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UNIVERSITY OF SOUTHERN CALIFORNIA
UNIVERSITY PARK
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INTERQUANTILE DIFFERENCES, INCOME INEQUALITY MEASUREMENT AND THE GINI CONCENTRATION INDEX

Z.M. BERREBI*

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MRG WORKING PAPER #M8606

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ABSTRACT

This paper studies the properties of a new income inequality index, I_s, which is equal to the sum of the squares of the symmetric interquantile share differences. I_s verifies most of the desired properties of an income inequality index. It will also be shown that it is closely related to Gini's concentration index. In a section at the end of the paper, an illustration is presented using data on the income distribution in various countries in 1968.

*Department of Economics, Bar Ilan University, Ramat Gan, ISRAEL, 50200.

**Department of Economics, The University of Southern California, Los Angeles; on leave from Bar Ilan University, ISRAEL.

INTRODUCTION

Most of the various indices which have been proposed to measure income inequality (see Atkinson (1970) or Sen (1973) for a review of the various indices) do not take into account the ranking of the various individuals in the income distribution. This is however not true for one well known and widely used index, the Gini index of concentration.

In this paper we propose a new income inequality index, I_s, which also takes into account the ranking of the individuals in the income distribution. I_s is defined as the sum of the squares of the symmetric interquantile share differences and it is shown in the first section that I_s verifies all of the properties of an income inequality index. In the second section we indicate that the Gini concentration index is closely related to I_s whereas in the third section an illustration is presented using data on the distribution of household incomes in various countries in 1968.

I. DEFINITION AND PROPERTIES OF I_s

Let us rank the incomes so that $y_1 \geq y_2 \geq \ldots \geq y_i \geq \ldots \geq y_n \geq 0$.

DEFINITION OF I_s: If s_i is equal to the share $(y_i/n\bar{y})$ of individual i in the total income of the population ($\bar{y}$ is the average income), then I_s is defined as

$$I_s = \sum_{i=1}^{(n-1)/2} (s_i - s_{n-i+1})^2 \quad \text{when } n \text{ is odd} \tag{1}$$

$$I_s = \sum_{i=1}^{n/2} (s_i - s_{n-i+1})^2 \quad \text{when } n \text{ is even.[1]} \tag{2}$$

PROPERTIES OF I_s:

1) It is easily verifed that when all incomes are equal, $I_s = 0$ whereas if one individual earns the whole income, $I_s = 1$.

2) I_s, being a function of income shares only, is obviously invariant to proportional income changes.

3) The value of I_s decreases when all incomes increase by the same amount a.

PROOF: The new value I'_s of I_s is then given by

$$I'_s = \sum_{i=1}^{n/2} \left[\frac{(y_i + a) - (y_{n-i+1} + a)}{\sum_{i=1}^{n/2} (y_i + a)} \right]^2$$

$$\leftrightarrow I'_s = \sum_{i=1}^{n/2} \frac{(y_i - y_{n-i+1})^2}{(\sum_{i=1}^{n/2} y_i + na)^2} < I_s = \sum_{i=1}^{n/2} \frac{(y_i - y_{n-i+1})^2}{(\sum_{i=1}^{n/2} y_i)^2} \tag{3}$$

4) The value of I_s decreases when income is transferred from a rich to a poor individual.

PROOF: Let us assume that a sum is transferred from individual i to individual $j (y_i > y_j)$. The new incomes z_i and z_j of individuals i and j are therefore defined as $z_i = y_i - a$ and $z_j = y_j + a$ whereas $z_k = y_k \ \forall\, k \neq i, j$.

Since total income did not change, the new shares w_i and w_j are equal to $w_i = (y_i - a)/n\bar{y}$ and $w_j = (y_i + a)/n\bar{y}$.

Different possibilities may arise:

*y_i and y_j are both smaller or greater than the median income. (e.g., $i < j < (n+1)/2$):

If I'_s is the new value of the income inequality index, we can write:

$$I'_s = (w_i - s_{n-i+1})^2 + (w_j - s_{n-j+1})^2 + \sum_{k \neq i,j}^{n/2} (s_k - s_{n-k+1})^2$$

$$\leftrightarrow I'_s = \frac{\sum_{k \neq i,j}^{n/2} (y_k - y_{n-k+1})^2 + (y_i - a - y_{n-i+1})^2 + (y_j + a - y_{n-j+1})^2}{(n\bar{y})^2}$$

$$\leftrightarrow I'_s = I_s + \frac{2a^2 - 2a[(y_i - y_j) + (y_{n-j+1} - y_{n-i+1})]}{(n\bar{y})^2} \tag{4}$$

By assumption, the transfer of a sum a did not affect the relative ranking of individuals i and j, that is $a < (y_i - y_j)/2$. Since $y_{n-j+1} > y_{n-i+1}$, it is clear that the expressions within brackets in (4) is greater than 2a so that $I'_s < I_s$.

*the median income is included in the interval $\{y_i, y_j\}$. (e.g., $i < (n+1)/2$ and $j > (n+1)/2$) with $j \neq n-i+1$:

The proof is the same as in the previous section since

$$y_i - y_j > 2a \quad \text{and} \quad y_{n-j+1} > y_{n-i+l}$$

*$j = n-i+1$. In this case

$$I'_s = \frac{\sum_{k \neq i}^{n/2} (y_k - y_{n-k+1})^2 + (y_i - a - y_j - a)^2}{n\bar{y}}$$

$$\leftrightarrow I'_s = \frac{\sum_{k}^{n/2} (y_k - y_{n-k+1})^2 + 4a^2 - 4a(y_i - y_j)}{n\bar{y}}$$

and therefore $I'_s < I_s$ since $a < (y_i - y_j)$

5) When the income distribution is symmetric, I_s is a linear transformation of the Herfindahl index, $\sum_{i=1}^{n} s_i^2$ (Stigler, 1968).

PROOF: If the distribution is symmetric,

$$s_i - s_{n-i+1} = 2\left(s_i - \frac{1}{n}\right) \forall i \quad \text{and}$$

$$I_s = \sum_{i=1}^{n/2} \left[2\left(s_i - \frac{1}{n}\right)\right]^2 = 2\left[\sum_{i=1}^{n}\left(s_i - \frac{1}{n}\right)^2\right] = 2\left[\sum_{i=1}^{n} s_i^2 - \left(\frac{1}{n}\right)\right] \tag{5}$$

One may also notice that since $var(s_i) = \frac{1}{n}\sum_{i=1}^{n}(s_i - \frac{1}{n})^2$, I_s is equal to (2n) times the variance of the income shares, when the distribution is symmetric.

II. RELATIONS BETWEEN I_s AND THE GINI CONCENTRATION INDEX I_G

Let us define a new index I_N which differs from I_s in (2) insofar as instead of squaring the relative share differences we multiply them by the relative rank difference $[(n - i +$

$1)/n - i/n]$. (This rank difference like the share difference $(s_i - s_{n-i+1})$, is defined as positive). In other words

$$I_N = \sum_{i=1}^{n/2}(s_i - s_{n-i+1})\left[\left(\frac{n-i+1}{n}\right) - \frac{i}{n}\right]$$

$$\leftrightarrow I_N = \frac{1}{n}\sum_{i=1}^{n} s_i(n-2i+1) = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{n\bar{y}}[n-(2i-1)] \qquad (6)$$

so that

$$I_N = 1 - \frac{y_E}{\bar{y}} \quad \text{where } y_E = \sum_{i=1}^{n}\frac{(2i-1)}{n^2}y_i \qquad (7)$$

The expression in (7) is just another expression for the Gini concentration index and was suggested by Donaldson and Weymark (1980) and extended by Berrebi and Silber (1981). The Gini concentration index I_G is therefore equal to the sum of the product of the relative ranks and income shares symmetric differences, a property which shows its similarity to the inequality index I_s. (For some implications of this formulation of the Gini coefficient, see Berrebi and Silber, 1985).

When the income distribution is symmetric it is easily shown that the Gini Index I_G may be written as

$$\frac{2}{n}\sum_{i=1}^{n}\left(s_i - \frac{1}{n}\right)\left(\frac{n+1}{2} - i\right) \qquad (8)$$

that is I_G is then equal to 2 times the covariance between the ranks and income shares.

III. EMPIRICAL ILLUSTRATION

Using Jain's (1975) data on the size distribution of incomes in various countries, we have computed the value of I_s and I_G for the year 1968. The data refer to the distribution of household incomes and were collected at the national level. The results are presented in Table 1 where we indicated also the value of the Gini coefficient obtained by Jain, using the estimated parameters of the Lorenz curve (cf. Kakwani and Podder (1973)) and the value of Theil's entropy index I_T (Theil (1967)). The number in parentheses in Table 1 give the ranking of the various countries, by decreasing inequality, for each of the indices presented.

The following observations may be made. Firstly the value of the Gini coefficient obtained using the algorithm of expression (6) is very similar to that obtained by Jain in his computation. Secondly, the ranking of the various countries according to the value of the inequality index, is practically identical whichever index is used. The lowest level of inequality is that of South Korea and the highest that of Honduras or Mexico (which have very similar inequality indices). Thirdly as Table 2 indicates, there are significant differences in the estimation of relative differences in inequality, the latter being computed as the ratio of the index of a given country to the value taken by the corresponding index for South Korea. For example, inequality in Honduras is twice as high as inequality in South Korea, according to the Gini index, but more than three and a half times as high according to Theil's entropy and four and a half times according to our index I_s. It is therefore impossible to evaluate inter country differences in inequality without referring to a specific index whereas ranking comparisons, as indicated, seemed independent of the index used. Finally, although I_G, I_s and I_T vary always between zero and one, as a rule I_s takes lower values than I_g and even than I_T.

Conclusion. In this study a new index of Inequality, equal to the sum of the squares of the symmetric interquantile differences, has been proposed. It has been shown that such an index has all of the desirable properties an inequality index should have. Moreover it appears that the Gini index of concentration is closely related to the index proposed here. An empirical section based on data on the income distribution of household incomes in various countries in 1968 compares the results which are obtained when the Gini index, Theil's entropy and the index presented here are used.

Table 1

Inequality Indices* of Household Incomes**
in 1968 in Various Countries

Country	G	I_G	I_S	I_T
Australia	.319 (2)	.311 (2)	.062 (2)	.161 (2)
Chile	.507 (8)	.483 (8)	.175 (8)	.406 (8)
Germany (Federal Republic)	.386 (6)	.378 (6)	.093 (6)	.230 (6)
Honduras	.625 (12)	.607 (12)	.277 (12)	.552 (11)
India	.478 (7)	.463 (7)	.149 (7)	.347 (7)
(South) Korea	.305 (1)	.298 (1)	.058 (1)	.149 (1)
Malaysia	.555 (9)	.530 (9)	.210 (9)	.462 (9)
Mexico	.611 (11)	.570 (11)	.268 (11)	.560 (12)
Pakistan	.336 (3)	.327 (3)	.072 (4)	.187 (4)
Turkey	.568 (10)	.546 (10)	.219 (10)	.473 (10)
United Kingdom	.339 (4)	.333 (4)	.070 (3)	.176 (3)
Yugoslavia	.347 (5)	.341 (5)	.074 (5)	.187 (4)

*Four indices are presented in this table:

1) G is the Gini inequality index as computed by Shail Jain (1975), using Kakwani and Podder's (1973) method.
2) I_G is an estimation of the Gini coefficient using the algorithm presented in this paper.
3) I_S is the new inequality index based on symmetric inter-quantile differences, presented in this paper.
4) I_T is Theil's Entropy Index as computed by Jain (1975).

**The data cover household incomes at the national level. The number in parentheses gives the ranking of the various countries, by decreasing inequality, for each index.

Table 2

Relative Differences in Inequality* in 1968,
According to the Various Indices

Country	I_G	I_S	I_T
Australia	1.044	1.069	1.081
Chile	1.621	3.017	2.725
Germany (Federal Republic)	1.268	1.603	1.544
Honduras	2.037	4.776	3.705
India	1.554	2.569	2.329
(South) Korea	1.000	1.000	1.000
Malaysia	1.779	3.621	3.101
Mexico	1.913	4.621	3.758
Pakistan	1.097	1.241	1.255
Turkey	1.832	3.776	3.174
United Kingdom	1.117	1.207	1.181
Yugoslavia	1.144	1.276	1.255

*Each number in this table is equal to the ratio of the corresponding inequality index given in Table 1 to that of South Korea, the country with the lowest inequality index.

NOTES

1. In the following pages it will always be assumed that n is even. It should however be clear that all of the proofs will hold if n is odd.

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