hence also θ^*. However, dQ/dn cannot be signed in general.

The next result is interesting because it differs from the standard monopsony result. In the standard monopsony model, a change in the level of fixed costs has no effect on the short run supply curve faced by the monopsonist, which depends on the marginal costs of the firms supplying the monopsonist. Thus, there is no effect on the level of output or payments made by the monopsonist. In the present auction model, a change in the level of fixed costs affects the bids of the firms; therefore it affects the marginal firm and the level of output.

PROPOSITION 2: *A change in the level of fixed costs (assumed to be the same for all firms unless otherwise stated) affects the optimal level of output and the marginal firm. Specifically, for θ^* sufficiently close to $\bar{\theta}$, $d\theta^*/dK < 0$ and $dQ/dK > 0$.*

The proof is straightforward and given in Appendix A3.

If there is asymmetry of information regarding fixed costs, but not regarding marginal cost, the buyer behaves like a first degree price discriminator by choosing the socially optimal output level. The bidding process then serves to allocate the division of the surplus between the buyer and the eventual supplier.

PROPOSITION 3: *If the only asymmetry of information is regarding fixed costs of the firms, i.e., the cost functions are of the form*

$$C(Q,\theta) = K(\theta) + \int_0^Q c(\tilde{Q})d\tilde{Q},$$

then the optimal fixed quantity of output procured by the buyer is ex post socially optimal.

We have discussed fixed quantity procurement in some detail, because this method of procurement is the standard model in the literautre and is employed in practice. However, as we argue in the next section, fixed-quantity procurement is not optimal.

3. The Quantity Schedule

We now wish to consider the possibility that the buyer, instead of procuring a fixed quantity of output, procures according to a variable quantity schedule $Q = Q(\theta)$ which relates the quantity procured to the cost θ of the lowest cost firm. We shall prove that if, instead of the optimal fixed quantity, the buyer procures according to a schedule $Q = Q(\theta)$ which has a *small* negative slope, then his net gain can be improved.

PROPOSITION 4: *There exists a variable quantity schedule $Q = Q(\theta)$ such that if the buyer procures $Q(\theta)$ when the lowest cost firm is θ, then his net gain is strictly greater than if he procured according to the fixed quantity schedule $Q = Q^*$, where Q^* is given by eq. (9).*

PROOF: To prove the theorem, we first derive the expected net buyer gain making use of the necessary conditions for incentive compatibility. It is shown in Appendix A4 that the expected buyer gain is given by:

$$\begin{aligned} g(\tilde{Q}(\cdot),\theta^*) =& n\int_{\underline{\theta}}^{\theta^*} (1-F(\theta))^{n-1}F'(\theta)\Big\{V(Q(\theta)) \\ &- C(Q(\theta),\theta) - C_\theta(Q(\theta),\theta)\frac{F(\theta)}{F'(\theta)}\Big\}d\theta \end{aligned} \tag{10}$$

It may be noted that this reduces back to the expression obtained in the fixed quantity case, eq. (5) when Q is fixed, by virtue of the relation $1-\left(1-F(\theta)\right)^{n} = n\int_{\underline{\theta}}^{\theta^*}(1-F(\theta))^{n-1}F'(\theta)d\theta$. Now, referring back to eq. (9), we have observed that $V'(Q^*)$ is a weighted average of $C_Q(Q^*,\theta)+C_{Q\theta}(Q^*,\theta)F(\theta)/F'(\theta)$ in the interval $[\underline{\theta},\theta^*]$. This rules out the following:

(a) $V'(Q^*) > C_Q(Q^*,\theta)+C_{Q\theta}(Q^*,\theta)F(\theta)/F'(\theta)$ *for all* $\theta\in[\underline{\theta},\theta^*]$,

(b) $V'(Q^*) < C_Q(Q^*,\theta)+C_{Q\theta}(Q^*,\theta)F(\theta)/F'(\theta)$ *for all* $\theta\in[\underline{\theta},\theta^*]$.

Hence, $\exists\ \hat{\theta}\in[\underline{\theta},\theta^*]$ such that

$$V'(Q^*) = C_Q(Q^*,\hat{\theta})+C_{Q\theta}(Q^*,\hat{\theta})F(\hat{\theta})/F'(\hat{\theta}).$$

Now, given our assumptions about the cost function *e.g.*, $C_\theta>0$. $C_{\theta\theta}\geqq 0$, and also that $F(\theta)/F'(\theta)$ is increasing in θ, it follows that

$$V'(Q^*)-C_Q(Q^*,\theta)-C_{Q\theta}(Q^*,\theta)F(\theta)/F'(\theta)\gtreqless 0 \quad\text{as}\quad \theta\lesseqgtr\hat{\theta}. \tag{11}$$

Now, let us define

$$Q(\theta) = Q^*+\beta(\theta-\hat{\theta}) \tag{12}$$

where β is either zero or negative and small. This implies that

$$g\left(\tilde{Q},\theta^*\right)\equiv J(\beta) = n\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)\Big[V\big(Q^*+\beta(\theta-\hat{\theta})\big)$$
$$-C\big(Q^*+\beta(\theta-\hat{\theta}),\theta\big)-C_\theta\big(Q^*+\beta(\theta-\hat{\theta}),\theta\big)\frac{F(\theta)}{F'(\theta)}\Big]d\theta.$$

If β changes, then θ^* would have to change in order to maximise buyer gain. Keeping θ^* unchanged when β changes would therefore underestimate the increase in buyer gain. Then, we obtain

$$J_\beta(0)\big|_{\theta^*\text{ fixed}} = n\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)\left[V'(Q^*)-C_Q(Q^*,\theta)-C_{Q\theta}(Q^*,\theta)\frac{F(\theta)}{F'(\theta)}\right][\theta-\hat{\theta}]d\theta$$

$$= n\int_{\underline{\theta}}^{\hat{\theta}}\left(1-F(\theta)\right)^{n-1}F'(\theta)\left[V'(Q^*)-C_Q(Q^*,\theta)-C_{Q\theta}(Q^*,\theta)\frac{F(\theta)}{F'(\theta)}\right][\theta-\hat{\theta}]d\theta$$

$$+ n\int_{\hat{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)\left[V'(Q^*)-C_Q(Q^*,\theta)-C_{Q\theta}(Q^*,\theta)\frac{F(\theta)}{F'(\theta)}\right][\theta-\hat{\theta}]d\theta,$$

which is negative by eq. (11). Hence, a small negative value of β resulting in the negatively sloped schedule eq. (12) will increase buyer gain even when θ^* is not adjusted. Q.E.D.

4. The Optimal Schedule

In this section, we consider the question of the *optimal* quantity schedule $Q = Q(\theta)$ in the variable-quantity procurement, again confining our attention to the case in which it is optimal to have only one firm produce the output. Formally, the problem is to choose $Q = Q(\theta)$ and θ^{**} such that the expected buyer gains

$$n \int_{\underline{\theta}}^{\theta^{**}} (1 - F(\theta))^{n-1} F'(\theta) \left\{ V(Q) - C(Q,\theta) - C_\theta(Q,\theta)\frac{F(\theta)}{F'(\theta)} \right\} d\theta$$

is maximized.

This is a free horizon calculus of variation problem. First of all, we may note that the integrand is concave in Q, since $V''(Q) - C_{QQ}(Q,\theta) - C_{QQ\theta}(Q,\theta)F(\theta)/F'(\theta) < 0$. Since concavity is satisfied, the following conditions are sufficient for a maximum, where J denotes the integrand.

(i) $J_{Q'} = 0$ at $\theta = \underline{\theta}$ and $\theta = \theta^{**}$

(ii) $J = 0$ at $\theta = \theta^{**}$

(iii) $J_Q = \frac{dJ_{Q'}}{d\theta}, \quad \underline{\theta} \leqq \theta \leqq \theta^{**}$

(i) is trivially satisfied in the present case. From (ii), we have

$$V(Q(\theta^{**})) = C(Q(\theta^{**}), \theta^{**}) + C_\theta(Q(\theta^{**}), \theta^{**})F(\theta^{**})/F'(\theta^{**}) \tag{13}$$

and from (iii),

$$V'(Q) = C_\theta(Q,\theta) + C_{Q\theta}(Q,\theta)F(\theta)/F'(\theta), \quad \underline{\theta} \leqq \theta \leqq \theta^{**} \tag{14}$$

Eq. (13) gives the optimal θ^{**}. Its similarity with (7) may be noted. The optimal quantity schedule $Q = Q(\theta)$ is given by (14). Given our assumptions, it can be checked that $Q'(\theta) < 0$.

Eqs. (13) and (14) show that the optimum variable-quantity auction is not in general *ex post* efficient. It is again possible, as in the fixed quantity case, that the lowest θ satisfies $\theta^{**} < \theta < \tilde{\theta}$, where $V(Q(\tilde{\theta})) = C(Q(\tilde{\theta}), \tilde{\theta})$. This implies that the procurement auction fails even though one of the firms could produce a level of output at a cost less than the buyers valuation of the output. Eq. (14) shows that the optimal quantity schedule results in a level of output that is not *ex post* optimal, unless the winning firm happens to be the lowest-cost firm ($i.e.$, $\theta = \underline{\theta}$). This happens with probability one if an infinite number of firms are bidding. The distortion from *ex post* optimality, represented by the term $C_{Q\theta}(Q(\theta), \theta)F(\theta)/F'(\theta)$, is on account of the information asymmetry. In order to prevent lower-cost firms from overstating their costs (or bidding too high), higher-cost firms have to be penalized by a reduction in output below the *ex post* optimal level. Since $C_{Q\theta}(Q, \theta)$ is nondecreasing in θ and $F(\theta)/F'(\theta)$ is increasing in θ, the quantity distortions become more severe the higher is the lowest reported cost.

Eq. (14) also makes clear the relation between the fixed-quantity and the variable-quantity auction. In section (2), an attempt was made to interpret the optimality rule (9) which gives the optimal level of output under fixed-quantity procurement. A better interpretation can now be given. Noting that $n(1 - F(\theta))^{n-1}F'(\theta)$ is the density of the lowest-order statistic, we see that the right hand side of (9) replaces the right hand side of (14) by its expected value over the lowest θ, conditional on there being a winning firm. This brings out very clearly the second-best nature of the fixed-quantity rule. It is as though the buyer were following the optimal variable-quantity rule but for some reason choose to ignore the information conveyed by the winning bid. However, the advantage of the variable-quantity auction over the fixed-quantity auction disappears as the number of bidding firms becomes very large. This is stated formally in proposition 5.

PROPOSITION 5: *Let the number of potential suppliers, n, approach infinity. Then the following are true: (a) Both the fixed and variable quantity auctions result in ex post socially optimal outcomes. (b) The expected profit of all firms is zero. (c) Under both* *auctions the expected net gain to the buyer is given by* $V(Q(\underline{\theta})) - C(Q(\underline{\theta}), \underline{\theta})$.

The proof follows readily from the fact that as $n \to \infty$, the density of the lowest order statistic converges with probability one to the discrete density which places all the probability at $\theta = \underline{\theta}$.

A greater number of firms may participate in the variable-quantity auction than in the fixed quantity case.

PROPOSITION 6: *The variable-quantity auction promotes more bidding competition than the fixed-quantity auction, i.e., the marginal firm has higher cost under the variable-quantity auction* ($\theta^{**} > \theta^{*}$).

The proof is sketched in the appendix.

Unlike the fixed-quantity auction, the level of fixed costs does not affect the optimal variable quantity schedule, except that it affects the domain of the schedule in that θ^{**} is affected. One can show the following:

PROPOSITION 7: *Under optimal variable quantity procurement, an increase in the level of fixed costs leads to a lower cost marginal firm, i.e.,* $d\theta^{**}/dK < 0$.

The proof is given in Appendix A3.

The following result has been observed in the related framework of Riordan and Sappington (1987).

PROPOSITION 8: *In the variable-quantity auction, both the optimal quantity schedule and the marginal firm are independent of the number of firms.*

Note that this result also contrasts with the fixed quantity one.

The fact that the optimum quantity schedule is independent of the number of bidders is one of the most interesting features of the procurement problem. It explains the similarity between the schedule we have derived, and those derived by Baron and Myerson (1982) in the context of regulating a monopolist with unknown costs. Riordan and Sappington

(1987) also discuss a similar schedule in the context of the procurement problem of a buyer who is unable to precommit himself. It is *because* the quantity schedule we derive has the property that the number of bidders does not matter that the Baron-Myerson and Riordan-Sappington schedules are much the same as ours. Thus, so long as costs are independent and only one winner is chosen, the optimal procurement schedule is the same as the Baron-Myerson schedule. With multiple winners, the problem changes as will be seen below.

It is worth pointing out that while the quantity schedule derived by Riordan and Sappington (1986, 1987) in their two stage problem *with commitment* has similarities to ours, its implications are quite different. This is perhaps best brought out by the example provided in Riordan and Sappington. They consider cost functions of the form $C(Q,\theta) = \theta Q + K$. The variable θ is only observed *after* costs K have been sunk. Prior to that, each firm observes a signal t about θ. It is assumed that the (conditional) density functions for θ are spanned by two linear functions of the form $f^j(\theta) = 1 + \alpha^j\left[\frac{1}{2}(\underline{\theta} + \bar{\theta}) - \theta\right]$, $j = L, H$, $\alpha^L > \alpha^H$, $\theta \in [1,2]$. If a signal $t \in [0,1]$ is observed, the conditional distribution of θ is given by $F(\theta|t) = tF^L(\theta) + (1-t)F^H(\theta)$, where $F^j(\theta) = \int_1^\theta f^j(\tilde{\theta})d\tilde{\theta}$, $j = L, H$. Riordan and Sappington show that for the buyer's utility function $V(Q) = rQ - \frac{1}{2}\delta Q^2$, and for the particular parameter values $\alpha^L = 0.75$ and $\alpha^H = 0.25$, $\gamma = 4$, $\delta = 0.02$, their optimal quantity schedule implies deviations from the *ex post* optimal output levels of at most 3 percent. Thus they conclude that the optimal mechanism may be closely approximated by the simpler mechanisms proposed by Loeb and Magat (1979).

We show below that these same parameter values lead to distortions that are much more significant in the context of the procurement model that we have discussed. We can dispense with the conditional distribution function $F(\theta|t)$ since private costs are known with certainty. The density $f(\theta)$ is given by $f(\theta) = 1 + \alpha\left[\frac{1}{2}[\underline{\theta} + \bar{\theta}] - \theta\right]$, and we choose $\alpha = 0.5$ (*i.e.*, the average of α^L and α^H in the Riordan Sappington example).

Let $Q^*(\theta)$ denote the *ex post* optimal level of output, and $\hat{Q}(\theta)$ the optimal procurement schedule. Then it can be shown that

$$\frac{\hat{Q}(\theta)}{Q^*(\theta)} = 1 - \frac{g(\theta)}{\gamma - \theta}$$

where

$$g(\theta) = \frac{F(\theta)}{f(\theta)} = \frac{\frac{7}{4}\theta - \frac{1}{4}\theta^2 - \frac{3}{2}}{\frac{7}{4} - \frac{1}{2}\theta}$$

The maximum distortion corresponding to $\theta = \bar{\theta} = 2$ is given by

$$\frac{\hat{Q}(\bar{\theta})}{Q^*(\bar{\theta})} = 1 - \frac{4}{3 \cdot 2} = 1 - \frac{2}{3} = \frac{1}{3}$$

Thus, the distortion is as high as 66%.

For $\theta = 1.5$, we get $\frac{\hat{Q}(1.5)}{Q^*(1.5)} = \frac{29}{40}$. Thus, the distortion is 27.5%. We therefore find that the deviations from optimality are far more serious for the procurement schedule we discuss in contrast to the Riordan-Sappington schedule. Simple schemes like the one suggested by Loeb and Magat will not work well in this situation.

5. IMPLEMENTATION

We have shown in Appendix 4 that if the buyer announces the following expected payment function and allocation rule, then truth-telling is optimal for each firm:

$$E_{\theta_{-i}}B(\theta_i) = C\big(Q,(\theta_i)\big)\big(1 - F(\theta_i)\big)^{n-1} + \int_{\theta_i}^{\theta^{**}} \big(1 - F(\theta)\big)^{n-1} C_\theta(Q,\theta)d\theta;$$

also, $Q = Q(\theta)$ if $\theta_i = \min(\theta_1, \ldots, \theta_n)$ and $Q = 0$ otherwise.

It can be readily verified that the expected payment to firm θ_i, $E_{\theta_{-i}}B(\theta_i)$, is nonincreasing in θ_i. However, the payment conditional on winning, $\tilde{B}(\theta_i) = \frac{E_{\theta_{-i}}B(\theta_i)}{\left(1-F(\theta_i)\right)^{n-1}}$, may either increase or decrease with θ_i.

The issue of implementation is now addressed. In the previous section, we had derived the optimal quantity schedule $Q(\theta)$ in the one-winner case. We now describe a simple implementation game that will result in the allocation $Q(\theta)$ as a Bayes-Nash equilibrium outcome. This implementation game consists of the buyer announcing a payment-quantity schedule $B = B(Q)$. Given this schedule, firms choose the output levels that they want to produce. The firm choosing the highest output level is then given the contract. It is readily seen that the schedule $B(Q) = \tilde{B} \cdot Q^{-1}(Q)$, where $\tilde{B}$ and Q are as defined above, implements $Q = Q(\theta)$ as a Bayes-Nash equilibrium of the implementation game. Given the schedule $B(Q)$, firm i chooses $\hat{Q}_i$ to maximize expected profits, which is given by $B(Q_i)\big(1 - F \cdot Q^{-1}(Q)\big)^{n-1}$ *under the assumption that other firms are playing* $Q = Q(\theta)$. Then by virtue of the result reported at the beginning of the sections, a Nash-equilibrium response for firm i is to choose $Q(\theta_i)$. Note that the implementation game described above is well-defined since $B(Q)\big(1 - F \cdot Q^{-1}(Q)\big)^{n-1}$ is increasing in Q. We therefore have the following result on implementation.

PROPOSITION 9: *Let $Q = Q(\theta)$ be the optimal quantity schedule in the one winner case. Then the following implementation game results in the allocation $Q = Q(\theta)$ as a Bayes-Nash equilibrium outcome:*

(a) The buyer, as first mover, preannounces a payment schedule, $B(Q) = \tilde{B} \cdot Q^{-1}(Q)$ (where $\tilde{B}$ is as defined previously), conditional on winning with a "bid" of Q. It is also announced that the firm that bids the highest Q wins the auction.

(b) Given this mechanism, firms choose the output levels they want to produce.

(c) The firm that bids the highest output is chosen.

The implementation discussed above can be contrasted with the "menu auction" implementation of Riordan and Sappington. To implement their two stage game, Riordan-Sappington suggest the following procedure. Initially, the buyer announces a menu of rank-ordered payment schedules. Each firm is then asked to choose a point on one of these schedules. The firm choosing the highest rank-ordered schedule wins. In contrast, the winner in our implementation game is the firm bidding the highest *output* on the unique payment schedule preannounced by the buyer.

It is important to point out in this context that unless restrictive assumptions are imposed on the cost functions, the usual auction method of choosing the firm that bids the lowest *per unit supply price* fails to implement the optimal schedule $Q(\theta)$. A per unit bid-quantity schedule $Q = Q(b)$ cannot be designed in general that would implement the optimal schedule $Q = Q(\theta)$ when the lowest per unit bid is chosen. The problem arises because the per unit bid $b(\theta) = \frac{\tilde{B}(\theta)}{Q(\theta)}$ is no longer monotonic in general. This is so because while $Q(\theta)$ is monotonic, as pointed out above, $\tilde{B}(\theta)$ is not.

This is where the distinction between fixed quantity and the variable quantity procurement is particularly interesting. The fixed quantity procurement can be implemented as a regular auction in which firms bid prices and the lowest bidder is chosen. The variable quantity auction cannot in general be so implemented. To implement the variable quantity auction, the buyer has to offer the firms a payment-output schedule, on which the firms choose the output levels they want to produce. The firm choosing the highest output level wins.

However, restrictive assumptions on the cost function can give implementability via per unit price auctions. It can be shown that if the cost functions exhibit increasing returns to scale over the relevant range of output, then a per unit bid-quantity schedule can be designed such that the optimal allocation $Q = Q(\theta)$ results when firms choose points on this schedule and the lowest unit price is selected. This is stated formally in proposition 10.

PROPOSITION 10: *Let an output schedule $Q = Q(\theta)$ be given for $\theta \in [\underline{\theta}, \theta^{**}]$ with $Q'(\theta) < 0$. Then, given increasing returns to scale in the relevant output range, there exists an equilibrium per unit bid function $b(\theta)$ increasing in θ and an associated bid-output schedule $Q = \tilde{Q}(b)$ such that in a low-bid procurement auction, the allocation $Q = Q(\theta)$ results.*

PROOF: Let $b(\theta) = \frac{C\left(Q(\theta),\theta\right)}{Q(\theta)} + \frac{\int_{\theta}^{\theta^{**}} C_{\theta}\left(Q(\tilde{\theta}),\theta\right)\left(1-F(\tilde{\theta})\right)^{n-1} d\tilde{\theta}}{\left(1-F(\theta)\right)^{n-1} Q(\theta)}$. It can be readily verified that under the assumption of IRS, $b'(\theta) > 0$. Hence $b(\theta)$ is monotonic in θ. Let $\tilde{Q}(b) = Q \cdot b^{-1}(b)$ be the bid-output schedule. The expected payment to firm i if it bids $b(x)$ is given by

$$\left(1-F(x)\right)^{n-1}\left[Q\cdot b^{-1}\left(b(x)\right)\right]\cdot b(x)=\int_x^{\theta^{**}} C_\theta\left(Q(\tilde{\theta}),\tilde{\theta}\right)\left(1-F(\tilde{\theta})\right)^{n-1}d\tilde{\theta} + C\left(Q(x),x\right)\left(1-F(x)\right)^{n-1}.$$

by lemma 2 in Appendix 4, it follows that expected profits are maximized at $x=\theta_i$. Hence $b(\theta_i)$ is an equilibrium bid function. Q.E.D.

The crucial issue in proposition 10 is the monotonicity of the per unit bid function. If the bid function is not monotonic, then $Q(b)=Q\cdot b^{-1}(b)$ cannot be defined. If IRS does not prevail, then lower cost firms may end up with higher per unit bids, since $Q'(\theta)<0$. IRS will prevail over the relevant range of output if fixed costs are sufficiently high.

However, it should be reiterated that this is quite a restrictive assumption, so that the usual auction method of implementation would fail in general to implement the optimal variable quantity procurement.[5]

6. Multiple Winners

Up to now, we have restricted our analysis to the case in which it is optimal to choose only one firm to produce the entire final output. Implicitly, this presupposes that either fixed costs are high in relation to the interval $[\underline{\theta},\bar{\theta}]$, or that the cost functions satisfy $C_{QQ}(Q,\theta)=0$ (*e.g.*, $C=\theta Q+K$). In general, however, it might be optimal to have more than one firm produce the final output. Rob (1986) has considered this problem but his thrust is different. We argue that "dual or multiple sourcing" may be advantageous (if fixed costs are not too high) because it allows the possibility of equalizing the marginal cost of production across firms. Rob, on the other hand, advocates dual sourcing on the ground that after a *fraction* of a project is awarded to an initial contractor, costs become known, so that the remaining fraction can be competitively awarded. This is known as an "educational buy". We do not pursue sequential procurement here. In our framework, the buyer *simultaneously* procures from two or more firms.

6a. The Buyer's Problem.

The general problem from the point of view of the buyer can be regarded as one of designing output allocation schemes $q_i^* = q_i(\theta_1, \ldots, \theta_n)$ and payment schemes $P(\theta_1, \ldots, \theta_n)$ such that the incentive compatibility and individual rationality constraints are satisfied and his expected net gain is maximized.

The expected profit of firm i with cost parameter θ_i is given by

$$\pi_i(x, \theta_i) = \underset{\theta_{-i}}{E} \left[P(x) - C\big(q_i(x, \theta_{-i}), \theta_i\big)\right]$$

Proceeding exactly as in lemma 1 in Appendix 4, we have for any $\langle P_i(\ ), q_i(\ )\rangle_{i=1}^n$ the following necessary conditions for incentive compatibility:

$$\underset{\theta_{-i}}{E} P(\theta_i) = \underset{\theta_{-i}}{E} C\big(q_i(\theta_i, \theta_{-i}), \theta_i\big) + \int_{\theta_i}^{\theta^{**}} \underset{\theta_{-i}}{E} C_\theta\big(q_i(\theta_i, \theta - i), \theta_i\big) d\theta \tag{6.1}$$

For any $\theta_i, \theta_i' \in [\underline{\theta}, \theta^*]$,

$$\int_{\theta_i}^{\theta_i'} \underset{\theta_{-i}}{E} \left[C_\theta\big(q(\theta_i, \theta_{-i}), \tilde{\theta}\big) - C_\theta\big(q(\theta_i', \theta_{-i}), \tilde{\theta}\big)\right] d\tilde{\theta} \geqq 0 \tag{6.2}$$

A sufficient condition for (6.2) is that,

$$q_i(\theta_i, \theta_{-i}) \quad \text{is nonincreasing in} \quad \theta_i. \tag{6.3}$$

It can once again be proved as in lemma 2 of Appendix 4 that (6.1) and (6.3) are *sufficient* for incentive compatability. From (6.1), it can be shown after integration by parts that expected payment by the buyer is given by

$$R = \int_{\theta_i} \cdots \int_{\theta_n} \sum_{i=1}^n \left[\frac{F(\theta_i)}{F'(\theta_i)} C_\theta\big(q_i(\theta), \theta_i\big) + C\big(q_i(\theta), \theta_i\big)\right] dF(\theta_i) \ldots dF(\theta_n)$$

where $\theta = (\theta_1, \ldots, \theta_n)$.

Let us consider the integrand

$$\sum_{i=1}^n \left[\frac{F(\theta_i)}{F'(\theta_i)} C_\theta\big(q_i(\theta), \theta_i\big) + C\big(q_i(\theta), \theta_i\big)\right]$$

Let $Q = \sum_{i=1}^{n} q_i$. For each Q, the buyer's problem is

$$\min_{q_1,\dots,q_n} \sum_{i=1}^{n} \left[\frac{F(\theta_i)}{F'(\theta_i)} C_\theta(q_i, \theta_i) + C(q_i, \theta_i) \right] \qquad \text{s.t.} \; \sum_{i=1}^{n} q_i = Q. \tag{6.4}$$

It is worth reminding ourselves that in the presence of fixed costs, $\lim_{q \to 0} C(q, \theta_i) = K > C(0, \theta_i) = 0$. Since we are minimizing a discontinuous function, first-order conditions are neither necessary nor sufficient. However, since the discontinuity is only at $q_i = 0$, the q_i's that are positive in the optimal solution must satisfy

$$\frac{F(\theta_i)}{F'(\theta_i)} C_{\theta q}(q_i, \theta_i) + C_q(q_i, \theta_i) = \frac{F(\theta_j)}{F'(\theta_j)} C_{\theta q}(q_j, \theta_j) + C_q(q_j, \theta_j),$$

$$\text{for} \quad i \neq j, \; q_i, q_j > 0.$$

Hence we have the following result:

PROPOSITION 11: *If it is optimal to procure from more than one firm, then the optimal procurement involves the equalization of the "adjusted" marginal costs of these firms, but not necessarily their marginal costs.*

Returning to the buyer's problem (6.4), the solution yields

$$q_i = q_i(\theta, Q), \quad i = 1, \dots, n \tag{6.5}$$

Substituting this into the integrand gives the function

$$L(\theta, Q) = \sum_{i=1}^{n} \left[C\big(q_i(\theta, Q), \theta_i\big) + C_\theta\big(q_i(\theta, Q), \theta_i\big) \frac{F(\theta_i)}{F'(\theta_i)} \right]$$

Optimal procurement then implies solving the problem

$$\max_{Q} V(Q) - L(\theta, Q) \tag{6.6}$$

subject to (6.2)

If we solve (6.6) without regard to (6.2), the solution yields $Q = Q(\theta)$. Substituting in (6.5) gives:

$$q_i^* = q_i^*(\theta), \quad i = 1, \ldots, n.$$

It is shown in Appendix A5 that under the regulatory assumption that $\frac{F(\theta)}{F'(\theta)}$ is nondecreasing in θ, $q_i^*(\theta)$ is nonincreasing in θ_i, so that (6.2) is satisfied, and $q_i^*(\theta)$ represents the solution to the buyer's problem.

6b. Implementability.

We now turn to the question of implementation. Given the optimal allocation $q_i^*(\theta)$, $i = 1, \ldots, n$, we know that if the expected payment of firms is given by

$$\underset{\theta_{-i}}{E}\, P(\theta_i) = \underset{\theta_{-i}}{E}\, C(q_i^*(\theta), \theta_i) + \int_{\theta_i}^{\theta^{**}} \underset{\theta_{-i}}{E}\, C_\theta(q_i^*(\tilde{\theta}, \theta_{-i}), \tilde{\theta})\, d\tilde{\theta}$$

then it is optimal for the firms to truthfully report their cost parameters. Because all agents are assumed to be risk neutral, the buyer can announce the payment $E_{\theta_{-i}} P(\theta_i) = T(\theta_i)$ *whenever* a firm reports θ_i (in other words, the payment is not conditional on the firm being asked to produce any output). We can then state the following result.

LEMMA 12: *Let $q = q(\theta_i)$ be any monotonic functions of θ_i. Let $(q^n)^{-1} : R^n \to R^n \equiv (q^{-1}(q_1), \ldots, q^{-1}(q_n))$. Then, given the allocation rule $q_i^* = q_i^* \cdot (q^n)^{-1}\{q_1, \ldots, q_n\}$ and the payment function $T \cdot q^{-1}(q_i)$, a Nash equilibrium response of every firm is to choose $q_i = q(\theta_i)$, so that $q_i^* = q_i^*(\theta_1, \ldots, \theta_n)$.*

The lemma therefore suggest that instead of having the firms report their true cost parameters, they can be made to choose points on a schedule $\tilde{T}(q) = T \cdot q^{-1}(q)$, given a preannounced allocation rule $q_i^* = \tilde{q}_i(q_1, \ldots, q_n) \equiv q_i^* \cdot (q^n)^{-1}\{q_1, \ldots, q_n\}$, which makes the eventual allocation to firm i a function of the output levels chosen by all firms.

Clearly, allocation rules $\tilde{q}_i(q_1, \ldots, q_n)$ will in general be quite complicated. However, for the case in which fixed costs are so high that it is never optimal to have more than two firms produce the final output (or the case of $n = 2$), fairly simple allocation rules may be obtained. For example, suppose $V(Q) = bQ - \frac{1}{2}aQ^2$, for $b > 0$, $a > 0$, and $C(q_i, \theta_i) = K + \theta_i q_i^2$ for $q_i > 0$. Let $q = q'(\theta)$ be the optimal schedule in the one winner case. Then the following is true.

PROPOSITION 12: *An optimal procurement strategy of the buyer when cost and utility functions are quadratic is to announce the payment schedule*

$$\tilde{T}(q) = T \cdot (q')^{-1}(q)$$

where $q'(\ )$ is the optimal schedule in the one winner case.

Each firm chooses a point on this schedule. If only the firm bidding the highest output q_1 is chosen, then it is asked to produce q_1. If the firms bidding the two highest outputs q_1 and q_2 are chosen, then they are asked to produce output levels given by

$$\hat{q}_1 = q_1 - \frac{q_1 q_2 (ab - a^2 q_1^2)}{b^2 - a^2 q_1 q_2}$$

$$\hat{q}_2 = q_2 - \frac{q_1 q_2 (ab - a^2 q_2^2)}{b^2 - a^2 q_1 q_2}$$

PROOF: The proof merely consists of solving for the optimal allocations when two winners are chosen *i.e.*, solving for $q_1^*(\theta_1, \theta_2)$ and $q_2^*(\theta_1, \theta_2)$ and defining

$$\hat{q}_1(q_1, q_2) = q_i^*\left(q'^{-1}(q_1), q'^{-1}(q_2)\right).$$

In conclusion, dual or multiple sourcing provides a possible way of cost reduction by allocating output among firms on the basis of "adjusted marginal costs". However, the gains that are thus achievable may be outweighed by the additional fixed cost of having one more firm produce the output — in which case selecting one winner is optimal. While implementation is quite complex in general, fairly simple implementation rules can be designed for the two winners case.

Conclusion

We have considered the procurement problem of a buyer who faces a given number of potential suppliers, but does not have information about the firms' costs. We have shown that the common practice of preannouncing a fixed quantity to be procured, inviting sealed bids for the supply price and selecting the lowest bidder is suboptimal unless the number of bidders is infinite. The fixed quantity procurement can be improved upon by making the quantity to be procured a function of the cost of the lowest-cost firm. Under standard regularity assumptions, we derive the optimal procurement schedule. We show, however, that the optimal schedule cannot in general be implemented by the usual auction method of inviting per unit-bids from potential suppliers, selecting the lowest bidder, and allocating output on the basis of the bids. Implementation in the variable quantity case takes the form of the buyer announcing a payment-quantity schedule, firms choosing output levels on this schedule, and the buyer selecting the firm bidding the highest output as the winner.

In this context, it is interesting to note that the analysis would remain unchanged if we assumed that the buyer is procuring a *fixed* quantity of an object, but is also concerned about some feature like the speed of completion, blueprints, or other performance features. Let us call this characteristic "quality" (a la Mussa and Rosen, 1978). So long as total and marginal production costs are increasing in quality, optimal procurement consists of the buyer announcing a payment-quality schedule. *Firms* decide what quality they want to produce, and the firm choosing the highest quality is selected. Thus, optimal procurement does not involve any preannounced quality specification by the buyer.

Finally, we discussed dual or multiple sourcing, in which two or more firms are simultaneously awarded the contract. We noted that optimal procurement involved allocating output across firms on the basis of equal "adjusted" marginal cost — unless additional fixed costs wiped out the cost-reducing advantages of allocating output to additional firms. Implementation is quite complex in general, but at least for the two winners case, simple implementation mechanisms exist.

FOOTNOTES

1. For example Sherman (1981, p. 175) states that "Most government procurement seeks performance of work rather than acquisition of a preexisting item. Procurement may solicit the production of a material end item or the rendering of services." See Agapos (????) on defense procurement.

2. *Life-cycle costs* refer to the costs which will be incurred by the *buyer* during the life of the product. *Production designs* reflect anticipated production costs for the seller. These estimates allow the buyer to determine the costs of future variations in quantity of output.

3. See for example Holt (1980), and McAfee and McMillan (1985b). The sale of indivisible objects is considered in Myerson (1981), Riley and Samuelson (1981) and Maskin and Riley (1984a). See McAfee and McMillan (1985f) for a review and reference to the vast auction literature.

4. See for instance Harris and Raviv (1979), Myerson (1981), Riley and Samuelson (1981).

5. Other "variable quantity" auction models exist. Wilson (1979) considers a seller offering shares of a good. Harris and Raviv (1981) examine the case where each buyer wishes to buy at most one unit of the commodity, and the seller produces under conditions of unlimited capacity and constant returns to scale. Riordan and Sappington (1985) models the problem of the award of a monopoly franchise. Of these, the Riordan and Sappington paper comes closest to our work. The optimal quantity schedule we discuss in section 5 is similar to the output schedule in their two stage bidding-production problem.

6. The assumption of IRS is made throughout in Riordan and Sappington (1986, 1987).

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APPENDIX A1

Here, using methods similar to Riley and Samuelson (1981), we derive expressions for the bid function, expected profits of firms and the buyer's expected net gain in the fixed-quantity auction.

If firm i, whose cost parameter is θ^i, bids $b(x)$, where $b(\,)$ is the Nash-equilibrium bid function with $b'(\,) > 0$, then expected profits of firm i is given by

$$\tilde{\pi}(x,\theta^i) = R\big(b(x)\big) - C(Q,\theta^i) \times \text{Prob}[b(x) < b(\theta^j)\ \forall\, j \neq i]$$

Here, $R(\,)$ denotes the expected payment function. In case of firm i, the expectation is over $\theta^1,\ldots,\theta^{i-1},\ldots,\theta^{i+1},\ldots,\theta^n$.

Since $b(x)$ is increasing, we can write

$$\tilde{\pi}(x,\theta^i) = R\big(b(x)\big) - C(Q,\theta^i)\big(1 - F(x)\big)^{n-1}$$

Incentive compatibility requires

$$\tilde{\pi}(\theta,\theta) - \tilde{\pi}(\theta,\theta') \geqq \tilde{\pi}(\theta,\theta) - \tilde{\pi}(\theta',\theta') \geqq \tilde{\pi}(\theta',\theta) - \tilde{\pi}(\theta',\theta')\ \forall\ \theta,\theta' \in [\underline{\theta},\bar{\theta}]$$

or

$$\begin{aligned} &C(Q,\theta')\big(1-F(\theta)\big)^{n-1} - C(Q,\theta)\big(1-F(\theta)\big)^{n-1} \\ &\geqq \pi(\theta,\theta) - \tilde{\pi}(\theta',\theta') \\ &\geqq C(Q,\theta')\big(1-F(\theta')\big)^{n-1} - C(Q,\theta)\big(1-F(\theta;)\big)^{n-1} \end{aligned}$$

Dividing by $\theta - \theta'$ and taking limits as $\theta \to \theta'$, we get

$$\tilde{\pi}'(\theta) = -C_\theta(Q,\theta)\big(1-F(\theta)\big)^{n-1}$$

Thus, $\tilde{\pi}(\theta^i) = \tilde{\pi}(\theta^*) + \int_{\theta_i}^{\theta^*} C_\theta(Q,\theta)\big(1-F(\theta)\big)^{n-1} d\theta$. Setting $\tilde{\pi}(\theta^*) = 0$, we get

$$\tilde{\pi}(\theta^i) = \int_{\theta^i}^{\theta^*} C_\theta(Q,\theta)\big(1-F(\theta)\big)^{n-1} d\theta.$$

Now,

$$
\begin{aligned}
R\big(b(\theta^i)\big) &= \tilde{\pi}(\theta^i) + C(Q,\theta^i)\big(1 - F(\theta^i)\big)^{n-1} \\
&= \int_{\theta^i}^{\theta^*} C_\theta(Q,\theta)\big(1 - F(\theta)\big)^{n-1} d\theta + C(Q,\theta^i)\big(1 - F(\theta_i)\big)^{n-1}
\end{aligned}
$$

However, we also have

$$
\begin{aligned}
R\big(b(\theta^i)\big) &= b(\theta^i) \times Q \times \text{Prob}[b(\theta^i) \text{ is the lowest bid}] \\
&= b(\theta^i) \times Q \times \big(1 - F(\theta^i)\big)^{n-1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
& b(\theta^i) \times Q \times \big(1 - F(\theta^i)\big)^{n-1} \\
&= \int_{\theta^i}^{\theta^*} C_\theta(Q,\theta)\big(1 - F(\theta)\big)^{n-1} d\theta + C(Q,\theta^i)\big(1 - F(\theta^i)\big)^{n-1}
\end{aligned}
$$

Thus

$$
b(\theta^i) = \frac{\int_{\theta^i}^{\theta^*} C_\theta(Q,\theta)\big(1 - F(\theta)\big)^{n-1} d\theta}{Q\big(1 - F(\theta)\big)^{n-1}} + \frac{C(Q,\theta^i)}{Q} \tag{A1.1}
$$

We now derive expressions for expected buyer payment and expected buyer gain. Expected buyer payment to firm i is given by

$$
\int_{\bar{\theta}}^{\theta^*} \big[C(Q,\tilde{\theta})\big(1 - F(\tilde{\theta})\big)^{n-1} + \int_{\underline{\theta}}^{\theta^*} C_\theta(Q,\theta)\big(1 - F(\theta)\big)^{n-1} d\theta\big] F'(\tilde{\theta}) d\tilde{\theta}
$$

After integration by parts, this reduces to

$$
\int_{\underline{\theta}}^{\theta^*} C(Q,\tilde{\theta})\big(1 - F(\tilde{\theta})\big)^{n-1} F'(\tilde{\theta}) d\tilde{\theta} + \int_{\underline{\theta}}^{\theta^*} F(\tilde{\theta}) C_\theta(Q,\tilde{\theta})\big(1 - F(\tilde{\theta})\big)^{n-1} d\tilde{\theta}
$$

Given equal treatment of all n firms, expected buyer payment is given by

$$
n \int_{\underline{\theta}}^{\theta^*} \left\{ C(Q,\theta) + C_\theta(Q,\theta) \frac{F(\theta)}{F'(\theta)} \right\} \big(1 - F(\theta)\big)^{n-1} F'(\theta) d\theta \tag{A1.2}
$$

Expected net buyer gain is then

$$
g(Q,\theta^*) = V(Q)\Big(1 - \big(1 - F(\theta^*)\big)^n\Big)
$$

$$
- n \int_{\underline{\theta}}^{\theta^*} \left\{ C(Q,\theta) + C_\theta(Q,\theta) \frac{F(\theta)}{F'(\theta)} \right\} \big(1 - F(\theta)\big)^{n-1} F'(\theta) d\theta
$$

APPENDIX A2

CLAIM 1: *The optimal level of Q in the maximization of (5) lies in some open interval $(0, Q_1)$, where $Q_1 < \infty$, and the optimal level of θ^* lies in the interval $(\underline{\theta}, \bar{\theta}]$ if the following additional assumption is made:*

ASSUMPTION (A2.1): $\exists\, \hat{\theta} \in [\underline{\theta}, \bar{\theta}]$ *and* $Q > 0$ *such that*

$$V(Q) \geqq C(Q, \hat{\theta}) + C_\theta(Q, \hat{\theta}) \frac{F(\hat{\theta})}{F'(\hat{\theta})}$$

PROOF: Given our regularity assumptions, it follows immediatley from (A2.1) that the expected buyer gain $g(Q, \hat{\theta})$ is higher than $g(0, \theta^*)$ or $g(Q, \underline{\theta})$. Thus Q_{opt} is bounded away from zero, and θ^* is bounded away from $\underline{\theta}$.

Moreover,

$$g_Q(Q, \theta^*) = \left[V'(Q) - \frac{\int_{\underline{\theta}}^{\theta^*} (1 - F(\theta))^{n-1} \left\{ C_Q(Q, \theta) + C_Q\theta(Q, \theta) \frac{F(\theta)}{F'(\theta)} \right\} F'(\theta) d\theta}{\int_{\underline{\theta}}^{\theta^*} (1 - F(\theta))^{n-1} F'(\theta) d\theta} \right]$$

$$\times \left[1 - (1 - F(\theta^*))^n \right]$$

Let Q_1 be defined by

$$V'(Q_1) = C_Q(Q_1, \underline{\theta}).$$

Such a Q_1 exists given (A2.1). Since Q_{opt} is bounded away from zero, we must have

$$g_Q(Q_{opt}, \theta^*) \geqq 0 \Longrightarrow V'(Q_{opt}) > C_Q(Q_{opt}, \underline{\theta}).$$

Given that $V'' < 0$, $C_{QQ} > 0$, it follows that

$$Q_{opt} < Q_1$$

CLAIM 2: *Let*

$$B(Q) = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \left\{ C_Q(Q, \theta) + C_{Q\theta}(Q, \theta) \frac{F(\theta)}{F'(\theta)} \right\} (1 - F(\theta))^{n-1} F'(\theta) d\theta}{\int_{\underline{\theta}}^{\bar{\theta}} (1 - F(\theta))^{n-1} F'(\theta) d\theta}$$

Moreover, let $\hat{Q}$ be defined by

$$V'(\hat{Q}) = B'(\hat{Q})$$

and $\hat{\hat{Q}}$ be defined by

$$V(\hat{\hat{Q}}) = C(\hat{\hat{Q}},\bar{\theta}) + C_\theta(\hat{\hat{Q}},\bar{\theta})$$

If

$$\hat{Q} > \hat{\hat{Q}}, \quad then \quad \theta^*_{opt} < \bar{\theta}$$

PROOF: $\hat{Q}$ is the optimum level of output for $\theta = \bar{\theta}$. However, since

$$g_{\theta^*}(Q,\theta^*) = n\big(1 - F(\theta^*)\big)^{n-1} F'(\theta^*)\left[V(Q) - C(Q,\theta^*) - C_\theta(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}\right]$$

it follows from the above conditions that for ϵ sufficiently small, $g_{\theta^*}(Q,\bar{\theta}-\epsilon) < 0$ where Q is optimally chosen, given $\theta^* = \bar{\theta} - \epsilon$. This shows that θ^*_{opt} must be bounded away from $\bar{\theta}$.

Q.E.D.

APPENDIX A3

Proof of Proposition 2.

Let us write the cost function as

$$C(Q,\theta) = K + \tilde{C}(Q,\theta)$$

where $\tilde{C}(Q,\theta) = \int_0^Q \tilde{c}(Q,\theta)d\tilde{Q}$. From the first-order condition (8) we get

$$V''(Q)\frac{dQ}{dK} = \frac{\left(1-F(\theta^*)\right)^{n-1}F'(\theta^*)}{\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)d\theta}\Big[C_Q(Q,\theta^*) + C_{Q\theta}(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)} - V'(Q)\Big]\frac{d\theta^*}{dK}$$

Again, from (7), we get

$$\frac{dQ}{dK}\left[V'(Q) - C_Q(Q,\theta^*) - C_{Q\theta}(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}\right]$$
$$= 1 + \frac{d\theta^*}{dK}\left[C_\theta(Q,\theta^*) + C_\theta(Q,\theta^*)\frac{d}{d\theta}\left(\frac{F(\theta)}{F'(\theta)}\right)\Big|_{\theta=\theta^*} + C_{\theta\theta}(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}\right]$$

Substituting for $\frac{dQ}{dK}$, we get

$$\left[\frac{1}{(-V''(Q))}\frac{\left(1-F(\theta^*)\right)^{n-1})}{\int_{\underline{\theta}}^{\theta^*}\left(1-F(\theta)\right)^{n-1}F'(\theta)d\theta}\left\{V'(Q) - C_Q(Q,\theta^*) - C_{Q\theta}(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}\right\}^2\right.$$
$$\left. - \left\{C_\theta(Q,\theta^*) + C_\theta(Q,\theta^*)\frac{d}{dQ}\left(\frac{F(\theta)}{F'(\theta)}\right)\Big|_{\theta=\theta^*} + C_{\theta\theta}(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}\right\}\right]\frac{d\theta^*}{dK} = 1$$

Thus, $\frac{d\theta^*}{dK}$ cannot be signed in general because both terms within the bracket are positive. However, for θ^* sufficiently close to $\bar{\theta}$, the first term is very small, and $\frac{d\theta^*}{dK} < 0$.

Proof of Proposition 6.

We have

$$V\big(Q(\theta^{**})\big) = C\big(Q(\theta^{**}),\theta^{**}\big) + C_\theta\big(Q(\theta^{**}),\theta^{**}\big)\frac{F(\theta^{**})}{F'(\theta^{**})} \tag{13}$$

From (14) we get

$$V'(Q(\theta^*)) = C_Q(Q(\theta^*),\theta^*) + C_{Q\theta}(Q(\theta^*),\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}$$

However, from (9), it follows that the optimal fixed-quantity output satisfies $V'(Q_{opt}) < C_Q(Q_{opt'}\theta^* + C_Q(Q_{opt'}\theta^*)F(\theta^*)/F'(\theta^*)$. Hence, given our assumptions, $Q_{opt} > Q(\theta^*)$. However, Q_{opt} also satisfies (7). Hence

$$V(Q(\theta^*)) > C(Q(\theta^*),\theta^*) + C_\theta(Q(\theta^*),\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}, \tag{A3.1}$$

Since $V'(Q) \leqq C_Q(Q,\theta^*) + C_{Q\theta}(Q,\theta^*)\frac{F(\theta^*)}{F'(\theta^*)}$ for $Q(\theta^*) \leqq Q \leqq Q_{opt}$. Comparing (13) and (A3.1), and making use of the fact that $\frac{d}{d\theta}[V(Q(\theta)) - C(Q(\theta),\theta) - C_Q(Q,\theta)\frac{F(\theta)}{F'(\epsilon)}] < 0$ it follows that $\theta^{**} > \theta^*$. Q.E.D.

Proof of Proposition 11.

We can write (10) as

$$V(Q(\theta^{**})) = K + \tilde{C}(Q(\theta^{**}),\theta^{**}) + \tilde{C}_\theta(Q(\theta^{**}),\theta^{**})\frac{F(\theta^{**})}{F'(\theta^{**})}$$

as in the proof of proposition 3. After differentiating with respect to K and simplication, we get

$$\frac{d\theta^{**}}{dK}\Bigg[\tilde{C}_\theta(Q(\theta^{**}),\theta^{**}) + \tilde{C}_\theta(Q(\theta^{**},\theta^{**})\frac{d}{d\theta}\left(\frac{F(\theta)}{F'(\theta)}\right)\Big|_{\theta=\theta^{**}}$$

$$+ C_{\theta\theta}(Q(\theta^{**}),\theta^{**})\frac{F(\theta^{**})}{F'(\theta^{**})}\Bigg] = -1$$

Hence $\frac{d\theta^{**}}{dK} < 0$. Q.E.D.

APPENDIX A4

We shall first derive the expression for the expected buyer gain in the variable quantity case in the context of a direct revelation game in which each firm is assumed to submit to the buyer a report of its cost parameter θ. In the single winner case, the lowest reported θ is assigned an output $Q(\theta)$, where $Q(\,)$ is preannounced by the buyer. Firms whose reports are higher produce nothing.

LEMMA 1: *Let θ^{**} be the cost parameter of the marginal firm. Firms whose cost parameters lie in the interval $[\underline{\theta}, \theta^{**}]$ will report their parameters truthfully only if the following hold*

(i) Expected payments to all firms with $\theta_i \leqq \theta^{**}$ from a truthful report is given by

$$\mathop{E}_{\theta_{-i}} B(\theta_i) = \big(1 - F(\theta_i)\big)^{n-1} C\big(Q(\theta_i), \theta_i\big) + \int_{\theta_i}^{\theta^{**}} \big(1 - F(\theta)\big)^{n-1} C_\theta\big(Q(\theta), \theta\big)\, d\theta$$

where $Q(\theta)$ is the announced output schedule that allocates $Q(\theta)$ to the lowest report θ.

(ii) For any $\theta, \theta' \in [\underline{\theta}, \theta^{**}]$,

$$\big(1 - F(\theta)\big)^{n-1} \int_{\theta}^{\theta'} C_\theta\big(Q(\theta), \tilde{\theta}\big) d\tilde{\theta} \geqq \big(1 - F(\theta')\big)^{n-1} \int_{\theta}^{\theta'} C_\theta\big(Q(\theta'), \tilde{\theta}\big) d\tilde{\theta}$$

A sufficient condition for (ii) is

(iii) $\quad Q'(\theta) \leqq 0 \;$ *for* $\; \theta \in [\underline{\theta}, \theta^{**}]$.

PROOF: Expected profits of a firm with cost parameter θ_i which reports x is given by

$$\pi(x, \theta_i) = \mathop{E}_{\theta_{-i}} B(x) - C\big(Q(x), \theta_i\big)\big(1 - F(x)\big)^{n-1}$$

Incentive compatibility requires, for $\theta, \theta' \in [\underline{\theta}, \theta^{**}]$,

$$\pi(\theta, \theta) - \pi(\theta, \theta') \geqq \pi(\theta, \theta) - \pi(\theta', \theta') \geqq \pi(\theta', \theta) - \pi(\theta', \theta')$$

i.e.,

$$\big(1 - F(\theta)\big)^{n-1} \left[C\big(Q(\theta), \theta'\big) - C\big(Q(\theta), \theta\big)\right] \geqq \pi(\theta, \theta) - \pi(\theta', \theta')$$

$$\geqq \big(1 - F(\theta')\big)^{n-1} \left[C\big(Q(\theta'), \theta'\big) - C\big(Q(\theta'), \theta\big)\right]$$

i.e.,

$$\left(1-F(\theta)\right)^{n-1}\int_{\theta}^{\theta'} C_\theta\left(Q(\theta),\tilde{\theta}\right)d\tilde{\theta} \geqq \pi_i(\theta,\theta)-\pi_i(\theta',\theta')$$

$$\geqq \left(1-F(\theta')\right)^{n-1}\int_{\theta}^{\theta'} C_\theta\left(Q(\theta'),\tilde{\theta}\right)d\tilde{\theta} \qquad \text{(A4.1)}$$

which establishes necessary condition (ii). It is also clear that a sufficient condition for (ii) to hold is that $Q(\theta)$ be nonincreasing in θ. This establishes (iii).

Moreover, defining $\pi(\theta)=\pi(\theta,\theta)$, dividing (A4.1) throughout by $\theta-\theta'$ and taking limits as $\theta'\to\theta$, we get

$$\frac{d\pi_i(\theta)}{d\theta} = -\left(1-F(\theta)\right)^{n-1}C_\theta\left(Q(\theta),\theta\right).$$

Hence, using the fact that $\pi(\theta^{**})=0$, we get

$$\pi(\theta_i)=\int_{\theta_i}^{\theta^{**}} \left(1-F(\theta)\right)^{n-1}C_\theta\left(Q(\theta),\theta\right)d\theta$$

Thus,

$$\begin{aligned}\underset{\theta_{-i}}{E}\, B(\theta_i) &= \pi(\theta_i)+C\left(Q(\theta_i),\theta_i\right)\left(1-F(\theta_i)\right)^{n-1}\\ &= \int_{\theta_i}^{\theta^{**}} \left(1-F(\theta)\right)^{n-1}C_\theta\left(Q(\theta),\theta\right)d\theta + C\left(Q(\theta_i),\theta_i\right)\left(1-F(\theta_i)\right)^{n-1}\end{aligned}$$

This establishes necessary condition (i). Q.E.D.

From the point of view of the buyer, θ_i is a random variable. Hence, expected buyer payment to firm i is given by

$$P_i=\int_{\underline{\theta}}^{\theta^{**}}\left[\int_{\theta_i}^{\theta^{**}} \left(1-F(\theta)\right)^{n-1}C_\theta\left(Q(\theta),\theta\right)d\theta + C\left(Q(\theta_i),\theta_i\right)\left(1-F(\theta_i)\right)^{n-1}\right]F'(\theta_i)d\theta_i$$

After integration by parts, this reduces to

$$P_i=\int_{\underline{\theta}}^{\theta^{**}} \left(1-F(\theta)\right)^{n-1}F'(\theta)\left[C\left(Q(\theta),\theta\right)+C_\theta\left(Q(\theta),\theta\right)\frac{F(\theta)}{F'(\theta)}\right]d\theta.$$

Given equal treatment of n firms, the expected buyer payment is given by

$$R = n \int_{\underline{\theta}}^{\theta^{**}} \left(1 - F(\theta)\right)^{n-1} F'(\theta) \left[C\left(Q(\theta), \theta\right) + C_\theta\left(Q(\theta), \theta\right) \frac{F(\theta)}{F'(\theta)} \right] d\theta.$$

Hence, the expected buyer gain is given by

$$G(Q(\;), \theta^{**}) = n \int_{\underline{\theta}}^{\theta^{**}} \Big\{ V\left(Q(\theta)\right) - C\left(Q(\theta), \theta\right)$$

$$- C_\theta\left(Q(\theta), \theta\right) \frac{F(\theta)}{F'(\theta)} \Big\} \left(1 - F(\theta)\right)^{n-1} F'(\theta) d\theta$$

where we have made use of the fact that the density of the lowest θ is given by $n\left(1 - F(\theta)\right)^{n-1} F'(\theta)$.

LEMMA 2: *Let $Q(\theta)$ be an output allocation rule and $E_{\theta_{-i}} B(\theta_i)$ the expected payment function in the one winner case. For truthful revelation, it is sufficient that*

(i) $E_{\theta_{-i}} B(\theta_i) = C\left(Q(\theta_i), \theta_i\right) \left(1 - F(\theta_i)\right)^{n-1} + \int_{\theta_i}^{\theta^{**}} \left(1 - F(\theta)\right)^{n-1} C_\theta\left(Q(\theta), \theta\right) d\theta$

and

(ii) $Q'(\theta) \leqq 0 \quad \forall \;\; \theta \in [\underline{\theta}, \theta^{**}]$.

PROOF: Expected profits from a report of x by firm θ_i is given by

$$\pi(x, \theta_i) = \underset{\theta_{-i}}{E} B(x) - C\left(Q(x), \theta_i\right) \left(1 - F(x)\right)^{n-1}$$

$$= C\left(Q(x), x\right) \left(1 - F(x)\right)^{n-1} - C\left(Q(x), \theta_i\right) \left(1 - F(x)\right)^{n-1}$$

$$+ \int_x^{\theta^{**}} \left(1 - F(\theta)\right)^{n-1} C_\theta\left(Q(\theta), \theta\right) d\theta$$

$$\therefore \quad \frac{\partial \pi(x, \theta_i)}{\partial x} = \left[C_Q\left(Q(x), x\right) - C_Q\left(Q(x), \theta_i\right) \right] Q'(x) \left(1 - F(x)\right)^{n-1}$$

$$- (n-1)\left(1 - F(x)\right)^{n-2} F'(x) \left[C\left(Q(x), x\right) - C\left(Q(x), \theta_i\right) \right]$$

$$\text{Given (ii), this is} \quad \begin{cases} > 0 & \text{if} \quad x < \theta_i \\ = 0 & \text{if} \quad x = \theta_i \\ < 0 & \text{if} \quad x > \theta_i \end{cases}$$

Hence, a maximum is attained at $x = \theta_i$. Therefore, truthtelling is optimal. Q.E.D.

APPENDIX A5

We prove below that in the many winner case, $q_i^*(\theta)$ is nonincreasing in θ_i given the regularity condition that $\frac{F(\theta_i)}{F'(\theta_i)}$ is increasing in θ_i.

PROOF: Consider the following first order conditions implied by (6.6).

$$V'(\sum q_j^*) = \frac{F(\theta_j)}{F'(\theta_j)} C_{q\theta}(q_j^*, \theta_j) + C_q(q_j^*, \theta_j) \ \forall\, j \text{ such that } q_j > 0 \qquad \text{(A5.1)}$$

(a) Let us consider first the case in which $q_i^* > 0$. If θ_i is raised, with θ_{-i} remaining unchanged, then the following possibilities arise:

(i) Suppose $\sum q_j$ increases or remains unchanged. Then either the ith firm produces zero output, or given our assumptions, it follows from (A5.1) that it produces a lower positive level of output. In either case, $q_i^*(\theta)$ is lower.

(ii) Suppose $\sum q_j$ decreases. Then from (A5.1) it follows that every firm other than the ith that continues to produce a positive level of output must now produce a lower level of output. Thus either the output produced by the ith firm is lower, or there must be fewer firms producing positive levels of output. However, the latter is not possbile. Consider, for instance, the highest cost firm among the firms that were producing a positive level of output before θ_i increased. Let this firm be θ_h. Let us denote output levels in the "old" configuration of costs by q_j, and those in the "new" configuration by q_j'. For the buyer to buy any output at all from θ_h in the old configuration of costs, it must be the case that

$$V\left(\sum_{j\neq h} q_j + q_h\right) - V\left(\sum_{j\neq h} q_j\right)$$

$$> \int_0^{q_h} \left[C_{\theta q}(\tilde{q}, \theta_h) \frac{F(\theta_h}{F'(\theta_n h} + C_q(\tilde{q}, \theta_n)\right] d\tilde{q} + K$$

Hence, in the new configuration, if the optimal q_h is zero, we would have

$$V\left(\sum_{j\neq h} q'_j + q_h\right) - V\left(\sum_{j\neq h} q'_j\right)$$

$$> V\left(\sum_{j\neq h} q_j + q_h\right) - V\left(\sum_{j\neq h} q_j\right)$$

$$> \int_0^{q_h} \left[C_{\theta q}(\tilde{q}, \theta_h)\frac{F(\theta_h)}{F'(\theta_h)} + C_q(\tilde{q}, \theta_h)\right] d\tilde{q} + K \qquad \text{(A5.2)}$$

by the concavity of $V(\ )$, since $\sum_{j\neq h} q'_j < \sum_{j\neq h} q_j$. But if (A5.2) holds, q'_h cannot be zero, since producing q_h is better.

Since the output of the highest cost firm that was producing a positive level of output cannot be reduced to zero, neither can that of any other firm. Consequently, the number of firms producing positive levels of output cannot decrease.

(b) This is the case in which $q^*_i = 0$ initially. It is clear from an argument similar to that in (ii) above that q'_i would be positive only if $\sum q_j$ is lower. This implies that at least one of the firms must drop out. But this is not possible since the firms that were initially producing positive levels of output (when $q_i = 0$) are all lower cost firms.

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