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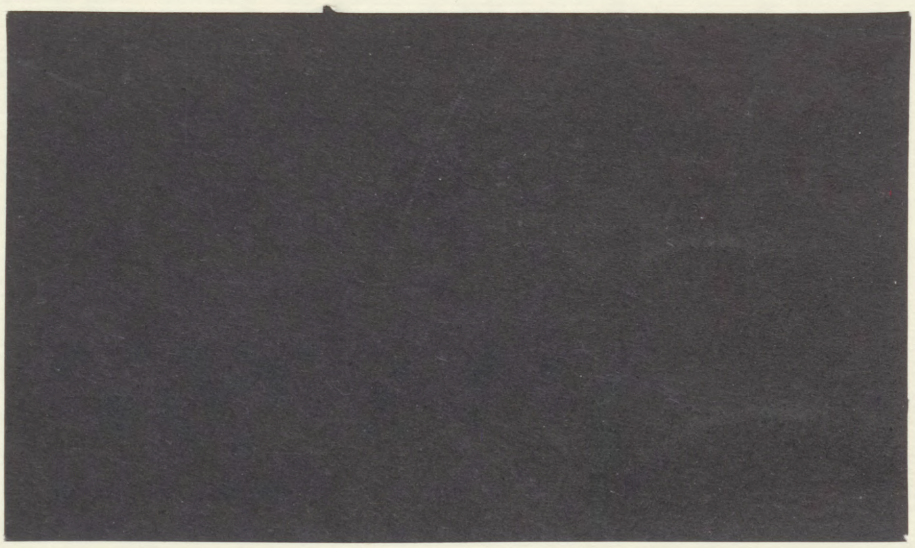

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EQUILIBRIUM WITH INCOMPLETE MARKETS OVER AN INFINITE HORIZON

Michael Magill*
University of Southern California

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Abstract

This paper shows how the general equilibrium model with incomplete markets (GEI) can be extended to an open-ended future, thereby providing a natural setting for analyzing problems in macroeconomics. To obtain a concept of equilibrium which respects the incompleteness of markets and ensures that agents' plans are node consistent, a transversality condition must be imposed on every subtree. Conditions for the existence of an equilibrium as well as certain properties of an equilibrium are derived. The prices of infinite-lived securities in zero net supply are shown to permit speculative bubbles which affect (do not affect) the equilibrium allocation if markets are incomplete (complete). The prices of securities in positive net supply (for example equity contracts) cannot have speculative bubbles, thus limiting the extent of speculation in the GEI model.

1. Introduction

The analysis of equilibrium on a sequence of markets in which agents correctly anticipate future prices was first introduced in an abstract setting by Radner (1972). This model has recently evolved into the model of *general equilibrium with incomplete markets* (GEI for short). The analysis of the GEI model (which is surveyed in Magill-Shafer (1991)) suggests that it may provide a valuable framework for discussing many issues in macroeconomics. To provide such a framework however the model, which has so far been restricted to economies with a finite horizon, needs to be extended to the more natural setting of an open-ended future.

There are two natural ways of extending the analysis of an economy to an infinite horizon. The first is to assume that there are a finite number of agents (families) who are infinitely lived; the second is to assume that all agents are finitely lived and are succeeded by their children in an indefinite sequence of overlapping generations. The models that arise from these two approaches have become the basic workhorses of modern macroeconomics (see Blanchard-Fischer (1989)). In this paper we explore the first type of extension: we study an exchange economy with a finite number of infinitely lived agents who use spot markets for the current exchange of goods and a limited array of financial markets for redistributing their income across time periods and uncertain events. Such a model provides an extension of the representative agent models (for example Lucas (1978)) to an economy with heterogeneous agents in which markets can be incomplete. When the markets are incomplete and agents are heterogeneous many phenomena can arise which cannot occur in the representative agent (complete market) version of the model.

In a model with a sequence of markets over an infinite horizon a new problem arises which has no counterpart in a finite horizon model: if agents are permitted to borrow, they may seek to postpone the payment of their debts by rolling them over indefinitely from one period to the next and if such Ponzi schemes are permitted there is no solution to an agent's decision problem. Broadly speaking two approaches are used in the macroeconomics literature to limit the rate at which agents can accumulate debts: one is to place *a priori* bounds on debts, the other is to require that the present value of debts goes to zero. Since *a priori* bounds may introduce an additional imperfection over and above the incompleteness of the financial markets, we seek a condition based on the latter approach.

Consider an economy in which every agent is rational and impatient. We argue that if every agent knows and recognizes that every other agent is rational and impatient then all agents in the economy will be led to limit the rate of growth of any debt they might plan to finance through the security markets. No impatient agent will allow his lending to grow so fast that it acquires a positive present value at infinity — a natural transversality condition for an impatient agent. If agents recognize that lenders limit the credit that they are willing to extend in this way, then they will not plan to finance a debt which has a negative present value at infinity. When markets are complete (so that agents are not constrained in redistributing their income accross the event-tree) if agents restrict their financial plans so that the present value of their debts from the perspective of the initial node tends to zero, then a consistent concept of equilibrium is obtained. However (as we show in section 4) if markets are incomplete (so that agents are limited in their ability to transfer income across the event-tree) then *the same condition must hold from the perspective of every node in the event-tree, if agents plans are to be node consistent* — a condition akin to subgame perfection for Nash equilibria.

When markets are incomplete there is a potential ambiguity involved in present value calculations, since there are many present value processes which are compatible with the no-arbitrage equations on the financial markets. Optimality of an agent's consumption-portfolio plan requires that the present value of his credit at infinity be *non-positive* when evaluated with the agent's *personal* present value vector. Since in a large economy in which trade is anonymous, agents cannot be expected to have information regarding the present value vectors of other agents, we assume that each agent uses his personal present value vector when limiting the present value of his debt at infinity to be *non-negative.* Thus an agent's budget set is defined by the requirement that purchases on the spot markets be compatible with earnings on the financial markets at every date-event, plus the asymptotic condition that from the perspective of every date-event, the present value of the agent's debt converges to zero as the horizon goes to infinity.

The financial markets are a key element of the GEI model. We consider *real* securities, namely securities whose returns are the value (under current spot prices) of specified bundles of goods. A model with *nominal* securities (i.e. securities whose returns are specified in units of account) in which price levels are not tied down, as studied by Balasko-Cass (1989) and Geanakoplos-Mas-Colell (1989), can always be converted into a family of models with real securities. The model covers securities in *zero* net supply such as bonds or futures contracts and securities in *positive* net supply such as equity contracts of firms and permits both *short* and *long-lived* securities.

To prove existence of equilibrium for the infinite horizon economy we use a method similar to that introduced by Bewley (1972) which consists in taking limits of equilibria of truncated (finite horizon) economies. We take the commodity space to be the space of bounded sequences (ℓ_∞) and the crucial assumption on agents' preferences is that they be continuous in the Mackey topology. As is well-known (see Brown-Lewis (1981) and Mas-Colell-Zame (1991)), this latter assumption is a precise abstract way of formalizing the idea that agents are impatient and it is this assumption (or more precisely a slight strengthening of it to uniform impatience) which permits the equilibria of finite horizon economies to "approximate" the equilibria of an infinite horizon economy.

In section 6 we show that if the securities are short-lived and have payoffs in a numeraire commodity then a GEI equilibrium exists for every infinite horizon economy (Theorem 6.1). As soon as there are securities whose payoffs depend on the spot prices of two or more commodities or as soon as there are long-lived securities an equilibrium may not exist: in this case for the finite horizon GEI model it has been shown that an equilibrium exists generically (Duffie-Shafer (1985, 1986)). We extend this result to the infinite horizon economy as follows: first we show that a pseudoequilibrium always exists (Theorem 7.2), then we show that for a dense set of asset structures a pseudoequilibrium is a GEI equilibrium (Theorem 7.4).

In view of the way equilibria are constructed in Theorem 7.2 as limits of equilibria of truncated economies, in any such equilibrium the price of an infinitely lived security is equal to the present value of its future income stream for every agent —the so-called *fundamental value* of the security. It is natural to enquire whether this property is true for all infinitely lived securities in all GEI equilibria: since no terminal condition can automatically be attached to the system of stochastic difference equations (the no-arbitrage equations for each agent) which must be satisfied by a security price, it is not a priori clear that the price of an infinitely lived security will equal its fundamental value.

The phenomenon of *speculative bubbles* has been the subject of much interest in macroeconomics (see Blanchard-Fischer (1989, chapter 5)). It is sometimes argued that bubbles cannot arise in an economy with a finite number of infinite-lived agents: in section 8 we show that this statement needs to be qualified. As in Tirole (1982) and Santos-Woodford (1992) we find that there cannot be a speculative bubble on infinitely lived securities in *positive* net supply: since equity contracts constitute a significant segment of the capital market, this places a bound on the extent to which the GEI model predicts the occurence of speculation. However the prices of infinitely lived securities in *zero* net supply behave quite differently: they admit substantial amounts of speculation. In Proposition 8.4 we show that a speculative bubble can always be added to the price of such a security without affecting the real equilibrium allocation. There is thus a significant nominal indeterminacy in the prices of infinitely lived assets in zero net supply. But there is a qualitative difference between speculative bubbles that can arise in a GEI equilibrium depending on whether the markets are complete or incomplete. If the markets are complete speculative bubbles only introduce a nominal indeterminacy; if the markets are incomplete a speculative bubble can have a real effect in the sense that the same equilibrium allocation cannot be supported by a system of asset prices without a speculative bubble.

Finally we mention some recent related papers. Hernandez-Santos (1991) study the problem of existence of equilibrium with short-lived nominal securities. Levine (1989) and Levine-Zame (1992) study conditions on a system of debt constraints under which equilibria of truncated economies converge to an equilibrium of an infinite economy. In a framework which is broad enough to also include the overlapping generation model (and is akin to that studied by Levine (1989)) Santos-Woodford (1992) study conditions under which prices of infinitely lived securities in positive net supply may or may not involve a speculative bubble.

2. Characteristics of the Economy

We consider an economy with time and uncertainty over an infinite horizon. Let $\mathbf{T} = \{0, 1, \ldots\}$ denote the set of time periods and let $\mathbf{S}$ be a countable set of states of nature. The revelation of information is described by a sequence of partitions of $\mathbf{S}$

$$\mathbb{F} = (\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_t, \ldots)$$

where the number of subsets in $\mathbb{F}_t$ is finite and $\mathbb{F}_t$ is finer than the partition $\mathbb{F}_{t-1}$ (i.e. $\sigma \in \mathbb{F}_t$, $\sigma' \in \mathbb{F}_{t-1} \implies \sigma \subset \sigma'$ or $\sigma \cap \sigma' = \emptyset$) for all $t \geqq 1$. At date 0 we assume that there is no information so that $\mathbb{F}_0 = \mathbf{S}$. The information available at time t (for $t \in \mathbf{T}$) is assumed to be the same for all agents in the economy (*symmetric information*) and is described by the subset σ of the partition $\mathbb{F}_t$ in which the state of nature lies. A pair $\xi = (t, \sigma)$ with $t \in \mathbf{T}$ and $\sigma \in \mathbb{F}_t$ is called a *date-event* or *node* and $t(\xi) = t$ is the *date* of node ξ. The set $\mathbf{D}$ consisting of all date-events (or nodes) is called the *event-tree* induced by $\mathbb{F}$

$$\mathbf{D} = \bigcup_{\substack{t \in \mathbf{T} \\ \sigma \in \mathbb{F}_t}} (t, \sigma)$$

It is convenient at this point to introduce certain subsets of $\mathbf{D}$ that reappear frequently throughout the paper. The unique node $\xi_0 = (0, \sigma)$ with $\sigma = \mathbf{S}$ is called the *initial* node. ξ' is said to *succeed* ξ (strictly) if $\xi' = (t', \sigma')$, $\xi = (t, \sigma)$ satisfy $t' \geqq t$ $(t' > t)$, $\sigma' \subset \sigma$ and we write $\xi' \geqq \xi$ $(\xi' > \xi)$. For any node $\xi \in \mathbf{D}$ the set of all nodes which succeed ξ is called the *subtree* $\mathbf{D}(\xi)$ starting at ξ,

$$\mathbf{D}(\xi) = \{ \xi' \in \mathbf{D} \mid \xi' \geqq \xi \}$$

$\mathbf{D}^+(\xi)$ denotes the set of all *strict* successors of ξ, $\mathbf{D}^+(\xi) = \{ \xi' \in \mathbf{D} \mid \xi' > \xi \}$, and ξ^+ denotes the set of *immediate successors* of ξ

$$\xi^+ = \{ \xi' \in \mathbf{D}^+(\xi) \mid t(\xi') = t(\xi) + 1 \}$$

Let $\#\xi^+$ denote the number of elements of ξ^+: this number which is denoted by $b(\xi)$ is finite since for each $t \in \mathbf{T}$ the number of subsets in $\mathbb{F}_{t+1}$ is finite and is called the *branching number* of the event-tree $\mathbf{D}$ at ξ, $b(\xi) = \#\xi^+, \xi \in \mathbf{D}$. For any $\xi \in \mathbf{D}$ and any $\tau \in \mathbf{T}$ with $\tau > t(\xi)$, $\mathbf{D}_\tau(\xi)$ denotes the subset of nodes of $\mathbf{D}(\xi)$ at date τ, while $\mathbf{D}^\tau(\xi)$ denotes the subset of nodes between dates $t(\xi)$ and τ

$$\mathbf{D}_\tau(\xi) = \{ \xi' \in \mathbf{D}(\xi) \mid t(\xi') = \tau \}, \ \mathbf{D}^\tau(\xi) = \{ \xi' \in \mathbf{D}(\xi) \mid t(\xi) \leq t(\xi') \leq \tau \}$$

When the subtree originates at the initial node ξ_0 it will simplify the notation if we write

$$\mathbf{D} = \mathbf{D}(\xi_0), \ \mathbf{D}^+ = \mathbf{D}^+(\xi_0), \ \mathbf{D}^\tau = \mathbf{D}^\tau(\xi_0), \ \mathbf{D}_\tau = \mathbf{D}_\tau(\xi_0)$$

For each $\xi \in \mathbf{D}^+$, $\xi = (t, \sigma)$ there is a unique $\sigma' \in \mathbb{F}_{t-1}$ such that $\sigma' \supset \sigma$; the node $\xi^- = (t-1, \sigma')$ is called the *predecessor* of ξ.

The economy consists of a finite collection of infinitely lived consumers (families) $\mathbf{I} = \{1, \ldots, I\}$ who purchase commodities on spot markets and trade securities at every node in the event-tree $\mathbf{D}$ described above. There is a set $\mathbf{L} = \{1, \ldots, L\}$ of commodities at each node: the set consisting of all commodities indexed over the event-tree is thus

$$\mathbf{D} \times \mathbf{L} = \{ (\xi, \ell) \mid \xi \in \mathbf{D}, \ \ell \in \mathbf{L} \}$$

Let $\mathbb{R}^{\mathbf{D}\times\mathbf{L}}$ denote the vector space of all maps $x : \mathbf{D} \times \mathbf{L} \longrightarrow \mathbb{R}$ and let $\ell_\infty(\mathbf{D} \times \mathbf{L})$ denote the subspace of $\mathbb{R}^{\mathbf{D}\times\mathbf{L}}$ consisting of all *bounded* maps (sequences)

$$\ell_\infty(\mathbf{D} \times \mathbf{L}) = \{ x \in \mathbb{R}^{\mathbf{D}\times\mathbf{L}} \mid \sup_{(\xi,\ell)\in\mathbf{D}\times\mathbf{L}} \mid x(\xi,\ell) \mid < \infty \}$$

The norm $\|\cdot\|_\infty$ of $\ell_\infty(\mathbf{D} \times \mathbf{L})$ is defined by $\| x \|_\infty = \sup_{(\xi,\ell)\in\mathbf{D}\times\mathbf{L}} \mid x(\xi,\ell) \mid$. As in Bewley (1972) we take the commodity space to be $\ell_\infty(\mathbf{D} \times \mathbf{L})$. Each agent $i \in \mathbf{I}$ has an *initial endowment* process given by $\omega^i = (\omega^i(\xi,\ell), (\xi,\ell) \in \mathbf{D} \times \mathbf{L})$ which is assumed to lie in the non-negative orthant $\ell_\infty^+(\mathbf{D} \times \mathbf{L})$. Let $\omega^i(\xi) = (\omega^i(\xi,\ell), \ell \in \mathbf{L}) \in \mathbb{R}^L$ denote the agent's endowment of the L goods at node ξ. Agent i chooses a *consumption process* $x^i = (x^i(\xi,\ell), (\xi,\ell) \in \mathbf{D} \times \mathbf{L})$ which must lie in his consumption set $X^i = \ell_\infty^+(\mathbf{D} \times \mathbf{L})$; $x^i(\xi) = (x^i(\xi,\ell), \ell \in \mathbf{L}) \in \mathbb{R}_+^L$ denotes the agent's consumption at node ξ. Note that this description of the commodity space assumes that each good is perfectly divisible and is perishable (no storable or durable goods) and that the supply of goods does not grow without bound. The agent's preference among consumption processes in X^i is expressed by a *preference ordering* $\underset{i}{\succsim}$.

At each date-event there are spot markets on which the L commodities are traded. Let

$$p = (p(\xi,\ell), (\xi,\ell) \in \mathbf{D} \times \mathbf{L}) \in \mathbb{R}^{\mathbf{D}\times\mathbf{L}}$$

denote the *spot price process* and let $p(\xi) = (p(\xi,\ell), \ell \in \mathbf{L})$ denote the vector of spot prices for the L goods at node ξ.

A rich variety of financial assets can be considered in a model of this kind (for a discussion see Magill-Shafer (1991)). We restrict our attention to the class of real securities: a financial asset is said to be a *real* security if its return at each node ξ after its node of issue is the value under the spot prices at node ξ of a specified bundle of the L goods. As noted in the introduction, a model with nominal securities can be converted in a family of models with real securities (see Geanakoplos-Mas-Colell (1989)).

Let $\mathbf{J} \subset \mathbb{N}$ (the positive integers) denote the set of (real) securities. Security $j \in \mathbf{J}$ which is issued at node $\xi(j) \in \mathbf{D}$ is a promise to deliver a dividend process $\{ V(\xi,j) \in \mathbb{R} \mid \xi \in \mathbf{D}^+(\xi(j)) \}$ at all nodes strictly succeeding its node of issue $\xi(j)$ where $V(\xi,j) = p(\xi)A(\xi,j)$ is the value of a prespecified bundle $A(\xi,j) \in \mathbb{R}^L$ of the L commodities under the spot prices $p(\xi)$. Each security is assumed to have a non-trivial dividend stream, that is, it makes a non-zero payment at some node after its node of issue ($A(\xi',j) \neq 0$ for some $\xi' > \xi(j)$ for each $j \in \mathbf{J}$). Contract j is first traded at its node of issue and is then retraded until a maturity node is reached, namely a node beyond which it makes no payment: after such a maturity node the security is no longer traded.

For every node ξ in the subtree $\mathbf{D}(\xi(j))$ prior to a maturity node let $q(\xi,j)$ denote the price of one unit of security j at node ξ after its dividend $V(\xi,j)$ at that node has been paid. Security j with node of issue $\xi = \xi(j)$ is said to be *short-lived* if it only pays dividends at the immediate successors of its node of issue, namely if $A(\xi',j) = 0 \quad \forall\ \xi' \notin \xi^+(j)$. In all other cases security j is said to be *long-lived.* A security whose commodity payoff is exclusively in good 1 ($A(\xi,\ell,j) = 0$ if $\ell \neq 1$) is called a *numeraire* security. If for every $t \geqq t(\xi(j))$ there exists a node ξ with $t(\xi) > t$ such that $A(\xi,j) \neq 0$ then security j is said to be *infinite-lived.*

To summarise the returns (dividends and prices) on the different securities in a compact form it is convenient to extend the definition of the dividend and price process of each security j from its subtree $\mathbf{D}(\xi(j))$ to the whole event-tree $\mathbf{D}$. Thus we define $A(\xi,j) = 0$ for all $\xi \notin \mathbf{D}^+(\xi(j))$. Similarly we set $q(\xi,j) = 0$ after a maturity node of asset j and for all $\xi \notin \mathbf{D}(\xi(j))$, $j \in \mathbf{J}$. Let

$$A(\xi) = (A(\xi,j),\ j \in \mathbf{J}),\ q(\xi) = q(\xi,j),\ j \in \mathbf{J}$$

denote the commodity payoffs and prices of the securities at node ξ.

Definition 2.1: A *security structure* is defined by a triple $(\mathbf{J},\zeta,A)$ where $\mathbf{J}$ is the set of securities, $\zeta = (\xi(j), j \in \mathbf{J})$ is the set of nodes of issue of the securities and $A \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ is the commodity payoff process of the securities which satisfies $A(\xi,j) = 0$ if $\xi \notin \mathbf{D}^+(\xi(j))$.

Let $J(\xi)$ denote the set of active securities at node ξ

$$J(\xi) = \{\, j \in \mathbf{J} \mid \xi \in \mathbf{D}(\xi(j)),\ \exists \xi' \in \mathbf{D}^+(\xi) \text{ with } A(\xi',j) \neq 0 \,\} \tag{2.1}$$

and let $j(\xi) = \#J(\xi)$ denote the number of active securities at this node: we assume $j(\xi) < \infty$ for every $\xi \in \mathbf{D}$. With this assumption even if $\mathbf{J}$ is infinite $A(\xi)$ and $q(\xi)$ have at most a finite number of non-zero components.

Let $z^i(\xi,j) \in \mathbb{R}$ denote the number of units of the j^{th} security purchased (if $z^i(\xi,j) > 0$) or sold (if $z^i(\xi,j) < 0$) by agent i at node ξ: each security is assumed to be perfectly divisible and can be bought and sold in unlimited amounts (no short-sales constraints). If contract j is not active at node ξ then we adopt the convention $z^i(\xi,j) = 0$. Let $z^i = (z^i(\xi,j),(\xi,j) \in \mathbf{D} \times \mathbf{J}) \in \mathbb{R}^{\mathbf{D}\times\mathbf{J}}$

denote the i^{th} agent's portfolio process. z^i is chosen from the *space of portfolios* defined by

$$Z = \{\, z^i \in \mathbb{R}^{\mathbf{D}\times\mathbf{J}} \mid z^i(\xi,j) = 0 \text{ if } j \notin J(\xi) \,\}$$

Let q denote the *security price* process $q = (q(\xi,j),(\xi,j) \in \mathbf{D}\times\mathbf{J})$ and let Q denote the space of security prices

$$Q = \{\, q \in \mathbb{R}^{\mathbf{D}\times\mathbf{J}} \mid q(\xi,j) = 0 \text{ if } j \notin J(\xi) \,\}$$

It is convenient to summarise the characteristics of the economy that we have described above. Let

$$\succsim = (\succsim_1, \ldots, \succsim_I),\ \omega = (\omega^1, \ldots, \omega^I)$$

denote the profile of preferences and endowments of the I agents and let $(\mathbf{J},\zeta,A)$ denote the security structure then $\mathcal{E}_\infty(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta,A))$ denotes the associated economy over the event-tree $\mathbf{D}$.

3. Assumptions

In this section we describe the assumptions that we impose on the characteristics of the economy $\mathcal{E}_\infty(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta,A))$. The crucial assumption required to obtain the existence of equilibrium in an infinite horizon economy is the choice of a topology in which agents' preference orderings are continuous. Let $\ell_1(\mathbf{D}\times\mathbf{L})$ denote the subspace of $\mathbb{R}^{\mathbf{D}\times\mathbf{L}}$ consisting of all *summable* sequences

$$\ell_1(\mathbf{D}\times\mathbf{L}) = \left\{ P \in \mathbb{R}^{\mathbf{D}\times\mathbf{L}} \,\middle|\, \sum_{(\xi,\ell)\in\mathbf{D}\times\mathbf{L}} \mid P(\xi,\ell) \mid < \infty \right\}$$

For $P \in \ell_1(\mathbf{D}\times\mathbf{L})$ and $x \in \ell_\infty(\mathbf{D}\times\mathbf{L})$ the scalar product is defined by

$$Px = \sum_{(\xi,\ell)\in\mathbf{D}\times\mathbf{L}} P(\xi,\ell)x(\xi,\ell)$$

The *Mackey topology* on $\ell_\infty(\mathbf{D}\times\mathbf{L})$ is the strongest locally convex topology such that the dual of $\ell_\infty(\mathbf{D}\times\mathbf{L})$ under this topology is $\ell_1(\mathbf{D}\times\mathbf{L})$. For a discussion of this topology see Bewley (1972) and Mas-Colell-Zame (1991).

A1 (*Event-tree*): For each node $\xi \in \mathbf{D}$ the branching number $b(\xi) = \#\xi^+$ is finite.

A2 (*Endowments*): There exists scalars m, M with $0 < m < M$ such that $\forall\ (\xi,\ell) \in \mathbf{D}\times\mathbf{L}$, $\omega^i(\xi,\ell) > m$, $\forall\ i \in \mathbf{I}$ and $\sum_{i\in\mathbf{I}} \omega^i(\xi,\ell) < M$.

Let $w = \sum_{i \in \mathbf{I}} \omega^i$ then A2 implies $\| w \|_\infty < M$. Thus a feasible consumption process x^i for agent i must lie in the set

$$F = \{ y \in \ell_\infty^+(\mathbf{D} \times \mathbf{L}) \mid \| y \|_\infty < M \}$$

A3 (*Preferences*): For $i \in \mathbf{I}$, $\underset{i}{\succsim}$ is a transitive, reflexive, complete preference ordering on $X^i = \ell_\infty^+(\mathbf{D} \times \mathbf{L})$ and $\underset{i}{\succsim}$ is monotone, strictly convex and continuous in the Mackey topology i.e. for all $\tilde{x}^i \in X^i$, $\{ x^i \in X^i \mid x^i \underset{i}{\succsim} \tilde{x}^i \}$ is strictly convex and closed in the Mackey topology and $\{ x^i \in X^i \mid x^i \underset{i}{\succ} \tilde{x}^i \}$ is open in the Mackey topology. $\underset{i}{\succsim}$ is monotone in the sense that for each $x^i \in X^i$ and for each $y \in \ell_\infty^+(\mathbf{D} \times \mathbf{L})$, $x^i + y \underset{i}{\succsim} x^i$.

Let $E \subset \mathbf{D}$ be a subset of nodes and let χ_E denote the characteristic function of E

$$\chi_E(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in E \\ 0 & \text{if} \quad \xi \notin E \end{cases}$$

For $x \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ define $x\chi_E = (x(\xi, l)\chi_E(\xi),\ (\xi, l) \in \mathbf{D} \times \mathbf{L})$. Let $e_\ell^\xi \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ denote the process which has all components 0 except for the component of good ℓ at node ξ which is 1

$$e_\ell^\xi(\xi', \ell') = \begin{cases} 1 & \text{if} \quad (\xi', \ell') = (\xi, \ell) \\ 0 & \text{if} \quad (\xi', \ell') \neq (\xi, \ell) \end{cases}$$

A4 (*Preferences*): There exists $\beta < 1$ such that for all $i \in \mathbf{I}$

$$x^i \chi_{\mathbf{D} \setminus \mathbf{D}^+(\xi)} + \beta x^i \chi_{\mathbf{D}^+(\xi)} + e_1^\xi \quad \underset{i}{\succ} \quad x^i \quad \forall\, x^i \in F,\ \forall\, \xi \in \mathbf{D},$$

A5 (*Securities*): Every security $j \in \mathbf{J}$ is a real security with commodity payoff $A(\cdot, j) \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ and the number of active securities $j(\xi)$ is finite at each node $\xi \in \mathbf{D}$.

Let $e_\ell \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ denote the commodity process consisting of one unit of good ℓ at each node

$$e_\ell(\xi, \ell') = \left\{ \begin{matrix} 1 & \text{if} & \ell' = \ell \\ 0 & \text{if} & \ell' \neq \ell \end{matrix} \right\} \quad \forall\, \xi \in \mathbf{D}$$

A6 (*Short-lived numeraire bond*): For each $\xi \in \mathbf{D}$ there exists $j_\xi \in J(\xi)$ such that $\xi(j_\xi) = \xi$ and

$$A(\cdot, j_\xi) = e_1 \chi_{\xi^+}$$

Remark: We have repeated A1 for completeness: it is essential that at each node ξ there are only a finite number of immediate successors. Assumption A2 asserts that the aggregate endowment process $w = \sum_{i \in \mathbf{I}} \omega^i$ is bounded above and hence that each individual endowment process ω^i is bounded above: in addition each agent has an endowment of each good which is uniformly positive across all nodes. Good 1 plays the role of a numeraire: by assumption A4 it is strictly desired by all agents at all nodes and thus has a positive price at each node.

Assumption A3 is classical since the paper of Bewley (1972). Mackey continuity of the preference ordering $\succsim_i$ expresses the idea that agent i prefers early to more distant consumption: it is a precise abstract way of formalising Irving Fisher's notion of impatience. Araujo (1985) has shown that all agents must have Mackey continuous preferences if an Arrow-Debreu equilibrium is to exist for a general class of economies. This assumption permits the infinite horizon economy to be approximated by truncated finite horizon economies since consumption in the very distant future is unimportant. The role of this assumption will become clear in section 6 when we establish the existence of an equilibrium for the economy $\mathcal{E}_\infty$ by taking limits of equilibria of truncated economies $\mathcal{E}_T$: roughly speaking impatience is a way of coping with the open-endedness of the future and ending up with an economy which in many respects behaves like a finite horizon economy (but with no pre-specified terminal date).

As Bewley (1972) has shown, $\succsim_i$ is Mackey continuous if it is represented by an additively separable utility function

$$u^i(x^i) = \sum_{\xi \in \mathbf{D}} \rho(\xi) \delta_i^{t(\xi)} v^i(x^i(\xi)) \tag{3.1}$$

where $\rho(\xi)$ is the probability of ξ (induced by a probability measure ρ on the measurable subsets of $\mathbf{S}$), $\delta_i \in (0,1)$ is a discount factor and $v^i : \mathbb{R}^L_+ \longrightarrow \mathbb{R}$ is a continuous, increasing concave function with $v^i(0) = 0$.

The assumption of strict convexity in A3 is needed for the case of general security structures so as to be able to invoke the currently available existence theorems for a finite horizon economy: in these theorems an agent's optimal consumption is assumed to lead to a demand function rather than a demand correspondence. Strict convexity and monotonicity imply strict monotonicity. It is likely that only the conditions of convexity and strict monotonicity with respect to one good are needed to obtain the existence results for a finite horizon economy, but we shall not enter into such refinements here.

Assumption A4 is a uniform condition on the impatience of each agent with respect to future consumption at each node. To understand what it means consider the following thought experiment.

Pick any feasible consumption process $x^i \in F$ and add one unit of commodity 1 at node ξ. If commodity 1 is desired at node ξ then the new consumption process is strictly preferred by agent i ($x^i + e_1^\xi \succ_i x^i$). By the Mackey continuity of $\succsim_i$ there exists $\beta_\xi < 1$ such that agent i still prefers the new consumption process even if it is scaled down by the factor β_ξ for all nodes that strictly succeed ξ i.e.

$$x^i \chi_{\mathbf{D} \setminus \mathbf{D}^+(\xi)} + \beta_\xi x^i \chi_{\mathbf{D}^+(\xi)} + e_1^\xi \succ_i x^i$$

Since F is bounded β_ξ can be chosen to be the same for all $x^i \in F$. The new requirement in A4 over and above Mackey continuity of $\succsim_i$ is that the coefficient β_ξ be uniformly bounded away from 1 when $t(\xi) \longrightarrow \infty$. Agent i thus exhibits *uniform impatience* in that at each node he is prepared to give up a small but positive fraction $(1 - \beta > 0)$ of his future consumption process in exchange for an additional unit of commodity 1 at that node. A4 is satisfied by a preference ordering represented by (3.1). A4 is essentially the only new assumption on preferences and endowments that we add to the assumptions made by Bewley (1972) in order to obtain the existence of an equilibrium in the case of incomplete markets.

A5 and A6 are assumptions on the type of securities available to agents on the financial markets and are thus specific to the GEI model. Since by A1 only a finite amount of uncertainty ($b(\xi) < \infty$) is resolved at each node ξ it seems reasonable to assume that only a finite number ($j(\xi) < \infty$) of securities are available for trading at each node. If $j(\xi) \geqq b(\xi)$, $\forall\ \xi \in \mathbf{D}$ then markets are essentially complete; if $j(\xi) < b(\xi)$ for some node $\xi \in \mathbf{D}$ then the financial markets are incomplete. The existence of a portfolio at each node which gives positive returns at each of the immediate successors is a classical assumption in the analysis of financial markets: assuming the existence of a short-lived numeraire bond at each node (A6) is a convenient way of ensuring that this condition is satisfied.

4. Budget Sets and the Concept of Equilibrium

In order to arrive at a concept of equilibrium we need to specify how agents perceive their trading opportunities on the system of markets when faced with the spot and security market prices (p, q). Two elements are involved in the construction of an agent's budget set; the first is the usual condition which asserts than an agent's net expenditure on the spot markets must not exceed the income earned on the financial markets at each node; the second is a new element introduced by the sequential nature of trade combined with the open-endedness of the future.

Let $z^i(\xi) = (z^i(\xi, j), j \in \mathbf{J})$ denote the i^{th} agent's portfolio at node ξ. At the (unique) predecessor ξ^- the agent has chosen the portfolio $z^i(\xi^-)$. Since we assume that no transactions costs are involved in the purchase or sale of securities there is no loss of generality in assuming that at node ξ agent i liquidates the portfolio position $z^i(\xi^-)$ taken at ξ^-. The agent's budget constraint at node ξ is thus given by

$$p(\xi)x^i(\xi) \leqq p(\xi)\omega^i(\xi) + (p(\xi)A(\xi) + q(\xi))z^i(\xi^-) - q(\xi)z^i(\xi) \tag{4.1}$$

Note that $z^i(\xi_0^-)$ is not a choice variable for the agent: we assume $z^i(\xi_0^-) = 0$ so that agents do not inherit financial commitments from the past. The consumption-portfolio process (x^i, z^i) which is chosen must satisfy (4.1) at every node. (Since we have assumed that each agent's preference ordering is strictly monotone with respect to the first commodity at each node (A4) we may replace the inequality in (4.1) by an equality.)

If the agent is to have a solution to his consumption-portfolio choice problem then the prices (p, q) must not offer arbitrage opportunities at any node $\xi \in \mathbf{D}$ i.e. there must not exist a portfolio $z^i(\xi)$ such that

$$-q(\xi)z^i(\xi) \geqq 0$$

$$(p(\xi')A(\xi') + q(\xi'))z^i(\xi') \geqq 0, \quad \forall\, \xi' \in \xi^+$$

with at least one strict inequality. This condition has been extensively discussed in the finite horizon incomplete markets literature and in the theory of finance (see for example Magill-Shafer (1991)) and is equivalent to the existence of a process $\pi = (\pi(\xi), \xi \in \mathbf{D})$ of positive *node (present value) prices* such that

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')(p(\xi')A(\xi') + q(\xi')), \; \forall\, \xi \in \mathbf{D} \tag{4.2}$$

In view of the open-endedness of the future even if the prices (p, q) do not offer arbitrage opportunities there will not be a solution to the agent's choice problem if a further restriction is not placed on the portfolio processes which the agent is permitted to consider. For any no-arbitrage prices (p, q) with $q \neq 0$ the agent can change any given portfolio ($z^i \longrightarrow z^i + \Delta z^i$) so as to obtain one more unit of income at the initial node and can roll over his debt ad infinitum thereafter. More formally a change Δz^i in the portfolio such that

$$\begin{aligned} 1 &= -q(\xi_0)\Delta z^i(\xi_0) \\ 0 &= (p(\xi')A(\xi') + q(\xi'))\Delta z^i(\xi'^-) - q(\xi')\Delta z^i(\xi'), \; \forall\, \xi' \in \mathbf{D}^+ \end{aligned} \tag{4.3}$$

is always feasible from any chosen portfolio $z^i \in Z$ and is preferred by agent i if his preference for consumption goods is monotone. A portfolio Δz^i satisfying these conditions is called a Ponzi scheme (for a discussion of this see Levine (1989, section 3) and Blanchard-Fischer (1989) p. 49). Thus some form of borrowing constraint which limits the amount of debt that an agent can plan to incur is necessary if his consumption-portfolio choice problem is to have a solution.

One approach, often adopted in the macroeconomics literature, which eliminates Ponzi schemes and leads to a well-defined concept of equilibrium is to impose *a priori* bounds on the indebtedness of the agents. Adopting such an approach for the GEI model would however essentially amount to reverting to Radner's approach of placing *a priori* bounds on agents' portfolios. As Hart (1975) pointed out such bounds are necessarily ad-hoc — furthermore they may add another imperfection to the model and much of the simplicity of the finite horizon GEI model arises from the fact that the only imperfections modelled are those arising from the incompleteness of the financial markets.

Since we focus on the GEI model *without default*, the most natural borrowing constraint is that agents should not borrow more that they can pay back: the problem is to give a precise meaning to this statement. The idea we want an equilibrium to express is the following. In an economy in which every agent is rational, impatient and prefers more (consumption), an agent will not be maximizing if he postpones spending so as to become *lender at infinity.* Since for every borrower there must be a lender, if it is assumed that every agent recognizes that every other agent is rational, impatient and prefers more, no agent should seek to be a *borrower at infinity.* Together these two conditions require that every agent be neither borrower nor lender at infinity, namely that the present value of the debts at infinity be zero.

This can be made more precise as follows. If agent i has an optimal consumption-portfolio plan (x^i, z^i) subject to the budget equation (4.1) and an appropriate growth condition on his debt, then there is associated with this plan a present value vector $\pi^i = (\pi^i(\xi),\ \xi \in \mathbf{D})$ where $\pi^i(\xi)$ is the multiplier (dual variable) induced by the budget equation (4.1) at node ξ. The vector π^i describes how agent i translates (discounts) a stream of income in the future to date 0. If the plan (x^i, z^i) is optimal then π^i must satisfy the adjoint equations (4.2) which express the fact that for agent i the marginal cost of each security at each node is equal to the marginal benefit of its return at the following nodes. In addition (π^i, z^i) must satisfy the *transversality condition*

$$\limsup_{T \longrightarrow \infty} \sum_{\xi \in \mathbf{D}_T} \pi^i(\xi) q(\xi) z^i(\xi) \leqq 0 \tag{4.4}$$

which asserts that an optimal portfolio does not leave value (make agent i a lender) at infinity. If (4.4) was not satisfied, agent i could find a preferred consumption stream by decreasing his lending (which is always possible even if markets are incomplete) thereby increasing earlier consumption.

Even if trade is anonymous and agents do not know more about the characteristics of other traders than that they are rational, impatient and prefer more, no agent should count on finding lenders on the markets who would finance a porfolio z^i that permits him to be a borrower at infinity

$$\liminf_{T \longrightarrow \infty} \sum_{\xi \in \mathbf{D}_T} \pi^i(\xi) q(\xi) z^i(\xi) < 0 \tag{4.5}$$

since this would oblige some other traders to be lenders at infinity. Strictly speaking, since markets are incomplete, agents on the other side of the transaction will evaluate their lending with different present value vectors, for when markets are incomplete the no-arbitrage equations (4.2) admit many solutions. But if markets are large and anonymous agent i cannot be expected to know the present value vectors of all other agents. In such circumstances it has become usual in the incomplete markets literature to make the assumption of *competitive perceptions* introduced by Grossman-Hart (1979): an agent uses his own present value vector to fill in the information regarding valuations which cannot be deduced from observed or anticipated prices. Using this convention, agent i will not attempt to finance a portfolio satisfying (4.5). Thus (recalling that (4.4) must be satisfied) a candidate growth condition on an agent i's debt is given by

$$\lim_{T \longrightarrow \infty} \sum_{\xi \in \mathbf{D}_T} \pi^i(\xi) q(\xi) z^i(\xi) = 0 \tag{4.6}$$

It can be shown that this growth condition sufficiently restricts the debt that an agent can plan to incur so that an equilibrium (which we call a *commitment equilibrium*) exists in which the budget sets of the agents are defined by (4.1) and (4.6) and markets clear. As the following example shows, if markets are incomplete, such equilibria may involve trading plans that agents would only carry out if they had made *binding commitments* at date 0 regarding their future trades.

Example A. Suppose that the event-tree $\mathbf{D}$ is such that the only uncertainty is at date 1, the future after date 1 being infinite but certain. Let (ξ_1, ξ_1') denote the two nodes at date 1, the node following ξ_1 (resp. ξ_1') at date t being denoted by ξ_t (resp. ξ_t') for $t \geqq 1$. (See Figure 1.)

We assume that the two nodes at date 1 are equally probable $(\rho(\xi_1)) = (\rho(\xi_1') = \frac{1}{2})$. Suppose the economy has an equal number of agents of two types and one good (income). Every agent has

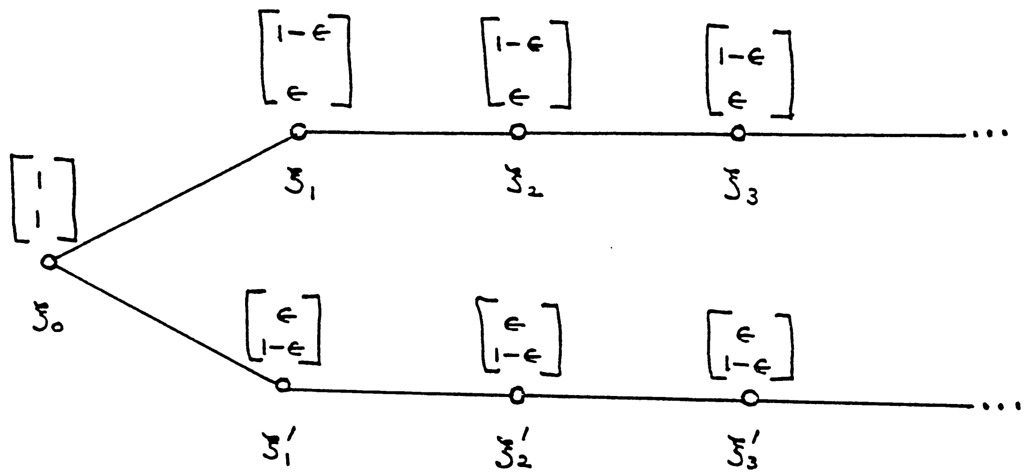

Figure 1:

the same additively separable utility function (3.1) with

$$u^i(x) = \sum_{\xi\in\mathbf{D}} \rho(\xi)\delta^{t(\xi)}\sqrt{x(\xi)} = \frac{1}{2}\sum_{t=0}^{\infty}\delta^t\sqrt{x(\xi_t)} + \frac{1}{2}\sum_{t=0}^{\infty}\delta^t\sqrt{x(\xi_t')}$$

The endowments of the two types of agents are shown in Figure 1: both types have 1 unit of income at node ξ_0 and if nature chooses node ξ_1, type 1 agents have $1-\epsilon$ for ever and type 2 agents have ϵ for ever; if nature chooses node ξ_1' the incomes of the two types are reversed.

In an Arrow-Debreu equilibrium for this economy the agents' budget sets would be defined by a vector of date 0 prices $\pi \in \ell_1(\mathbf{D})$ (Bewley (1972, Theorem 2))

$$\{x^i \in \ell_\infty^+(\mathbf{D}) \mid \sum_{\xi\in\mathbf{D}} \pi(\xi)(x^i(\xi) - w^i(\xi) = 0\}$$

By symmetry the only equilibrium is such that both types of agents consume 1 at node ξ_0 and $\frac{1}{2}$ at all subsequent nodes regardless of the branch chosen by nature. The Arrow-Debreu equilibrium is given by

$$\bar{x}^i(\xi_0) = 1, \quad \bar{x}^i(\xi_t) = \bar{x}^i(\xi_t') = \frac{1}{2}, \ \ \forall\, t \geqq 1,\ i = 1,2 \tag{4.7}$$

$$\bar{\pi}(\xi_0) = 1, \quad \bar{\pi}(\xi_t) = \bar{\pi}(\xi_t') = \frac{\delta^t}{\sqrt{2}}, \ \ \forall\, t \geqq 1 \tag{4.8}$$

It is easy to check that the equilibrium allocation in (4.7) can also be achieved by a system of complete financial markets consisting of two insurance contracts at node ξ_0 (paying 1 unit at node

ξ_1 (resp. ξ_1') and 0 at node ξ_1' (resp. ξ_1) and the short-lived numeraire bond at all subsequent nodes, in which agents have budget sets defined by (4.1) and (4.6).

More surprisingly, (4.7) can also be acheived as a commitment equilibrium allocation with incomplete markets: suppose that at each node (in particular at node ξ_0) the only contract is the short-lived numeraire bond, then the bond prices and strategies are given by

$$q(\xi) = \begin{cases} \sqrt{2}\delta \text{ if } \xi = \xi_0 \\ \delta \text{ if } \xi \neq \xi_0 \end{cases}, \; z^1(\xi) = \begin{cases} 0 \text{ if } \xi = \xi_0 \\ \frac{1}{2}(\frac{1}{\delta} + \ldots + \frac{1}{\delta^t}), \text{ if } \xi = \xi_t, \; t \geqq 1 \\ -\frac{1}{2}(\frac{1}{\delta} + \ldots + \frac{1}{\delta^t}), \text{ if } \xi = \xi_t', \; t \geqq 1 \end{cases}, \; z^2 = -z^1 \tag{4.9}$$

Agents of type 1 commit to lend for ever if ξ_1 occurs in exchange for the right to borrow for ever if ξ_1' occurs: on the upper branch agents of type 1 accumulate a credit (growing like $\frac{1}{\delta^t}$) which is balanced against a debt which grows at the same rate on the lower branch (with the reverse for agents of type 2). Such trading strategies are not however node consistent: if ξ_1 (resp. ξ_1') occurs it is not in the interest of type 1 (2) agents to lend. In the language of game theory the commitment equilibrium defined by (4.9) is not subgame perfect.

While the concept of a commitment equilibrium may prove to be of interest for exploring the social value of rules or commitments[1] it is not in the spirit of an equilibrium with incomplete markets. In the remark following Proposition 4.2 we show that an Arrow-Debreu equilibrium can always be realized as a commitment equilibrium even when the financial markets are extremely incomplete (the only security at each node is the short-lived numeraire bond). The binding commitments made by agents at date zero regarding their future trades on the financial markets serve as a collection of new contracts which, when combined with the existing securities, complete the markets.

In the above example an agent of type 1 (2) can foresee from date 0 that if node ξ_1 (ξ_1') occurs it will not be in his interest to lend for ever if there is no binding commitment to do so. It is clear that if we are to obtain a concept of equilibrium which respects the incompleteness of markets and does not introduce extraneous binding commitments then additional conditions must be imposed on the trading plans of the agents.

In the spirit of subgame perfection it should now be clear what these conditions are. In order

[1] One might argue that in early stages of society when there were no financial instruments beyond borrowing and lending, the social norms which put pressure on more fortunate individuals to lend (often with a zero interest) to those who had suffered unavoidable losses, were an attempt to use commitments to replace missing insurance markets. In the macroeconomics literature the value of commitments forms part of an extensive literature (rules versus dicretion) (see Kydland-Prescott (1977) and Blanchard-Fischer (1989), chapter 11)

for an agent's plan to be optimal, it must be optimal when restricted to the subtree $\mathbf{D}(\xi)$ for every node ξ in the event-tree. Thus an agent's portfolio must not imply that he is a lender at infinity when viewed from any node in the event tree. Similarly, respecting the fact that other traders plans must have the same property, the agent should not attempt to be a borrower at infinity when viewed from any node $\xi \in \mathbf{D}$. Thus the condition which makes agents' trading plans node consistent throughout the event-tree is the requirement that for each agent i

$$\lim_{T \longrightarrow \infty} \sum_{\xi' \in \mathbf{D}_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0, \ \forall \ \xi \in \mathbf{D} \tag{4.10}$$

We are thus led to define the budget set of agent i as follows

$$\mathcal{B}_\infty(p,q,\pi^i,\omega^i,A) = \left\{ x^i \in \ell^+_\infty(\mathbf{D} \times \mathbf{L}) \left| \begin{array}{l} \exists \ z^i \in Z, \text{ such that } \forall \xi \in \mathbf{D} \\ \lim\limits_{T \longrightarrow \infty} \sum\limits_{\xi' \in \mathbf{D}_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0 \\ p(\xi)(x^i(\xi) - \omega^i(\xi)) = (V(\xi))z^i(\xi^-) - q(\xi)z^i(\xi) \end{array} \right. \right\}$$

where $V(\xi) = p(\xi)A(\xi) + q(\xi)$ is the vector of returns at the successor ξ for the $j(\xi^-)$ securities traded at ξ^-. In equilibrium we will require that π^i be the present value vector of agent i. If (x^i, z^i) is a pair satisfying the constraints in $\mathcal{B}_\infty(p,q,\pi^i,\omega^i,A)$ then z^i is said to *finance* x^i and with a slight abuse of notation we write

$$(x^i; z^i) \in \mathcal{B}_\infty(p,q,\pi^i,\omega^i,A)$$

$(\bar{x}^i; \bar{z}^i)$ is said to be $\underset{i}{\succsim}$ *maximal* in $\mathcal{B}_\infty(p,q,\pi^i,\omega^i,A)$ if $\bar{z}^i$ finances $\bar{x}^i$ and $\bar{x}^i \underset{i}{\succsim} x^i$ for all $(x^i; z^i) \in \mathcal{B}_\infty(p,q,\pi^i,\omega^i,A)$.

These considerations thus lead us to the following concept of equilibrium.

Definition 4.1: A GEI *equilibrium* of the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ is a pair

$$((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}}) \in \ell^+_\infty(\mathbf{D} \times \mathbf{L} \times \mathbf{I}) \times Z^{\mathbf{I}} \times \mathbf{R}^{\mathbf{D} \times \mathbf{L}} \times Q \times \ell^+_1(\mathbf{D} \times \mathbf{I})$$

where $(\bar{x}, \bar{z}) = (\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I)$ such that

(i) $(\bar{x}^i; \bar{z}^i)$ is $\underset{i}{\succsim}$ maximal in $\mathcal{B}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$, for each $i \in \mathbf{I}$

(ii) for each $i \in \mathbf{I}$

(a) $\bar{\pi}^i(\xi) > 0, \ \forall \ \xi \in \mathbf{D}$ and $\bar{P}^i \in \ell^+_1(\mathbf{D} \times \mathbf{L})$ where $\bar{P}^i = (\bar{P}^i(\xi), \xi \in \mathbf{D}) = (\bar{\pi}^i(\xi)\bar{p}(\xi), \xi \in \mathbf{D})$

(b) $\bar{x}^i$ is $\underset{i}{\succsim}$ maximal in $B_\infty(\bar{P}^i,\omega^i) = \{x^i \in \ell_\infty^+(\mathbf{D} \times \mathbf{L}) | \ \bar{P}^i(x^i - \omega^i) \leqq 0\}$

(c) $\bar{\pi}^i(\xi)\bar{q}(\xi,j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')(\bar{p}(\xi')A(\xi',j) + q(\xi',j)), \ \forall \ j \in J(\xi), \ \forall \ \xi \in \mathbf{D}$

(iii) $\sum_{i \in \mathbf{I}} (\bar{x}^i - \omega^i) = 0$ (iv) $\sum_{i \in \mathbf{I}} \bar{z}^i = 0$

Remark: Condition (ii) characterises the equilibrium present value vector $\bar{\pi}^i$ of agent i: $\bar{\pi}^i$ must satisfy the agent's adjoint equations (c), while (b) asserts that the induced date 0 commodity price vector $\bar{P}^i$ supports the preferred set of agent i at $\bar{x}^i$ or equivalently that $\bar{x}^i$ is the agent's most preferred bundle in his induced Arrow-Debreu budget set $B_\infty(\bar{P}^i,\omega^i)$. (a) requires that the supporting price $\bar{P}^i$ lie in $\ell_1(\mathbf{D} \times \mathbf{L})$, a property which is implied by the Mackey continuity of the preferences as we shall see in the proof of Theorem 7.1. With an appropriate normalization of the spot prices — since good 1 is the numeraire good we adopt the normalisation $p(\xi,1) = 1, \ \forall \ \xi \in \mathbf{D}$— this implies that $\bar{\pi}^i$ lies in $\ell_1(\mathbf{D})$.

In the above definition all securities are taken to be in zero net supply: this is not a restriction since by an appropriate change of variable an economy in which assets are in positive net supply can be transformed into an economy in which assets are in zero net supply as we show later. Taking assets in zero net supply is convenient for the proofs.

Let $V(\xi^+) = [V(\xi')]_{\xi' \in \xi^+}$ denote the $j(\xi) \times b(\xi)$ matrix of returns at the successors $\xi' \in \xi^+$ for the $j(\xi)$ active securities traded at node ξ. The (financial) markets are said to be *complete* if rank $V(\xi^+) = b(\xi), \ \forall \ \xi \in \mathbf{D}$ and *incomplete* if this condition is not satisfied.

If markets are incomplete, then typically the set of GEI equilibrium allocations of Definition 4.1 does not include the Arrow-Debreu equilibrium allocations. To see this let $(\bar{x},\bar{P})$ be an Arrow-Debreu equilibrium of an economy with characteristics $(\mathbf{D},\succsim,\omega)$ satisfying A1-A3. In order that the equilibrium price process $\bar{P} = (\bar{P}(\xi,l), \ (\xi,l) \in \mathbf{D} \times \mathbf{L})$ be the present value of the spot price process $\bar{p} = (\bar{p}(\xi,l), \ (\xi,l) \in \mathbf{D} \times \mathbf{L})$ (with $\bar{p}(\xi,1) = 1$), discounted by the process $\bar{\pi} = (\bar{\pi}(\xi), \ \xi \in \mathbf{D})$ we must have

$$\bar{\pi}(\xi)\bar{p}(\xi,l) = \bar{P}(\xi,l), \quad \bar{\pi}(\xi) = \bar{P}(\xi,1), \ \forall \ (\xi,l) \in \mathbf{D} \times \mathbf{L} \tag{4.11}$$

Let $(\mathbf{J},\zeta,A)$ be a security structure satisfying A5-A6. If the prices of the securities are equal to the discounted value of their future dividends[2] then we must have

[2]The case where the price of an infinite-lived security differs from its fundamental value by a speculative bubble is considered below.

$$\bar{P}(\xi,1)\bar{q}(\xi) = \sum_{\xi' \in \mathbf{D}^+(\xi)} \bar{P}(\xi')A(\xi'),\ \forall\ \xi \in \mathbf{D} \tag{4.12}$$

Given $P \in \ell_1(\mathbf{D} \times \mathbf{L})$ and $y \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ it is convenient to have a notation for the present value of the commodity stream $y\chi_{\mathbf{D}(\xi)}$, namely the restriction of y to the subtree $\mathbf{D}(\xi)$. We thus define

$$P \underset{\xi}{\cdot} y = \sum_{\xi' \in \mathbf{D}(\xi)} P(\xi')y(\xi')$$

Proposition 4.2: *Let $(\bar{x}, \bar{P})$ be an Arrow-Debreu equilibrium of an economy $\mathcal{E}(\mathbf{D}, \succsim, \omega)$ satisfying A1-A3 and let $(\mathbf{J}, \zeta, A)$ be a security structure satisfying A5-A6, then $\bar{x}$ can be obtained as the allocation of a GEI equilibrium $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in\mathbf{I}}))$ with prices $(\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in\mathbf{I}}$ defined by (4.11),(4.12) and $\bar{\pi}^i = \bar{\pi}$, $\forall\ i \in \mathbf{I}$ if and only if $\forall\ \xi \in \mathbf{D}$, $\forall\ i \in \mathbf{I}$*

$$\left(\bar{P} \underset{\xi'}{\cdot} (\bar{x}^i - \omega^i)\right)_{\xi' \in \xi^+} \in \ < [\bar{P} \underset{\xi'}{\cdot} A(\cdot, j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} > \tag{4.13}$$

where $< [\bar{P} \underset{\xi'}{\cdot} A(\cdot, j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} >$ denotes the subspace of $\mathbf{R}^{b(\xi)}$ spanned by the returns (measured by the discounted value of the dividends) at the successors ξ^+ of ξ of the $j(\xi)$ active securities at node ξ.

Proof: Let $(\bar{x}, \bar{P})$ be an Arrow-Debreu equilibrium. Since agents' preferences are Mackey continuous, $\bar{P} \in \ell_1(\mathbf{D} \times \mathbf{L})$ (Bewley (1972), Theorem 2). If the prices $(\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in\mathbf{I}})$ are defined by (4.11), (4,12) and $\bar{\pi}^i = \bar{\pi}$, $\forall\ i \in \mathbf{I}$ then (ii) and (iii) in Definition 4.1 are satisfied. Thus $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^I)$ can be obtained as the allocation of a GEI equilibrium (with the above prices) if and only if there exist portfolios $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^I)$ such that $\bar{z}^i$ finances $\bar{x}^i$, $\forall\ i \in \mathbf{I}$ and $\sum_{i\in\mathbf{I}} \bar{z}^i = 0$.

Consider all the portfolios $z = (z^1, \ldots, z^I)$ such that the budget equations (4.1) hold for all $\xi \in \mathbf{D}$, for each agent i and $\sum_{i\in\mathbf{I}} z^i = 0$. Such portfolios always exist and can be constructed by forward induction.[3] Since the riskless bond has a nonzero (positive) price at each node, there exists a portfolio $z^i(\xi_0)$ at node ξ_0 such that

$$\bar{p}(\xi_0)(\bar{x}^i(\xi_0) - \omega^i(\xi_0)) = -\bar{q}(\xi_0)z^i(\xi_0)$$

and by forward induction there exists a portfolio $z^i(\xi)$ such that

$$\bar{p}(\xi)(\bar{x}^i(\xi) - \omega^i(\xi)) = (\bar{p}(\xi)A(\xi) + \bar{q}(\xi))z^i(\xi^-) - \bar{q}(\xi)z^i(\xi),\ \forall\ \xi \in \mathbf{D}^+$$

[3] We are grateful to an anonymous referee for pointing this out.

for all $i = 2, \ldots, I$. For agent 1 choose $z^1(\xi) = -\sum_{i=2}^{I} z^i(\xi)$, $\forall \xi \in \mathbf{D}$. Since $\sum_{i \in \mathbf{I}}(\bar{x}^i - \omega^i) = 0$, (4.1) is also satisfied for agent 1. Thus $\bar{x}$ can be obtained as a GEI equilibrium if and only if one such process of portfolios $z = (z^1, \ldots, z^I)$ satisfies (4.10) for each $i \in \mathbf{I}$. Consider any node ξ. Multiplying the budget equation at each node $\xi'' \in \mathbf{D}(\xi)$ by $\bar{\pi}(\xi'')$, summing from the immediate successors $\xi' \in \xi^+$ and using (4.11) and (4.12) gives for any $T > t(\xi)$

$$\sum_{\xi'' \in \mathbf{D}^T(\xi)} \bar{P}(\xi'')(\bar{x}^i(\xi'') - \omega^i(\xi'')) = (\bar{P}(\xi')A(\xi') + \bar{P}(\xi', 1)\bar{q}(\xi'))z^i(\xi) - \sum_{\xi'' \in \mathbf{D}_T(\xi')} \bar{\pi}(\xi'')\bar{q}(\xi'')z^i(\xi'') \tag{4.14}$$

z^i satisfies the transversality condition (4.10) if and only if

$$\left(\bar{P} \underset{\xi'}{\cdot} (\bar{x}^i - \omega^i)\right)_{\xi' \in \xi^+} = \left[\bar{P} \underset{\xi'}{\cdot} A(\cdot, j)\right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} z^i(\xi), \ \forall \ \xi \in \mathbf{D}$$

and such a portfolio process z^i exists for each i if and if (4.13) holds $\forall \ \xi \in \mathbf{D}$, $\forall \ i \in \mathbf{I}$.$\triangle$

Remark. Note that if the weaker transversality condition (4.6) replaces (4.10) in each agent's budget set, then Definition 4.1 defines what we earlier called a *commitment equilibrium.* If there is a numeraire bond at each node (A6) then the Arrow-Debreu allocation $\bar{x}$ can always been obtained as a commitment equilibrium. To see this note that (4.13) applied to node $\xi' = \xi_0$ implies that any process of portfolios $z = (z^1, \ldots, z^I)$ constructed by forward induction in the proof of Proposition 4.2 satisfies (4.6) for each i, since $\bar{P}(\bar{x}^i - \omega^i) = 0$. Thus for any such process of portfolios z, $((\bar{x}, z), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$ with $\bar{\pi}^i = \bar{\pi}$, $\forall \ i \in \mathbf{I}$ is a commitment equilibrium.

The spanning condition (4.13) is the same as in the finite horizon case (see Magill-Shafer (1991)), the only difference being that the discounted sums are infinite. The transversality condition (4.10) which formalizes the requirement that on each path of the event-tree $\mathbf{D}$ the present value of an agent's debt tends to zero, replaces the terminal condition $z^i(\xi) = 0$ for all terminal nodes in a T-period model. The spanning condition (4.13) can be used to prove[4] that if $I \geqq 2$ Arrow-Debreu equilibria can typically (generically) been obtained as GEI equilibria only if there are enough active securities at each node ξ to span the immediate contingencies ξ^+ (i.e. $j(\xi) \geqq b(\xi)$) for each node $\xi \in \mathbf{D}$.

[4]A generic argument can be constructed in the case where agents' preferences are represented by (3.1) using the finite equation method of analyzing infinite horizon Arrow-Debreu equilibria introduced by Kehoe-Levine (1985).

Example A *(continued)*. Consider the economy in Example A with the financial structure consisting of the short-lived numeraire bond at each node. Since $1=j(\xi_0) < b(\xi_0) =2$, the markets are incomplete at node ξ_0. The only GEI equilibrium is the no-trade equilibrium $(\bar{x}, \bar{z}, \bar{q}) = (\omega, 0, \bar{q})$ where

$$\bar{q}(\xi_t) = \bar{q}(\xi_t') = \begin{cases} \frac{\delta}{2}(\frac{1}{\sqrt{1-\epsilon}} + \frac{1}{\sqrt{\epsilon}}), \text{ if } t = 0 \\ \delta, \text{ if } t \geqq 1 \end{cases}$$

In Proposition 4.2 we assumed that the price of every security in a GEI equilibrium is equal to the discounted value of its dividends (its so-called *fundamental value*). As we show in section 8 in an infinite horizon economy the prices of infinite-lived securities in zero net supply can have a *speculative bubble* component. If $(\mathbf{J}, \zeta, \forall)$ contains such securities then (4.13) must be replaced by

$$\left(\bar{P}_{\underset{\xi'}{\cdot}}(\bar{x}^i - \omega^i)\right)_{\xi' \in \xi^+} \in \, < [\bar{P}_{\underset{\xi'}{\cdot}} A(\cdot, j) + \bar{P}(\xi', 1)\rho(\xi', j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} > \tag{4.13$'$}$$

for all $\xi \in \mathbf{D}$, for all $i \in \mathbf{I}$, where $\rho(\xi', j) \neq 0$ if security j has a speculative bubble, $\rho(\xi', j)$ being a solution of the homogeneous equation

$$\bar{P}(\xi, 1)\rho(\xi, j) = \sum_{\xi' \in \xi^+} \bar{P}(\xi', 1)\rho(\xi', j), \ \forall \, \xi \in \mathbf{D}$$

In this case it can be shown that if $I > b(\xi)$ for some node ξ for which $j(\xi) < b(\xi)$, then for some ω the I vectors $\{\bar{P}_{\underset{\xi'}{\cdot}}(\bar{x}^i - \omega^i)_{\xi' \in \xi^+}\}_{i \in \mathbf{I}}$ span $\mathbf{R}^{b(\xi)}$, so that (4.13′) cannot be satisfied at node ξ for all $i \in \mathbf{I}$. Thus the Arrow-Debreu allocation cannot be obtained as a GEI equilibrium allocation. (4.13′) does however suggest that speculative bubbles can affect the spanning opportunities offered by the financial markets. The next example shows that this is indeed possible.

Example B. Consider an economy with event-tree, preferences and endowments of Example A. The financial structure consists of two securities issued at node ξ_0. Security 1 (2) pays 1 unit at node $\xi_1 (\xi_1')$ and 0 elsewhere and is retraded at all nodes on the upper (lower) branch $(\xi_t \ (\xi_t'), \ t \geqq 1)$. There are two symmetric equilibria. In the first the securities are priced at their fundamental values

$$\bar{q}_1(\xi_0) = \bar{q}_2(\xi_0) = \frac{\delta}{\sqrt{2}}, \quad \bar{q}_1(\xi_t) = \bar{q}_2(\xi_t') = 0, \ t \geqq 1$$

Trading the securities thus only permits income transfers at date 1, the agents remaining at their initial endowments thereafter:

$$\bar{x}^i(\xi_0) = 1, \ \bar{x}^i(\xi_1) = \bar{x}^i(\xi_1') = \frac{1}{2}, \ \bar{x}^1(\xi_t) = \bar{x}^2(\xi_t') = 1 - \epsilon, \ \bar{x}^1(\xi_t') = \bar{x}^2(\xi_t) = \epsilon, \ t \geqq 2$$

In the second equilibrium security 1 (2) has a bubble on the upper (lower) branch

$$\bar{q}_1(\xi_0) = \bar{q}_2(\xi_0) = \frac{\delta}{\sqrt{2}}, \quad \bar{q}_1(\xi_t) = \bar{q}_2(\xi_t') = \frac{1}{\delta^t}, \ t \geqq 1$$

and the Arrow-Debreu equilibrium allocation (4.7) is achieved with the portfolios

$$\bar{z}_1^1(\xi_t) = -\bar{z}_2^1(\xi_t') = -\frac{\delta^t}{1-\delta}(\frac{1}{2} - \epsilon), \ t \geqq 0, \ \bar{z}^2 = -\bar{z}^1$$

Even though the security prices grow at the implicit interest rate ($\frac{1}{\delta}$), the agents' borrowing (lending) decreases at the rate δ so that the transversality condition (4.10) is satisfied

$$\lim_{T \longrightarrow \infty} \bar{\pi}(\xi_T)\bar{q}(\xi_T)\bar{z}^1(\xi_T) = -\lim_{T \longrightarrow \infty} \frac{\delta^T}{\sqrt{2}} \frac{1}{\delta^T} \frac{\delta^T}{1-\delta}(\frac{1}{2} - \epsilon) = 0$$

In this example the securities have value after date 1 only because each agent believes that other agents believe that they have value: the equilibrium thus depends on the beliefs of the agents, not on the fundamentals of the securities.

In the above example we have permitted an intrinsically worthless (zero dividend) asset to be traded. Strictly speaking this requires extending the definition of active securities given in (2.1). In sections 5-7 where we study the limits of truncated economies, we retain the convention (2.1), since in finite horizon economies an intrinsically worthless asset has a zero price. In the above example only the first equilibrium is a limit of equilibria of truncated economies: the second equilibrium is specific to the infinite horizon economy and has no counterpart in finite horizon economies.

Since, when markets are incomplete, GEI equilibria are typically different from Arrow-Debreu equilibria, the existence of a GEI equilibrium cannot be deduced from the existence theorem for an Arrow-Debreu equilibrium for an infinite economy (Bewley (1972)). Our next objective is thus to give conditions under which a GEI equilibrium exists[5]. The proof of existence of equilibrium is divided into two parts. The first part considers an economy in which all securities are short-lived and deliver commodity payoffs in the numeraire good (section 6); the second part (section 7) considers the general case where securities are short, long or infinite-lived and have arbitrary commodity payoffs. In the first part it is shown that an equilibrium always exists; in the second

[5]Existence of a GEI equilibrium cannot be deduced either from the theorems of existence of equilibrium for a finite horizon economy with a countable number (or a continuum) of states (Zame (1988), Hellwig (1991), Mas-Colell-Zame (1991), Mas-Colell-Monteiro (1991)) since in these models the number of securities is finite. Nor can it be deduced from the existence result of Brown-Werner (1992) who consider an economy with an infinite number of securities in a two-period, one-good model: they use the property of constrained optimality of a GEI equilibrium for such an economy but this property does not extend to a multiperiod or multigood model.

part it is shown that a pseudoequilibrium (defined in section 5) always exists. From this we deduce by a separate argument that an equilibrium exists for a dense subset of asset payoffs. Section 8 examines the possibility of speculative bubbles for the equilibrium prices of infinite-lived securities and whether or not they can affect the GEI equilibrium allocation.

5. The T-truncated Economy and Pseudoequilibrium

We prove existence of GEI equilibrium for the infinite horizon economy by taking limits of GEI equilibria in truncated economies in which trade stops at some finite date. Let $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ be an infinite horizon economy. The associated T-truncated economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ is the economy with the same characteristics as $\mathcal{E}_\infty$ in which agents are constrained to stop trading at date T. If $(p_T, q_T) \in \mathbb{R}^{\mathbf{D}\times\mathbf{L}} \times Q$ is a commodity and security price process then the budget set of agent i in the truncated economy $\mathcal{E}_T$ is defined by

$$\mathcal{B}_T(p_T, q_T, \omega^i, A) = \left\{ x^i \in \ell_\infty^+(\mathbf{D}\times\mathbf{L}) \;\middle|\; \begin{array}{r} \exists\, z^i \in Z,\ z^i(\xi) = 0 \text{ if } t(\xi) \geqq T \\ p_T(\xi)(x^i(\xi) - \omega^i(\xi)) = (p_T(\xi)A(\xi) + q_T(\xi))z^i(\xi^-) \\ - q_T(\xi)z^i(\xi) \text{ if } t(\xi) \leqq T \\ x^i(\xi) = \omega^i(\xi) \text{ if } t(\xi) > T \end{array} \right\}$$

Even though the consumption-portfolio process of an agent is defined over the whole event-tree, a T-truncated economy is essentially a finite horizon economy with $T+1$ periods since the consumption-portfolio process of an agent is fixed after date T.

Definition 5.1: A GEI *equilibrium* of the truncated economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ is a pair

$$((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T)) \in \ell_\infty^+(\mathbf{D}\times\mathbf{L}\times\mathbf{I})) \times Z^{\mathbf{I}} \times \mathbb{R}^{\mathbf{D}\times\mathbf{L}} \times Q$$

such that

(i) $(\bar{x}_T^i; \bar{z}_T^i)$ is $\underset{i}{\succsim}$ maximal in $\mathcal{B}_T(\bar{p}_T, \bar{q}_T, \omega^i, A)$, $\forall i \in \mathbf{I}$

(ii) $\sum_{i\in\mathbf{I}}(\bar{x}_T^i - \omega^i) = 0$ (iii) $\sum_{i\in\mathbf{I}} \bar{z}_T^i = 0$

(iv) $\bar{p}_T(\xi) = 0$ if $t(\xi) > T$, $\bar{q}_T(\xi) = 0$ if $t(\xi) \geqq T$

Since only the prices of the commodities and securities which are traded in $\mathcal{E}_T$ are well-determined, (iv) is a natural way of extending these prices to the whole event-tree.

Since in an equilibrium of the truncated economy the terminal condition $z_T^i(\xi) = 0$ for all ξ with $t(\xi) \geqq T$ replaces the asymptotic condition (4.7), the present value vectors of the agents do not appear explicitly in Definition 5.1. Each agent has however a well-defined present value vector in an equilibrium of $\mathcal{E}_T$ which is characterised as follows.

Proposition 5.2: *Under assumptions (A1-A4) if $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ is a GEI equilibrium of $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ then each agent $i \in \mathbf{I}$ has a present value vector $\bar{\pi}_T^i \in \mathbb{R}^{\mathbf{D}}$ satisfying*

(a) $\bar{\pi}_T^i(\xi) > 0$ *if* $t(\xi) \leqq T$, $\bar{\pi}_T^i(\xi) = 0$ *if* $t(\xi) > T$

(b) $\bar{x}_T^i$ *is* $\succsim_i$ *maximal in* $B_T(\bar{P}_T^i, \omega^i) = \left\{ x^i \in \ell_\infty^+(\mathbf{D} \times \mathbf{L}) \;\middle|\; \begin{array}{l} \bar{P}_T^i(x^i - \omega^i) \leqq 0 \\ x^i(\xi) = \omega^i(\xi) \text{ if } t(\xi) > T. \end{array} \right\}$
where $\bar{P}_T^i = (\bar{P}_T^i(\xi), \xi \in \mathbf{D}) = (\bar{\pi}_T^i(\xi)\bar{p}_T(\xi), \xi \in \mathbf{D})$

(c) $\bar{\pi}_T^i(\xi)\bar{q}_T(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}_T^i(\xi')(\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j)),\ \forall\, j \in \mathbf{J}(\xi), t(\xi) \leqq T - 1$

Proof: (see appendix).

It is well-known that not every truncated economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ has a GEI equilibrium. The problem arises from the fact that at node ξ the dimension of the subspace of $\mathbb{R}^{b(\xi)}$ spanned by the columns of the $b(\xi) \times j(\xi)$ returns matrix

$$[p_T(\xi')A(\xi', j) + q_T(\xi', j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \tag{5.1}$$

can change when the prices $(p_T(\xi'), q_T(\xi'))_{\xi' \in \xi^+}$ vary. There is however a convenient case where this difficulty can be avoided. If all securities are *short-lived numeraire assets* and if spot prices are normalised by using good 1 as the numeraire at each node

$$p_T(\xi, 1) = 1 \quad \forall\, \xi \in \mathbf{D} \tag{5.2}$$

then the returns matrix (5.1) is independent of both the spot prices and the security prices since $q(\xi', j) = 0$ for each $j \in \mathbf{J}(\xi)$ for all $\xi' \in \xi^+$. In this case a GEI equilibrium of a truncated economy $\mathcal{E}_T$ always exists (see for example Geanakoplos-Polemarchakis (1986)).

Theorem 5.3: *If assumptions A1-A3 are satisfied and if $(\mathbf{J}, \zeta, A)$ is a security structure consisting* *of short-lived numeraire assets then every truncated economy $\mathcal{E}_T(D, \succsim, \omega, (\mathbf{J}, \zeta, A))$ has a GEI equilibrium.*

When securities have payoffs involving two or more commodities or when there are long-lived securities then changes in the prices (p_T, q_T) can create discontinuities in agents' demands which may lead to the failure of existence of a GEI equilibrium for a truncated economy. The approach followed in the recent literature is to introduce the concept of a pseudoequilibrium: such an equilibrium exists for all truncated economies and for most economies a pseudoequilibrium is a GEI equilibrium. The idea is as follows: typically the matrix in (5.1) is of rank

$$a(\xi) \;=\; \min\;(b(\xi), j(\xi)) \tag{5.3}$$

but for some values of (p_T, q_T) the rank may fall. A pseudoequilibrium is an equilibrium of an economy in which agents are given an artificial subspace of income transfers of dimension $a(\xi)$ at node ξ (for each $\xi \in \mathbf{D}^T$) which contains the subspace of transfers achievable with the existing securities — but which is larger when the matrix in (5.1) has rank less than $a(\xi)$. Existence of a pseudoequilibrium for a two period economy has been proved by different methods by Duffie-Shafer (1985), Husseini-Lasry-Magill (1990), Hirsch-Magill-Mas-Colell (1990) and Geanakoplos-Shafer (1990); Duffie-Shafer (1986) and Magill-Shafer (1991) cover the multiperiod case.

In these papers a pseudoequilibrium is defined using a vector of discounted date 0 prices and an abstract subspace at each node. The subspaces are parametrised in a way which is convenient for proving existence. Here we adopt an alternative representation which is more convenient for the passage to the limit from the finite to the infinite case: this representation consists in defining the artificial subspace at each node by an orthogonal basis which may be interpreted as the returns on $a(\xi)$ short-lived numeraire assets issued at node ξ.

Definition 5.4: $(\mathbf{K}, \eta, \Gamma)$ is an *artificial short-lived numeraire asset structure* for the economy $\mathcal{E}(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ if

(i) $\mathbf{K} = (K(\xi), \xi \in \mathbf{D})$ where $K(\xi)$ consists of $a(\xi)$ short-lived numeraire securities issued at node ξ with good 1 payoff at the immediate successors $\xi' \in \xi^+$

$$k \in K(\xi) \implies \eta(k) = \xi \text{ and } \Gamma(\xi', \ell, k) = 0 \text{ if } \ell \neq 1 \text{ or } \xi' \notin \xi^+$$

(ii) at each node ξ the returns of the securities issued at node ξ are pairwise orthogonal

$$\sum_{\xi' \in \xi^+} \Gamma(\xi', 1, k)\Gamma(\xi', 1, k') = 0 \quad \forall\, k, k' \in K(\xi), k \neq k'$$

(iii) the payoff on each security is non-zero and is normalised so that

$$\max_{\xi' \in \xi^+} |\, \Gamma(\xi', 1, k)\,| = 1 \quad \forall\, k \in K(\xi), \quad \forall\, \xi \in \mathbf{D}$$

Definition 5.5: Let $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ be a GEI equilibrium of the truncated economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{K}, \eta, \Gamma))$ normalised by (5.2) where $\bar{\gamma}_T = (\bar{\gamma}_T^1, \ldots, \bar{\gamma}_T^I)$ is the vector of agents' trades in the securities of $\mathbf{K}$ and $\bar{\rho}_T$ is the vector of security prices and let π_T be any vector of no-arbitrage node prices

$$\pi_T(\xi)\bar{\rho}_T(\xi, k) = \sum_{\xi' \in \xi^+} \pi_T(\xi')\Gamma(\xi', 1, k), \quad \forall\, k \in K(\xi), \quad \forall\, \xi \in \mathbf{D}^{T-1} \tag{5.4}$$

Let $\bar{q}_T$ be defined by

$$\pi_T(\xi)\bar{q}_T(\xi, j) = \begin{cases} \displaystyle\sum_{\xi' \in \mathbf{D}^+(\xi)} \pi_T(\xi')\bar{p}_T(\xi')A(\xi', j), & \forall\, j \in J(\xi), \quad \forall\, \xi \in \mathbf{D}^{T-1} \\ 0 \quad \text{otherwise} \end{cases} \tag{5.5}$$

We say that $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ is a *short-lived numeraire asset pseudoequilibrium* of the economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ if

$$\left\langle \left[\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j)\right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \right\rangle \subset \left\langle \left[\Gamma(\xi', 1, k)\right]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \right\rangle, \quad \forall\, \xi \in \mathbf{D}^{T-1} \tag{5.6}$$

where for a matrix B, $< B >$ denotes the subspace spanned by the columns of the matrix.

Remark: When (5.6) holds, the definition of $\bar{q}_T$ in (5.5) is independent of the choice of the vector of node prices π_T satisfying (5.4). This can be seen as follows:

$$\begin{aligned}
\bar{q}_T(\xi, j) &= \frac{1}{\pi_T(\xi)} \sum_{\xi' \in \xi^+} \pi_T(\xi')(\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j)) \\
&= \frac{1}{\pi_T(\xi)} \sum_{\xi' \in \xi^+} \pi_T(\xi') \sum_{k \in K(\xi)} \alpha_k^j \Gamma_T(\xi', 1, k) \\
&= \sum_{k \in K(\xi)} \alpha_k^j \bar{\rho}(\xi, k)
\end{aligned}$$

where $(\alpha_k^j, k \in K(\xi))$ are the coordinates of the vector $(\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j))_{\xi' \in \xi^+}$ on the basis $(\Gamma_T(\cdot, 1, k), k \in K(\xi))$. The reverse calculation with any other vector of nodes prices $(\tilde{\pi}_T(\xi), \xi \in \mathbf{D}^T)$ satisfying (5.4) leads to (5.5) using the terminal condition $q_T(\xi, j) = 0$ if $t(\xi) = T$.

Theorem 5.6: *Under assumptions A1-A3, for any truncated economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ there exists an asset structure $(\mathbf{K}, \eta, \Gamma_T)$ satisfying (i)-(iii) in Definition 5.4 such that a GEI equilibrium $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ of $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{K}, \eta, \Gamma_T))$ is a short-lived numeraire asset pseudoequilibrium of $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$.*

Proof: It suffices to check that the existence of a short-lived numeraire asset pseudoequilibrium is equivalent to the existence of a pseudoequilibrium as defined in section 2.4 of Magill-Shafer (1991): the result follows by applying Theorem 16 in that section. $\triangle$

If in a pseudoequilibrium the returns matrix of the original securities $(\mathbf{J}, \zeta, A)$ has maximal rank $a(\xi)$ at each node $\xi \in \mathbf{D}^{T-1}$ then the inclusion in (5.6) is an equality. In this case trading in the artificial securities gives each agent access to the same opportunity set as trading in the original securities. Thus when the rank condition in (5.7) below is satisfied, up to converting the portfolios and security prices from the artificial to the original securities, a pseudoequilibrium is a GEI equilibrium.

Proposition 5.7: *Let $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ be a GEI equilibrium for $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{K}, \eta, \Gamma_T))$ which is a short-lived numeraire asset pseudoequilibrium of the economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ and let $\bar{q}_T$ be defined by (5.5). If*

$$rank\,[\,\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j)\,]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = a(\xi), \ \forall\, \xi \in \mathbf{D}^{T-1} \tag{5.7}$$

then there exists a vector of portfolios for the agents $\bar{z}_T = (\bar{z}_T^1, \ldots, \bar{z}_T^I)$ such that $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ is a GEI equilibrium of the economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$.

Proof: At each node $\xi \in \mathbf{D}^{T-1}$ let $V(\xi^+)$ and $\Gamma_T(\xi^+)$ denote the returns matrices on the left and right side of (5.6) respectively. When (5.7) holds agents have the same opportunity sets with the two asset structures. For agent $i \in \mathbf{I}$, $i \neq 1$, define the agent's portfolio $\bar{z}_T^i$ by

$$\begin{bmatrix} -\bar{q}_T(\xi) \\ V(\xi^+) \end{bmatrix} \bar{z}_T^i(\xi) \ = \ \begin{bmatrix} -\bar{\rho}_T(\xi) \\ \Gamma_T(\xi^+) \end{bmatrix} \bar{\gamma}^i(\xi), \quad \forall\, \xi \in \mathbf{D}^{T-1} \tag{5.8}$$

with $\bar{z}_T^i(\xi) = 0$ if $t(\xi) \geqq T$. Let $\bar{z}_T^1(\xi) = -\sum_{i=2}^{I} \bar{z}^i(\xi)$, $\forall\, \xi \in \mathbf{D}$ then the spot market clearing equations imply that $\bar{z}_T^1$ satisfies (5.8) and $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ is a GEI equilibrium. $\triangle$

6. Existence of Equilibrium with Short-lived Numeraire Assets

In this section we prove the existence of a GEI equilibrium for an infinite horizon economy in which securities are short-lived numeraire assets. The limit arguments that we develop for this case will be used to prove the existence of a pseudoequilibrium when the economy has a general security structure.

Theorem 6.1: *If assumptions A1-A6 are satisfied and if* $(\mathbf{J}, \zeta, A)$ *is a security structure consisting of short-lived numeraire assets then the economy* $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ *has a GEI equilibrium.*

Proof: In a GEI model with real securities, price levels play no role since the budget equation (4.1) at each node is a homogeneous function of the current prices $(p(\xi), q(\xi))$. Since by assumption A3 good 1 will always have a positive spot price a convenient normalisation is obtained by using good 1 as the numeraire at each node

$$p(\xi, 1) = 1 \quad \forall\, \xi \in \mathbf{D} \tag{6.1}$$

Let $A(\xi', 1) = (A(\xi', 1, j \in J(\xi))$ denote the vector of commodity 1 payoffs for the $j(\xi)$ securities issued at node ξ. Then the budget set of agent i is given by

$$\mathcal{B}_\infty(p, q, \pi^i, \omega^i, A) = \left\{ x^i \in \ell_\infty^+(\mathbf{D} \times \mathbf{L}) \;\middle|\; \begin{array}{l} \exists\, z^i \in Z \text{ such that } \forall\, \xi \in \mathbf{D} \\ \lim\limits_{T \longrightarrow \infty} \sum\limits_{\xi' \in \mathbf{D}_T(\xi)} \pi^i(\xi') q(\xi') z^i(\xi') = 0 \\ p(\xi)(x^i(\xi) - \omega^i(\xi)) = A(\xi, 1) z^i(\xi^-) - q(\xi) z^i(\xi) \end{array} \right\}$$

with the obvious modification for the budget set $\mathcal{B}_T(\cdot)$ for the T-truncated economy. With short-lived numeraire securities, redundant securities can be removed without changing the exchange opportunities of the agents. We may thus assume, without loss of generality, that the returns on the securities $j \in J(\xi)$ are linearly independent i.e.

$$\text{rank}\ [A(\xi', 1, j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = j(\xi) \leqq b(\xi) \tag{6.2}$$

The main steps in the proof are as follows. By Theorem 5.3 the truncated economy $\mathcal{E}_T$ has a GEI equilibrium $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ for every $T \in \mathbf{T}$. The first step consists in establishing uniform bounds (in T) on the truncated equilibria. The second step is to take an appropriate limit of these equilibria. The final step is to show that this limit is an equilibrium for $\mathcal{E}_\infty$.

Step 1: Uniform bounds. By Theorem 5.3 for every $T \in \mathbf{T}$ there exists a GEI equilibrium $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ for the truncated economy $\mathcal{E}_T$. The spot prices may be normalised by setting $p_T(\xi, 1) = 1, \quad \forall\, \xi \in \mathbf{D}^T$. For each $i \in \mathbf{I}$ let $(\bar{\pi}^i_T, \bar{P}^i_T)$ denote the present value vector and the vector of discounted prices of agent i defined in Proposition 5.2. Since the relations satisfied by $(\bar{\pi}^i_T, \bar{P}^i_T)$ are homogeneous we may normalise $\bar{P}^i_T$ by setting

$$\bar{P}^i_T \mathbb{1} \;=\; \sum_{(\xi,\ell)\in \mathbf{D}^T \times \mathbf{L}} \bar{P}^i_T(\xi,\ell) \;=\; 1, \quad \forall\, i \in \mathbf{I}, \quad \forall\, T \in \mathbf{T} \tag{6.3}$$

where $\mathbb{1} = (1, \ldots, 1, \ldots) \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ denotes the vector all of whose components are equal to 1. Since the securities are short-lived numeraire assets, the adjoint equations for agent i in Proposition 5.2(c) simplify to

$$\bar{\pi}^i_T(\xi)\bar{q}_T(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i_T(\xi') A(\xi', 1, j), \quad \forall\, j \in J(\xi), \quad \forall\, \xi \in \mathbf{D}^{T-1} \tag{6.4}$$

We shall now find bounds independent of T for $(\bar{x}^i_T(\xi,\ell), \bar{z}^i_T(\xi,j), \bar{\pi}^i_T(\xi))$ and $(\bar{p}_T(\xi,\ell), \bar{q}_T(\xi,j))$, $\forall\, i \in \mathbf{I}, \quad \forall\, (\xi, \ell, j) \in \mathbf{D} \times \mathbf{L} \times \mathbf{J}$.

(i) *bounds on* $\bar{x}^i_T(\xi,\ell)$: since $\bar{x}^i_T(\xi,\ell) \geqq 0$ and $\sum_{i\in\mathbf{I}} \bar{x}^i_T(\xi,\ell) = \sum_{i\in\mathbf{I}} \omega^i(\xi,\ell) \leqq M$, $0 \leq \bar{x}^i_T(\xi,\ell) \leqq M$, $\forall\, (\xi,\ell) \in \mathbf{D} \times \mathbf{L}$, $\forall\, i \in \mathbf{I}$.

(ii) *bounds on* $\bar{\pi}^i_T(\xi)$ *and* $\bar{p}_T(\xi,\ell)$: It suffices to consider $T \geqq t(\xi)$ since for $T > t(\xi)$, $\bar{\pi}^i_T(\xi) = 0 \quad \forall\ i \in \mathbf{I}$ and $\bar{p}_T(\xi,\ell) = 0, \quad \forall \ell \in \mathbf{L}$. Since $\bar{P}^i_T(\xi,1) = \bar{\pi}^i_T(\xi)$ and $\sum_{(\xi,\ell)\in\mathbf{D}^T\times\mathbf{L}} \bar{P}^i_T(\xi,\ell) = 1$, $0 \leqq \bar{\pi}^i_T(\xi) \leqq 1$. Let us show that $\bar{\pi}^i_T(\xi)$ is uniformly positive for $T \geqq t(\xi)$. This is a consequence of the continuity and strict monotonicity of the agents' preferences which imply the following property.

Lemma 6.2: *For each* $\xi \in \mathbf{D}$ *there exists* $\alpha_\xi < 1$ *such that* $\forall\, i \in \mathbf{I}$

$$\alpha_\xi x^i + e^\xi_1 \succ_i x^i, \quad \forall\, x^i \in F$$

Proof: (see appendix).

By Proposition 5.2 $\bar{x}^i_T$ is $\succsim_i$ maximal in $B_T(\bar{P}^i_T, \omega^i)$. Consider scaling down agent i's consumption up to date T to $\alpha_\xi \bar{x}^i_T$. This would free the income $(1-\alpha_\xi)\bar{P}^i_T \bar{x}^i_T = (1-\alpha_\xi)\bar{P}^i_T \omega^i \geqq (1-\alpha_\xi) m$ which could be converted into $\frac{(1-\alpha_\xi)m}{\bar{P}^i_T(\xi,1)}$ units of good 1 at node ξ. By Lemma 6.2 we must have for

$T \geqq t(\xi)$

$$\frac{(1-\alpha_\xi)m}{\bar{P}^i_T(\xi,1)} \leqq 1 \iff \bar{P}^i_T(\xi,1) \geqq (1-\alpha_\xi)m \iff \bar{\pi}^i_T(\xi) \geqq (1-\alpha_\xi)m \tag{6.5}$$

since otherwise the new consumption would be preferred to $\bar{x}^i_T$, contradicting the optimality of $\bar{x}^i_T$ in $B_T(\bar{P}^i_T, \omega^i)$.

Since $0 \leqq \bar{P}^i_T(\xi,\ell) = \bar{\pi}^i_T(\xi)\bar{p}_T(\xi,\ell) \leqq 1$, (6.5) implies

$$0 \leqq \bar{p}_T(\xi,\ell) \leqq \frac{1}{(1-\alpha_\xi)m}, \quad \forall\, \ell \in \mathbf{L} \tag{6.6}$$

(iii) *bounds on* $\bar{q}_T(\xi,j)$: since $\bar{q}_T(\xi) = 0$ for $T \leqq t(\xi)$ it suffices to consider $T > t(\xi)$. Since $\sum_{i\in\mathbf{I}} \bar{z}^i_T(\xi) = 0 \implies \sum_{i\in\mathbf{I}} \bar{q}_T(\xi)\bar{z}^i_T(\xi) = 0$ there exists at least one agent $i \in \mathbf{I}$ with $\bar{q}_T(\xi)\bar{z}^i_T(\xi) \geqq 0$. Consider the following change in the portfolio of this agent: he scales down the portfolio $\bar{z}^i_T$ from node ξ onwards

$$\bar{z}^i_T \longrightarrow \begin{cases} \bar{z}^i_T(\xi') & \forall\, \xi' \notin \mathbf{D}^+(\xi) \\ \beta\bar{z}^i_T(\xi') & \forall\, \xi' \in \mathbf{D}^+(\xi) \end{cases} \tag{6.7}$$

where $\beta < 1$ is the factor defined by A4. Agent i can still consume $\omega^i(\xi')$ if $t(\xi') > T$, $\bar{x}^i_T(\xi')$ if $\xi' \in \mathbf{D}\backslash\mathbf{D}^+(\xi)$ and $\beta\bar{x}^i_T(\xi')$ if $\xi' \in \mathbf{D}^+(\xi)$ with $t(\xi') \leq T$ since

$$\bar{p}_T(\xi)\omega^i(\xi) + A(\xi,1)\bar{z}^i_T(\xi^-) - \beta\bar{q}_T(\xi)\bar{z}^i_T(\xi) \geqq \bar{p}_T(\xi)\bar{x}^i_T(\xi)$$

and for all $\xi' \in \mathbf{D}^+(\xi)$ with $t(\xi') \leqq T$

$$\bar{p}_T(\xi')\omega^i(\xi') + A(\xi',1)\beta\bar{z}^i_T(\xi'^-) - \bar{q}_T(\xi')\beta\bar{z}^i_T(\xi') = \beta\bar{p}_T(\xi')\bar{x}^i_T(\xi') + (1-\beta)\bar{p}_T(\xi')\omega^i(\xi')$$

This change frees the income

$$(1-\beta)\bar{p}_T(\xi')\omega^i(\xi') \geqq (1-\beta)m$$

at each node $\xi' \in \mathbf{D}^+(\xi)$ with $t(\xi') \leqq T$ and in particular at each successor $\xi' \in \xi^+$. By going short (i.e. borrowing) $(1-\beta)m$ units of the numeraire bond j_ξ (which exists at each node ξ by A6) agent i can then increase his consumption of good 1 at node ξ by at least $\bar{q}_T(\xi,j_\xi)(1-\beta)m$. By A4 we must have

$$\bar{q}_T(\xi,j_\xi)(1-\beta)m \leqq 1 \iff \bar{q}_T(\xi,j_\xi) \leqq \frac{1}{(1-\beta)m} \tag{6.8}$$

since otherwise the new consumption would be preferred to $\bar{x}^i_T$, contradicting the optimality of $\bar{x}^i_T$ in $\mathcal{B}_T(\bar{p}_T, \bar{q}_T, \omega^i, A)$. (6.4) applied with $j = j_\xi$ and (6.8) then imply

$$\bar{q}_T(\xi, j_\xi) \;=\; \frac{\sum\limits_{\xi' \in \xi^+} \bar{\pi}^i_T(\xi')}{\bar{\pi}^i_T(\xi)} \;\leqq\; \frac{1}{(1-\beta)m}$$

Reapplying (6.4) to all $j \in J(\xi)$ gives

$$|\,\bar{q}_T(\xi, j)\,| \;\leqq\; \sum_{\xi' \in \xi^+} \frac{\bar{\pi}^i_T(\xi')}{\bar{\pi}^i_T(\xi)}\,|\,A(\xi', 1, j)\,| \;\leqq\; \frac{\delta(\xi)}{(1-\beta)m} \tag{6.9}$$

where

$$\delta(\xi) \;=\; \max\ \{\,|A(\xi', 1, j)|,\ \xi' \in \xi^+,\ j \in J(\xi)\,\}$$

(iv) *bounds on* $\bar{q}_T(\xi)\bar{z}^i_T(\xi)$: As before let $T > t(\xi)$. Consider an agent who is a net lender at node ξ i.e. $\bar{q}_T(\xi)\bar{z}^i_T(\xi) \geqq 0$. This agent can consider scaling down his portfolio as in (6.7): he can then consume at least $\beta \bar{x}^i_T(\xi')$ for $\xi' \in \mathbf{D}^+(\xi)$ and increase his consumption of good 1 at node ξ by $(1-\beta)\bar{q}_T(\xi)\bar{z}^i_T(\xi)$. By A4 this increase must be less than 1 so that

$$\bar{q}_T(\xi)\bar{z}^i_T(\xi) \;\geqq\; 0 \quad\Longrightarrow\quad \bar{q}_T(\xi)\bar{z}^i_T(\xi) \;\leqq\; \frac{1}{1-\beta}$$

Since $\sum\limits_{i \in \mathbf{I}} \bar{q}_T(\xi)\bar{z}^i_T(\xi) = 0$ agents who are net borrowers must find net lenders. Thus

$$\bar{q}_T(\xi)\bar{z}^i_T(\xi) \;\leqq\; 0 \quad\Longrightarrow\quad -\left(\frac{I-1}{1-\beta}\right) \;\leqq\; \bar{q}_T(\xi)\bar{z}^i_T(\xi)$$

so that

$$-\left(\frac{I-1}{1-\beta}\right) \;\leqq\; \bar{q}_T(\xi)\bar{z}^i_T(\xi) \;\leqq\; \frac{1}{1-\beta}, \quad \forall\, i \in \mathbf{I} \tag{6.10}$$

Note that these bounds do not depend on ξ.

(v) *bounds on* $\bar{z}^i_T(\xi, j)$: Let $T > t(\xi)$. For all $\xi' \in \xi^+$

$$A(\xi', 1)\bar{z}^i_T(\xi) = \bar{p}_T(\xi')\bar{x}^i_T(\xi') - \bar{p}_T(\xi')\omega^i(\xi') + \bar{q}_T(\xi')\bar{z}^i_T(\xi')$$

The inequalities $0 \leqq \bar{p}_T(\xi')\bar{x}^i_T(\xi') \leqq \frac{LM}{(1-\alpha_{\xi'})m}$, $0 \leqq \bar{p}_T(\xi')\omega^i(\xi') \leqq \frac{LM}{(1-\alpha_{\xi'})m}$ and (6.10) imply for each $\xi' \in \xi^+$

$$-\frac{LM}{(1-\alpha_{\xi'})m} - \frac{(I-1)}{1-\beta} \;\leqq\; A(\xi', 1)\bar{z}^i_T(\xi) \;\leqq\; \frac{LM}{(1-\alpha_{\xi'})m} + \frac{1}{1-\beta} \tag{6.11}$$

Since by (6.2) there are no redundant securities, (6.11) bounds $\bar{z}^i_T(\xi)$. This can be seen as follows. Let $\bar{u} \in \mathbb{R}^{b(\xi)}$ be defined by $\bar{u}(\xi') = A(\xi', 1)\bar{z}^i_T(\xi)$. Consider the system of equations

$$\sum_{j \in J(\xi)} A(\xi', 1, j) z(\xi, j) = \bar{u}(\xi'), \quad \xi' \in \xi^+$$

By (6.2) there exists a $j(\xi) \times j(\xi)$ determinant in $[\, A(\xi', 1, j)\,]_{\xi' \in \xi^+,\ j \in J(\xi)}$ which is not zero. Keeping the corresponding equations and applying Cramer's formula to compute $(\bar{z}^i_T(\xi, j),\ j \in J(\xi))$ shows that there exists a bound $\gamma(\xi)$ such that

$$|\bar{z}^i_T(\xi, j)| < \gamma(\xi), \quad j \in J(\xi)$$

Step 2: limits. For a convenient summary of the notation and results from functional analysis that we use in the rest of the proof the reader is referred to the mathematical appendix of Bewley (1972). Let $ba(\mathbf{D} \times \mathbf{L}) = \ell^*_\infty(\mathbf{D} \times \mathbf{L})$ denote the norm dual of $\ell_\infty(\mathbf{D} \times \mathbf{L})$ consisting of bounded finitely additive set functions on $\mathbf{D} \times \mathbf{L}$ and let $\| \cdot \|_{ba}$ denote the norm of $ba(\mathbf{D} \times \mathbf{L})$. The prices $(\bar{P}^i_T, T \in \mathbf{T})_{i \in \mathbf{I}}$ can be viewed as elements of $ba(\mathbf{D} \times \mathbf{L})$.

Let $\sigma(ba, \ell_\infty)$ denote the weak* topology of ba. Since $\bar{P}^i_T \mathbb{1} = \| \bar{P}^i_T \|_{ba} = 1 \quad \forall\, T \in \mathbf{T}, \quad \forall\, i \in \mathbf{I}$ and since by Alaoglu's theorem the unit sphere in $ba(\mathbf{D} \times \mathbf{L})$ is $\sigma(ba, \ell_\infty)$ compact there exists a directed set $(\Lambda, \geqq)$ and a subnet $\{(\bar{P}^i_{T_\lambda}, i \in \mathbf{I}), \lambda \in (\Lambda, \geqq)\}$ such that $\bar{P}^i_{T_\lambda}$ converges to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology, $\forall\, i \in \mathbf{I}$.

Let $Y = \mathbb{R}^{\mathbf{D} \times \mathbf{L} \times \mathbf{I}} \times \mathbb{R}^{\mathbf{D} \times \mathbf{J} \times \mathbf{I}} \times \mathbb{R}^{\mathbf{D} \times \mathbf{L}} \times \mathbb{R}^{\mathbf{D} \times \mathbf{J}} \times \times \mathbb{R}^{\mathbf{D} \times \mathbf{I}}$. In view of the bounds established in (i)-(v) of step 1 it follows from Tychonov's theorem (Dunford-Schwartz (1966), p. 32) that there exists a set $K \subset Y$ which is compact in the product topology on Y such that $\{(\bar{x}_T, \bar{z}_T, \bar{p}_T, \bar{q}_T, (\bar{\pi}^i_T)_{i \in \mathbf{I}}\}_{T \in \mathbf{T}} \subset K$. Thus by extracting an appropriate subnet, $((\bar{x}_{T_\lambda}, \bar{z}_{T_\lambda}, \bar{p}_{T_\lambda}, \bar{q}_{T_\lambda}, (\bar{\pi}^i_{T_\lambda})_{i \in \mathbf{I}})$ converges to $(\bar{x}, \bar{z}, \bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}})$ in the product topology. By Theorem 9 (p. 292) and Theorem 1 (p. 430) of Dunford-Schwartz (1966) the Mackey topology and product topology coincide on bounded subsets of $\ell_\infty(\mathbf{D} \times \mathbf{L})$. Since $x^i_{T_\lambda} \in F$, $\forall\, T_\lambda$, $\forall\, i \in \mathbf{I}$ it follows that $\bar{x}^i_{T_\lambda}$ converges to $\bar{x}^i$ in the Mackey topology $\forall\, i \in \mathbf{I}$.

Step 3: limit is an equilibrium. We begin by showing that $\bar{x}^i$ is $\succsim_i$ maximal in agent i's induced Arrow-Debreu budget set

$$B_\infty(\bar{P}^i, \omega^i) = \{\, x^i \in \ell^+_\infty(\mathbf{D} \times \mathbf{L}) \mid \bar{P}^i(x^i - \omega^i) \leqq 0 \,\}$$

To this end we first show that

$$x^i \in \ell^+_\infty(\mathbf{D} \times \mathbf{L}),\ x^i \succsim_i \bar{x}^i \implies \bar{P}^i x^i \geqq \bar{P}^i \omega^i \tag{6.12}$$

If $x^i \succsim_i \bar{x}^i$ then for any $\epsilon > 0$, $x^i + \epsilon \mathbb{1} \succ_i \bar{x}^i$. Since $\chi_{\mathbf{D} \setminus \mathbf{D}^T}$ converges to zero in the Mackey topology

$$(x^i + \epsilon \mathbb{1})\chi_{\mathbf{D}^T} + \omega^i \chi_{\mathbf{D}\setminus\mathbf{D}^T} \longrightarrow x^i + \epsilon \mathbb{1}$$

in the Mackey topology. Since $\bar{x}^i_{T_\lambda}$ converges to $\bar{x}^i$ (Mackey) there exists $\bar{\lambda} \in \Lambda$ such that $\lambda > \bar{\lambda}$ implies

$$(x^i + \epsilon \mathbb{1})\chi_{\mathbf{D}^{T_\lambda}} + \omega^i \chi_{\mathbf{D}\setminus\mathbf{D}^{T_\lambda}} \succ_i \bar{x}^i_{T_\lambda}$$

The consumption vector on the left side of this relation could be considered by agent i in the economy $\mathcal{E}_{T_\lambda}$ since it coincides with ω^i after date T_λ. Since by Proposition 5.2 $\bar{x}^i_{T_\lambda}$ is $\succsim_i$ maximal in $B_{T_\lambda}(\bar{P}^i_{T_\lambda}, \omega^i)$ we must have

$$\bar{P}^i_{T_\lambda} x^i + \epsilon > \bar{P}^i_{T_\lambda} \bar{x}^i_{T_\lambda} = \bar{P}^i_{T_\lambda} \omega^i, \quad \forall\, \lambda > \bar{\lambda}$$

Since $\bar{P}^i_{T_\lambda}$ converges to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology and since $x^i \in \ell^+_\infty(\mathbf{D} \times \mathbf{L})$,

$$\bar{P}^i x^i + \epsilon \geqq \bar{P}^i \omega^i$$

Letting $\epsilon \longrightarrow 0$ gives (6.12). It is now easy to show that

$$x^i \in \ell^+_\infty(\mathbf{D} \times \mathbf{L}),\ x^i \succ_i \bar{x}^i \implies \bar{P}^i x^i > \bar{P}^i \omega^i \tag{6.13}$$

For suppose $x^i \succ_i \bar{x}^i$ then by continuity of $\succsim_i$ there exists $\alpha < 1$ such that $\alpha x^i \succ_i \bar{x}^i$. By (6.12), $\alpha \bar{P}^i x^i \geqq \bar{P}^i \omega^i \implies \bar{P}^i x^i > \bar{P}^i \omega^i$.

Since for each $\xi \in \mathbf{D}$

$$p_{T_\lambda}(\xi)(\bar{x}^i_{T_\lambda}(\xi) - \omega^i(\xi)) = A(\xi, 1)\bar{z}^i_{T_\lambda}(\xi^-) - \bar{q}_{T_\lambda}(\xi)\bar{z}^i_{T_\lambda}(\xi)$$

involves only a finite number of terms, the equation is satisfied in the limit. Thus $(\bar{x}^i, \bar{z}^i)$ satisfy the budget equations

$$\bar{p}(\xi)(\bar{x}^i(\xi) - \omega^i(\xi)) = A(\xi, 1)\bar{z}^i(\xi^-) - \bar{q}(\xi)\bar{z}^i(\xi), \quad \forall\, \xi \in \mathbf{D} \tag{6.14}$$

For the same reason the adjoint equations for agent i

$$\bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')A(\xi', 1, j),\ j \in J(\xi), \quad \forall\, \xi \in \mathbf{D} \tag{6.15}$$

are satisfied in the limit.

Since $\bar{P}^i \in ba(\mathbf{D}\times\mathbf{L})$ and $\bar{P}^i \geqq 0$, it follows from the Yosida-Hewitt theorem that there exists a unique decomposition $\bar{P}^i = \bar{P}^i_c + \bar{P}^i_f$ where $\bar{P}^i_c \in \ell_1^+(\mathbf{D}\times\mathbf{L})$ and $\bar{P}^i_f$ is a non-negative purely finitely additive measure (sometimes called a pure charge). Furthermore $\bar{P}^i_f y = 0$ whenever $y \in \ell_\infty(\mathbf{D}\times\mathbf{L})$ has only a finite number of non-zero components. Since

$$\bar{P}^i_{T_\lambda} e^\xi_\ell = \bar{P}^i_{T_\lambda}(\xi,\ell) = \bar{\pi}^i_{T_\lambda}(\xi)\bar{p}_{T_\lambda}(\xi,\ell), \quad \forall\ (\xi,\ell) \in \mathbf{D}\times\mathbf{L}$$

passing to the limit gives

$$\bar{P}^i e^\xi_\ell = \bar{P}^i_c(\xi,\ell) = \bar{\pi}^i(\xi)\bar{p}(\xi,\ell), \quad \forall\ (\xi,\ell) \in \mathbf{D}\times\mathbf{L}$$

Multiplying the budget equation (6.14) for node ξ by $\bar{P}^i_c(\xi,1) = \bar{\pi}^i(\xi)$, adding the resulting equations for all nodes ξ with $t(\xi) \leqq T$ and using the adjoint equations (6.15) gives

$$\sum_{\xi\in\mathbf{D}^T} \bar{P}^i_c(\xi)(\bar{x}^i(\xi) - \omega^i(\xi)) = -\sum_{\xi\in\mathbf{D}_T} \bar{\pi}^i(\xi)\bar{q}(\xi)\bar{z}^i\xi) \tag{6.16}$$

By (6.10), $\bar{q}_{T_\lambda}(\xi)\bar{z}^i_{T_\lambda}(\xi)$ is bounded uniformly in T_λ and ξ so that $(\bar{q}\bar{z}^i) = (\bar{q}(\xi)\bar{z}^i(\xi), \xi \in \mathbf{D}) \in \ell_\infty(\mathbf{D})$. Since $\sum_{\xi\in\mathbf{D}} \bar{\pi}^i(\xi) = \sum_{\xi\in\mathbf{D}} \bar{P}^i_c(\ \xi,1) \leqq \sum_{(\xi,\ell)\in\mathbf{D}\times\mathbf{L}} \bar{P}^i_c(\xi,\ell) \leqq 1$ implies $\bar{\pi}^i \in \ell_1(\mathbf{D})$, the term on the right side of (6.16) tends to zero in the limit and

$$\bar{P}^i_c(\bar{x}^i - \omega^i) = 0 \tag{6.17}$$

Suppose $\bar{P}^i_f > 0$. Then $\omega^i \geqq m\mathbb{1}$ implies $\bar{P}^i_f\omega^i > 0$. By (6.17) $\bar{P}^i_c\bar{x}^i = \bar{P}^i_c\omega^i < \bar{P}^i\omega^i$. By the strict monotonicity and Mackey continuity of $\underset{i}{\succsim}$ for all $\alpha > 0$ there exists $T > 0$ such that

$$(\bar{x}^i + \alpha\mathbb{1})\chi_{\mathbf{D}^T} \underset{i}{\succ} \bar{x}^i \tag{6.18}$$

Choose $0 < \alpha \leqq \bar{P}^i_f\omega^i$ then

$$\bar{P}^i(\bar{x}^i + \alpha\mathbb{1})\chi_{\mathbf{D}^T} \leqq \bar{P}^i_c(\bar{x}^i + \alpha\mathbb{1}) \leqq \bar{P}^i_c\bar{x}^i + \alpha \leqq \bar{P}^i\omega^i \tag{6.19}$$

(6.18) and (6.19) contradict (6.13). Thus $\bar{P}^i_f = 0$ so that $\bar{P}^i = \bar{P}^i_c$ and by (6.17), $\bar{P}^i(\bar{x}^i - \omega^i) = 0$. By (6.13) $\bar{x}^i$ is $\underset{i}{\succsim}$ maximal in the agent's induced Arrow-Debreu budget set $B_\infty(\bar{P}^i, \omega^i)$ and $(\bar{\pi}^i, \bar{P}^i)$ satisfy (a)-(c) in Definition 4.1(ii).

Let us show that $(\bar{x}^i; \bar{z}^i)$ is $\underset{i}{\succsim}$ maximal in the GEI budget set $\mathcal{B}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$. $(\bar{q}, \bar{z}) \in \ell_\infty(\mathbf{D})$ and $\bar{\pi}^i \in \ell_1(\mathbf{D})$ implies $\lim_{T\longrightarrow\infty} \sum_{\xi'\in\mathbf{D}(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi')\bar{z}^i(\xi') = 0,\ \forall\ \xi \in \mathbf{D}$. Since the budget equations (6.14) are satisfied, $\bar{z}^i$ finances $\bar{x}^i$ and $\bar{x}^i \in \mathcal{B}_\infty(\bar{p},\bar{q},\bar{\pi}^i,\omega^i,A)$. Since for any $(x^i;z^i) \in \mathcal{B}_\infty(\bar{p},\bar{q},\bar{\pi}^i,\omega^i,A)$, $\lim_{T\longrightarrow\infty} \sum_{\xi\in\mathbf{D}_T} \bar{\pi}^i(\xi)\bar{q}(\xi)z^i(\xi) = 0$ replacing $(\bar{x}^i,\bar{z}^i)$ in (6.16) by (x^i,z^i) gives $\bar{P}^i(x^i-\omega^i) = 0$ so that

$$\mathcal{B}_\infty(\bar{p},\bar{q},\bar{\pi}^i,\omega^i,A) \subset B_\infty(\bar{P}^i,\omega^i), \quad \forall\, i \in \mathbf{I}$$

Thus $\bar{x}^i$ $\succsim_i$ maximal in $B_\infty(\bar{P}^i,\omega^i)$ is also $\succsim_i$ maximal in $\mathcal{B}_\infty(\bar{p},\bar{q},\bar{\pi}^i,\omega^i,A)$, $\forall\; i \in \mathbf{I}$.

Since in the limit

$$\sum_{i\in\mathbf{I}}(\bar{x}^i-\omega^i) = 0 \quad \text{and} \quad \sum_{i\in\mathbf{I}}\bar{z}^i = 0$$

the limit $((\bar{x},\bar{z}),(\bar{p},\bar{q},(\bar{\pi}^i)_{i\in\mathbf{I}})$ is an equilibrium of the economy $\mathcal{E}(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta,A))$ and the proof is complete. $\triangle$

7. Existence of Equilibrium for General Security Structures

In this section we draw on Theorem 5.6 and the limit arguments of the previous section to show that an infinite horizon economy with a general security structure has a pseudoequilibrium (Theorem 7.2). As in the finite horizon case, a pseudoequilibrium corresponding to an artificial asset structure is a GEI equilibrium if the subspace of income transfers generated by the original security structure has maximal dimension at each node. In the finite horizon case it has been shown (Duffie-Shafer (1986)) that this condition (equation (5.7)) is verified for all pseudoequilibria of an open dense set of economies parametrised by endowments and asset payoffs. In the infinite horizon case we establish (Theorem 7.4) a weaker result: for given characteristics $(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta))$ there exists a dense set of asset payoffs $\mathcal{A}^*$ such that for all $A \in \mathcal{A}^*$ the economy $\mathcal{E}_\infty(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta,A))$ has a GEI equilibrium.

The Definition 5.5 of a pseudoequilibrium for a finite horizon economy can be extended in the natural way to an infinite horizon economy.

Definition 7.1: Let $(\mathbf{K},\eta,\Gamma)$ be an artificial short-lived numeraire asset structure for the economy $\mathcal{E}_\infty(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta,A))$ as in Definition 5.4 and let $((\bar{x},\bar{\gamma}),(\bar{p},\bar{\rho},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ be a GEI equilibrium for $\mathcal{E}_\infty(\mathbf{D},\succsim,\omega,(\mathbf{K},\eta,\Gamma))$. We say that $((\bar{x},\bar{\gamma}),(\bar{p},\bar{\rho},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a *short-lived numeraire asset pseudoequilibrium* of the economy $\mathcal{E}_\infty(\mathbf{D},\succsim,\omega,(\mathbf{J},\zeta,A))$ if there exists $\bar{q} \in \mathbb{R}^{\mathbf{D}\times\mathbf{J}}$ such that

$$\bar{q}(\xi,j) = \frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi'\in\mathbf{D}^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi',j), \quad \forall\; j \in J(\xi), \quad \forall\; \xi \in \mathbf{D}, \quad \forall i \in \mathbf{I} \tag{7.1}$$

and

$$\left\langle \left[\bar{p}(\xi')A(\xi',j) + \bar{q}(\xi',j)\right]_{\substack{\xi'\in\xi^+ \\ j\in J(\xi)}} \right\rangle \subset \left\langle \left[\Gamma(\xi',1,k)\right]_{\substack{\xi'\in\xi^+ \\ k\in K(\xi)}} \right\rangle, \quad \forall\, \xi \in \mathbf{D} \tag{7.2}$$

Theorem 7.2: *Under assumptions A1-A6, for any infinite horizon economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ there exists an asset structure $(\mathbf{K}, \eta, \Gamma)$ satisfying (i) - (iii) in Definition 5.4 such that a GEI equilibrium $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i\in\mathbf{I}})$ of $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{K}, \eta, \Gamma))$ is a short-lived numeraire asset pseudoequilibrium of $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$.*

Proof: By Theorem 5.6 for every $T \in \mathbf{T}$ there exists an asset structure $(\mathbf{K}, \eta, \Gamma_T)$ satisfying (i) - (iii) in Definition 5.4 and a GEI equilibrium $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ of the artificial economy $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{K}, \eta, \Gamma_T))$ which is a short-lived numeraire asset pseudoequilibrium of $\mathcal{E}_T(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$. Let $\bar{q}_T$ denote the prices of the original securities defined by equation (5.5) and let $(\bar{\pi}^i_T, \bar{P}^i_T)_{i\in\mathbf{I}}$ denote the present value vectors and discounted prices of the agents associated with the equilibrium (Proposition 5.2). In view of the remark following Definition 5.5 for each $i \in \mathbf{I}$

$$\bar{\pi}^i_T(\xi)\bar{q}_T(\xi,j) \;=\; \sum_{\xi'\in\mathbf{D}^+(\xi)} \bar{\pi}^i_T(\xi')\bar{p}_T(\xi')A(\xi',j), \quad \forall\, j \in J(\xi), \quad \forall\, \xi \in \mathbf{D}^{T-1}$$

which can be written as

$$\bar{q}_T(\xi,j) = \frac{1}{\bar{P}^i_T(\xi,1)}\bar{P}^i_T A(\cdot,j)\chi_{\mathbf{D}^+(\xi)}, \quad \forall\, j \in J(\xi), \quad \forall\, \xi \in \mathbf{D}^{T-1} \tag{7.3}$$

since $\bar{\pi}^i_T(\xi) = \bar{P}^i_T(\xi,1) > 0$ if $\xi \in \mathbf{D}^{T-1}$ and $\bar{P}^i_T(\xi) = 0$ if $t(\xi) > T$. Applying the limit argument in the proof of Theorem 6.1 shows that there exists a directed set $(\Lambda, \geqq)$ such that $(\bar{x}_{T_\lambda}, \bar{\gamma}_{T_\lambda}, \bar{p}_{T_\lambda}, \bar{\rho}_{T_\lambda}, \Gamma_{T_\lambda}, (\bar{\pi}^i_{T_\lambda})_{i\in\mathbf{I}})$ converges to $(\bar{x}, \bar{\gamma}, \bar{p}, \bar{\rho}, \Gamma, (\bar{\pi}^i)_{i\in\mathbf{I}})$ in the product topology and $\bar{P}^i_{T_\lambda}$ converges to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology for all $i \in \mathbf{I}$. By step 3 of the proof of Theorem 6.1, $\bar{P}^i \in \ell_1^+(\mathbf{D}\times\mathbf{L})$ and $\bar{P}^i(\xi,\ell) = \bar{\pi}^i(\xi)\bar{p}(\xi,\ell)\;\; \forall\, (\xi,\ell) \in \mathbf{D}\times\mathbf{L}$. By (6.5) $\bar{\pi}^i_{T_\lambda}(\xi) \longrightarrow \bar{\pi}^i(\xi) = \bar{P}^i(\xi,1) \geqq (1-\alpha_\xi)m > 0$. Since $A(\cdot,j)\chi_{\mathbf{D}^+(\xi)} \in \ell_\infty(\mathbf{D}\times\mathbf{L})$ and since $\bar{P}^i_{T_\lambda}$ converges to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology.

$$\bar{P}^i_{T_\lambda}A(\cdot,j)\chi_{\mathbf{D}^+(\xi)} \longrightarrow \bar{P}^i A(\cdot,j)\chi_{\mathbf{D}^+(\xi)}$$

Thus (7.3) implies

$$\bar{q}_{T_\lambda}(\xi,j) \longrightarrow \frac{1}{\bar{P}^i(\xi,1)}\bar{P}^i A(\cdot,j)\chi_{\mathbf{D}^+(\xi)}, \quad \forall\, i \in \mathbf{I}, \quad \forall\, j \in J(\xi), \quad \forall\, \xi \in \mathbf{D}$$

Let $\bar{q}(\xi, j)$ denote this limit. Then $\bar{q} = (\bar{q}(\xi, j), \xi \in \mathbf{D}, j \in J(\xi))$ satisfies (7.1). Passing to the limit in (5.6) in the obvious way gives (7.2) and the proof is complete. $\triangle$

We now show that for given characteristics $(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta))$ there is a dense set $\mathcal{A}^*$ of asset payoffs such that for all $A \in \mathcal{A}^*$ every pseudoequilibrium of $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ is a GEI equilibrium. To this end we define the following space of commodity payoffs satisfying assumptions A5 and A6.

Definition 7.3: Let $(\mathbf{J}, \zeta, (J(\xi), \xi \in \mathbf{D}))$ be a countable set $\mathbf{J}$ of securities with nodes of issue $\zeta = (\xi(j), j \in \mathbf{J})$ and with active securities at node ξ given by a finite subset $J(\xi) \subset \mathbf{J}$ at each node, where ζ and $(J(\xi), \xi \in \mathbf{D})$ are compatible in the sense that

(i) $j \in J(\xi) \iff \xi \in \mathbf{D}(\xi(j))$

(ii) $\xi \in \mathbf{D}(\xi(j))$ and $j \notin J(\xi) \implies j \notin J(\xi') \quad \forall\, \xi' \geqq \xi$.

The set of *admissible security payoffs* $\mathcal{A}$ for $(\mathbf{J}, \zeta, (J(\xi), \xi \in \mathbf{D}))$ is the subset of $\ell_\infty(\mathbf{D} \times \mathbf{L} \times \mathbf{J})$ satisfying: $A \in \mathcal{A}$ if for all $\xi \in \mathbf{D}$

(1) $A(\xi, \ell, j) = 0$ if $\xi \notin \mathbf{D}(\xi(j))$

(2) $J_A(\xi) = \{j \in \mathbf{J} | \xi \in \mathbf{D}(\xi(j)),\ \exists\, \xi' \in \mathbf{D}^+(\xi) \text{ with } A(\xi', j) \neq 0\} \subseteq J(\xi)$,

(3) there exists $j_\xi \in J_A(\xi)$ such that $\xi(j_\xi) = \xi$ and $A(\cdot, j_\xi) = e_1 \chi_{\xi^+}$.

Since $\mathcal{A} \subset \ell_\infty(\mathbf{D} \times \mathbf{L} \times \mathbf{J})$ it is natural to endow $\mathcal{A}$ with the norm topology, the norm of a payoff process A being defined by

$$\| A \|_\infty = \sup_{(\xi, \ell, j) \in \mathbf{D} \times \mathbf{L} \times \mathbf{J}} |A(\xi, \ell, j)|$$

Note that $\mathcal{A}$ is a closed subset of $\ell_\infty(\mathbf{D} \times \mathbf{L} \times \mathbf{J})$.

Theorem 7.4: *Under assumptions A1-A6 there exists a dense subset $\mathcal{A}^* \subset \mathcal{A}$ such that if $A \in \mathcal{A}^*$ then the infinite horizon economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ has a GEI equilibrium.*

Proof: (see appendix)

The idea of the proof is to show that if we pick a payoff process $\bar{A} \in \mathcal{A}$ for which the rank condition

$$rank\,[\bar{p}(\xi')\bar{A}(\xi', j) + \bar{q}(\xi', j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = a(\xi),\ \forall\, \xi \in \mathbf{D}$$

is not satisfied in an associated pseudoequilibrium $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in \mathbf{I}})$ then for all $\epsilon > 0$, there exists a payoff process $A \in \mathcal{A}$ in the ball of radius ϵ around $\bar{A}$ such that $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in \mathbf{I}})$ is a pseudoequilibrium for $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ and the rank condition is satisfied for A.

8. Equilibrium Prices of Infinite-lived Securities

In any GEI equilibrium the prices of the securities $\bar{q}$ and each agent's present value vector $\bar{\pi}^i$ must satisfy the agent's adjoint equations

$$\bar{\pi}^i(\xi)\bar{q}(\xi, j) \;=\; \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')(\bar{p}(\xi')A(\xi', j) + \bar{q}(\xi', j)), \;\forall\, \xi \in \mathbf{D}, \quad \forall\, i \in \mathbf{I} \tag{8.1}$$

In a finite horizon economy or as here, in a T-truncated economy $\mathcal{E}_T$, "integrating" these adjoint equations (by successive substitution) and using the terminal condition

$$\bar{q}_T(\xi) \;=\; 0, \quad \forall\, \xi \in \mathbf{D}_T \tag{8.2}$$

gives

$$\bar{q}_T(\xi, j) \;=\; \frac{1}{\bar{\pi}^i_T(\xi)} \sum_{\xi' \in \mathbf{D}^+(\xi)} \bar{\pi}^i_T(\xi')\bar{p}_T(\xi')A(\xi', j), \;\forall\, \xi \in \mathbf{D}, \quad \forall\, i \in \mathbf{I} \tag{8.3}$$

so that the equilibrium price of each security is equal to the present value of its future income stream for each agent. The expression on the right side of (8.3) is called the *fundamental value for agent i* of asset j (at node ξ).

It is evident that in an infinite horizon economy (8.3) holds for any finite-lived asset. However for an infinite-lived security there is no terminal condition (8.2) that can be added to the adjoint equations (8.1) which would force the equilibrium price of the security to equal its fundamental value for each agent. This leads to the following definition.

Definition 8.1: Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in\mathbf{I}}))$ be a GEI equilibrium of the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \xi, A))$. Asset $j \in \mathbf{J}$ is said to be priced at its *fundamental value* if for all agents $i \in \mathbf{I}$

$$\bar{q}(\xi, j) = \frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi' \in \mathbf{D}^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi', j), \quad \forall\, \xi \in \mathbf{D}(\xi(j)) \tag{8.4}$$

Asset j is said to have a *speculative bubble* if for some agent $i \in \mathbf{I}$ (8.4) is not satisfied.

It is a natural consequence of the method used in the previous section to construct equilibria of an infinite horizon economy $\mathcal{E}_\infty$ that in these equilibria all assets are priced at their fundamental values: since in the truncated economies $\mathcal{E}_T$ (8.3) is satisfied, this property is transmitted to the equilibrium prices of the assets in the limit. To what extent are such equilibria typical? Do there exist equilibria in which some of the assets have speculative bubbles? To answer this we need to distinguish between economies in which assets are in zero and in positive net supply.

Let $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$ denote an economy which is identical in all respects to that considered in sections 2-7 except that securities issued at date 0 can have *positive initial supply* $\delta = (\delta_j,\ j \in J(\xi_0))$ where $\delta_j = \sum_{i \in \mathbf{I}} \delta^i_j$ and δ^i_j is agent i's initial holding of security j. (Securities issued after date 0 could also be permitted to be in positive initial supply, but this unnecessarily complicates the notation). A GEI equilibrium of the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$ is given by Definition 4.1 with the following modifications: the new budget set $\mathcal{B}(p, q, \pi^i, \omega^i, \delta^i, A)$ of agent i is identical to that defined in section 4 except for the equation at the initial node which becomes

$$p(\xi_0)(x^i(\xi_0) - \omega^i(\xi_0)) \;=\; q(\xi_0)(\delta^i - z^i(\xi_0)) \tag{8.5}$$

and the market clearing condition (iv) becomes

$$\sum_{i \in \mathbf{I}} \bar{z}^i(\xi, j) \;=\; \delta_j, \quad \forall\, \xi \in \mathbf{D}, \quad \forall\, j \in \mathbf{J} \tag{8.6}$$

where $\delta_j = 0$ if $j \notin J(\xi_0)$. With an economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$ in which assets are in positive initial supply we may associate an economy $\tilde{\mathcal{E}}_\infty(\mathbf{D}, \succsim, \underset{\sim}{\omega}, (\mathbf{J}, \zeta, A))$ in which assets are in zero initial supply and agents have the modified endowments

$$\underset{\sim}{\omega}^i(\xi) = \omega^i(\xi) + \sum_{j \in J(\xi_0)} \delta^i_j A(\xi, j), \quad \forall\, \xi \in \mathbf{D}, \tag{8.7}$$

If $((\bar{x}, \bar{\bar{z}}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$ is a GEI equilibrium of the induced economy $\tilde{\mathcal{E}}_\infty(\mathbf{D}, \succsim, \underset{\sim}{\omega}, (\mathbf{J}, \zeta, A))$ in which every asset is priced at its fundamental value and if for each $i \in \mathbf{I}$

$$\bar{z}^i(\cdot, j) = \bar{\bar{z}}^i(\cdot, j) + \delta^i_j, \quad \forall\, j \in \mathbf{J}$$

where $\delta^i_j = 0$ if $j \notin J(\xi_0)$, then $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$ is a GEI equilibrium of the original economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$. To see this it suffices to check that the two budget sets are the same. The budget equations at each node are clearly the same and if assets are priced at their fundamental values then for all $j \in \mathbf{J}$, $i \in \mathbf{I}$ and $\xi \in \mathbf{D}$

$$\lim_{T \longrightarrow \infty} \sum_{\xi' \in \mathbf{D}_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi', j) \;=\; \lim_{T \longrightarrow \infty} \sum_{\xi' \in \mathbf{D}(\xi) \setminus \mathbf{D}^T(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi', j) = 0$$

since $\bar{P}^i \in \ell_1(\mathbf{D} \times \mathbf{L})$ and $A(\cdot, j) \in \ell_\infty(\mathbf{D} \times \mathbf{L})$. Thus the transversality condition (4.10) is the same if it is expressed in the transformed variables $\tilde{z}^i$ or in the original variables z^i, for all $\xi \in \mathbf{D}$

$$\lim_{T \longrightarrow \infty} \sum_{\xi' \in \mathbf{D}_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi')z^i(\xi') = \lim_{T \longrightarrow \infty} \sum_{\xi' \in \mathbf{D}_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi')(\tilde{z}^i(\xi')+\delta^i) = \lim_{T \longrightarrow \infty} \sum_{\xi' \in \mathbf{D}_T(\xi)} \pi^i(\xi')\bar{q}(\xi')\tilde{z}^i(\xi')$$

To apply Theorem 7.4 to the transformed economy $\tilde{\mathcal{E}}_\infty$ the endowments $\underset{\sim}{\omega}^i$ must be uniformly positive: to ensure this we make the following assumption.

A7: If $\delta_j > 0$ then $\delta_j^i \geqq 0 \quad \forall\, i \in \mathbf{I}$ and $A(\cdot, j) \in \ell_\infty^+(\mathbf{D} \times \mathbf{L})$. If $\delta_j = 0$ then $\delta_j^i = 0 \quad \forall\, i \in \mathbf{I}$.

Securities in positive initial supply are required to have non-negative payoffs. Agents do not inherit any initial debt (or credit) but they can be initial owners of productive assets such as equity contracts of firms or tracts of land which yield rentals (crops) in the future.

Theorem 7.4 applies to economies $\tilde{\mathcal{E}}_\infty(\mathbf{D}, \succsim, \underset{\sim}{\omega}, (\mathbf{J}, \zeta, A))$ parametrised by payoff processes $A \in \mathcal{A}$. To apply the theorem to economies with assets in positive initial supply a perturbation in the agents' initial endowments $\omega = (\omega^1, \ldots, \omega^I)$ is necessary so that the induced endowments $\underset{\sim}{\omega}^i$ in (8.7) stay constant. Let $\Omega = \ell_\infty^{++}(\mathbf{D} \times \mathbf{L} \times \mathbf{I})$ be the space of initial endowments of the agents. The following result is a corollary of Theorem 7.4.

Proposition 8.2: *Under assumptions A1-A7 there exists a dense subset $\Delta \subset \Omega \times \mathcal{A}$ such that if $(\omega, A) \in \Delta$ then the infinite horizon economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$ has a GEI equilibrium in which every security is priced at its fundamental value.*

In the equilibria of Proposition 8.2 no security price has a speculative bubble. We shall now show that this is a typical property of the equilibrium price of any asset in positive net supply but is atypical of infinite-lived assets in zero net supply.

Proposition 8.3: *Under assumptions A1-A5 and A7 in any GEI equilibrium the price of every asset in positive net supply ($\delta_j > 0$) is equal to its fundamental value.*

Proof: This result could be deduced from Santos-Woodford (1992) but a simple proof can be obtained in the present context. The reasoning in step (iv) of the proof of Theorem 6.1 can be applied to any equilibrium $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$ of an economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$ and leads to the inequality

$$-\left(\frac{I-1}{1-\beta}\right) \leqq \bar{q}(\xi)\bar{z}^i(\xi) \leqq \frac{1}{1-\beta}, \quad \forall\, \xi \in \mathbf{D}, \quad \forall\, i \in \mathbf{I}$$

Thus $(\bar{q}\bar{z}^i) = (\bar{q}(\xi)\bar{z}^i(\xi), \xi \in \mathbf{D}) \in \ell_\infty(\mathbf{D})$. Since $\bar{q}(\xi)\sum_{i\in\mathbf{I}} \bar{z}^i(\xi) = \sum_{j\in J(\xi)} \bar{q}(\xi,j)\delta_j$ and since $\delta_j \neq 0$ implies $\delta_j > 0$ and $\bar{q}_j(\xi,j) \geqq 0$ (by A7) for all j such that $\delta_j > 0$, $\bar{q}(\cdot,j) \in \ell_\infty(\mathbf{D})$. By the adjoint equations (8.1) of agent i

$$\bar{\pi}^i(\xi)\bar{q}(\xi,j) = \sum_{\substack{\xi'\in\mathbf{D}^T(\xi)\\ \xi'>\xi}} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi',j) + \sum_{\xi'\in\mathbf{D}_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi')$$

Since $\bar{\pi}^i \in \ell_1(\mathbf{D})$ and $\bar{q}(\cdot,j) \in \ell_\infty(\mathbf{D})$, $\sum_{\xi'\in\mathbf{D}_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi') \longrightarrow 0$ as $T \longrightarrow \infty$, $\forall\, i \in \mathbf{I}$, so that (8.4) holds for each agent. $\triangle$

We shall now show that the equilibrium prices of infinite-lived assets in zero net supply are not tied to their fundamental values. It is always possible to add a bubble component to the equilibrium price of an infinite-lived security so that the resulting price remains an equilibrium price. There is however a striking difference between speculative bubbles in complete and incomplete markets. When financial markets are complete speculative bubbles are *trivial* in the sense that every equilibrium allocation can be supported by security prices which do not involve any speculative bubbles; removing the bubble component in the price of any infinite-lived asset does not alter the span of the markets and hence does not affect the real equilibrium allocation. When markets are incomplete there exist equilibria in which infinite-lived assets have speculative bubbles which are *non-trivial* in the sense that the same equilibrium allocation cannot be obtained if assets are priced at their fundamental values: the bubble components in the prices of the infinite-lived assets affect the span of the markets in such a way that they cannot be removed without altering the real equilibrium allocation.

Note that when the price of a security has a bubble component, the price ceases to be a linear functional on the space of income streams $\ell_\infty(\mathbf{D})$. Thus the bubbles that arise in the GEI model are not the same as the bubbles studied by LeRoy-Gilles (1992) which come from a pure charge component of a continuous linear functional on $\ell_\infty(\mathbf{D})$.

Proposition 8.4: *(a) Let $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, \delta, (\mathbf{J}, \zeta, A))$ be an economy satisfying assumptions A1-A7.*

(i) If $((\bar{x},\bar{z}), (\bar{p},\bar{q},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a GEI equilibrium and if $j \in J(\xi_0)$ is an infinite-lived asset in zero

net supply then there exists a positive solution ρ of the homogeneous system of equations

$$\bar{\pi}^i(\xi)\rho(\xi) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')\rho(\xi'), \quad \forall\, \xi \in \mathbf{D}, \quad \forall\, i \in \mathbf{I} \tag{8.8}$$

and a vector of portfolios $\tilde{z}$ such that $((\bar{x},\tilde{z}),(\bar{p},\tilde{q},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a GEI equilibrium where $\tilde{q}(\cdot,k) = \bar{q}(\cdot,k) + \rho(\cdot)$ if $k = j$ and $\tilde{q}(\cdot,k) = \bar{q}(\cdot,k)$ if $k \neq j$.

(ii) Conversely, if $((\bar{x},\bar{z}),(\bar{p},\bar{q},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a GEI equilibrium and if the financial markets are complete even without the infinite-lived assets in zero net supply then there exists a vector of portfolios and a vector of asset prices $\bar{q}$ under which every asset is priced at its fundamental value such that $((\bar{x},\bar{z}),(\bar{p},\bar{q},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a GEI equilibrium.

(b) There exist equilibria in which financial markets are incomplete and in which some infinite-lived asset has a speculative bubble such that the same real allocation can not be supported by a vector of asset prices under which every asset is priced at its fundamental value.

Proof: (a) (i). By A6 there exists a short lived numeraire bond at node ξ whose equilibrium price $\bar{q}(\xi,j_\xi)$ defines the one-period interest rate $\bar{r}(\xi)$ at node ξ, $\bar{q}(\xi,j_\xi) = \frac{1}{1+\bar{r}(\xi)}, \quad \forall\, \xi \in \mathbf{D}$. Let

$$\rho(\xi) = \alpha \prod_{\xi' \in [\xi_0,\xi^-]} (1+\bar{r}(\xi')), \quad \forall\, \xi \in \mathbf{D}$$

where $[\xi_0,\xi^-]$ denotes the path in $\mathbf{D}$ from ξ_0 to ξ^- and α is a positive constant. $\rho = (\rho(\xi),\xi \in \mathbf{D})$ satisfies the system of equations (8.8) since for all $\xi \in \mathbf{D}$ and for all $i \in \mathbf{I}$

$$\begin{aligned}\sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')\rho(\xi') = \Big(\sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')\Big) \prod_{\xi'' \in [\xi_0,\xi]} (1+\bar{r}(\xi'')) &= \bar{\pi}^i(\xi)\bar{q}(\xi,j_\xi) \prod_{\xi'' \in [\xi_0,\xi]} (1+\bar{r}(\xi'')) \\ = \bar{\pi}^i(\xi) \prod_{\xi'' \in [\xi_0,\xi^-]} (1+\bar{r}(\xi'')) = \bar{\pi}^i(\xi)\rho(\xi)\end{aligned}$$

Thus the asset prices $\tilde{q}(\cdot,k) = \bar{q}(\cdot,k) + \rho(\cdot)$ if $k = j$ and $\tilde{q}(\cdot,k) = \bar{q}(\cdot,k)$ if $k \neq j$ satisfy the adjoint equations (8.1). Let $V(q,\xi^+) = [\bar{p}(\xi')A(\xi',j) + q(\xi',j]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}}$ denote the returns matrix at node ξ and let $< V >$ denote the span of the columns of V. Since $\rho(\xi^+) = (\rho(\xi'),\xi' \in \xi^+)$ is collinear to the vector $(1,\ldots,1)$ and $(1,\ldots,1)^T \in < V(\bar{q},\xi^+) >$ it follows that $< V(\tilde{q},\xi^+) > = < V(\bar{q},\xi^+) >$. Since $\tilde{q}$ satisfies (8.1)

$$\langle \begin{bmatrix} -\tilde{q}(\xi) \\ V(\tilde{q},\xi^+) \end{bmatrix} \rangle = \langle \begin{bmatrix} -\bar{q}(\xi) \\ V(\bar{q},\xi^+) \end{bmatrix} \rangle, \quad \forall\, \xi \in \mathbf{D}$$

The subspace of income transfers is unchanged at each node and since no agent inherits debt at date 0 in the infinite-lived asset j ($\delta^i_j = 0$ by A7) $\bar{q}(\xi_0)\delta^i = \tilde{q}(\xi_0)\delta^i$ so that by (8.5) the agents wealth at date 0 is unchanged. Thus $\mathcal{B}(\bar{p},\bar{q},\bar{\pi}^i,\omega^i,\delta^i,A) = \mathcal{B}(\bar{p},\tilde{q},\bar{\pi}^i,\omega^i,\delta^i,A)$. For all agents $i \in \mathbf{I}, i \neq 1$ define the new portfolio $\tilde{z}^i$ by

$$\begin{bmatrix} -\tilde{q}(\xi) \\ V(\tilde{q},\xi^+) \end{bmatrix} \tilde{z}^i(\xi) = \begin{bmatrix} -\bar{q}(\xi) \\ V(\bar{q},\xi^+) \end{bmatrix} \bar{z}^i(\xi), \quad \forall\, \xi \in \mathbf{D} \tag{8.9}$$

Let $\tilde{z}^1(\xi) = -\sum_{i=2}^{I} \tilde{z}^i(\xi), \quad \forall\, \xi \in \mathbf{D}$ then the spot market clearing equations imply that $\tilde{z}^1(\cdot)$ satisfies (8.9). Thus $(\bar{x}^i;\tilde{z}^i)$ is $\succsim_i$ maximal in $\mathcal{B}(\bar{p},\tilde{z},\bar{\pi}^i,\omega^i,\delta^i,A), \quad \forall\, i \in \mathbf{I}$, $\sum_{i\in\mathbf{I}} \tilde{z}^i = 0$ and $\sum_{i\in\mathbf{I}}(\bar{x}^i - \omega^i) = 0$ imply $((\bar{x},\tilde{z})(\bar{p},\tilde{q},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a GEI equilibrium.

(ii). Let $\bar{\pi}^i = \bar{\pi}, \ \forall\, i \in \mathbf{I}$ denote the common present value vector of the agents and let $\tilde{q}$ denote the vector of asset prices defined by (8.4). Then $\bar{q}(\cdot,j) = \tilde{q}(\cdot,j)$ except possibly for some infinite-lived assets in zero net supply. Since $< V(\tilde{q},\xi^+) >=< V(\bar{q},\xi^+) >= \mathbb{R}^{b(\xi)}, \quad \forall\, \xi \in \mathbf{D}$ and since $\delta^i_j = 0$ for assets in zero net supply $\mathcal{B}(\bar{p},\tilde{q},\bar{\pi}^i,\omega^i,\delta^i,A) = \mathcal{B}(\bar{p},\bar{q},\bar{\pi}^i,\omega^i,\delta^i,A)$. By the same argument as in (i) there exists $\tilde{z}$ such that $((\bar{x},\tilde{z}),(\bar{p},\tilde{q},(\bar{\pi}^i)_{i\in\mathbf{I}}))$ is a GEI equilibrium.

(b) This property was exhibited in Example B of section 4. To keep the example as simple as possible the event-tree was chosen to have no uncertainty after date 1: thus A6 was not invoked, to permit the markets to be potentially incomplete. The property is however quite general: the span of the markets can differ depending on whether the prices of infinite-lived securities do or do not have bubble components. By altering the span of the markets, speculative bubbles can thus affect the equilibrium allocation.

Appendix

Proof of Proposition 5.2: For a vector $x^i \in \mathbb{R}^{\mathbf{D}\times\mathbf{L}}$ let $\hat{x}^i \in \mathbb{R}^{\mathbf{D}^T\times\mathbf{L}}$ denote the components of x^i up to date T. Since the two convex subsets of $\mathbb{R}^{\mathbf{D}^T\times\mathbf{L}}$

$$\mathcal{U}_T^i = \{\, \hat{x}^i \in \mathbb{R}_+^{\mathbf{D}^T\times\mathbf{L}} \mid \hat{x}^i\chi_{\mathbf{D}^T} + \omega^i\chi_{\mathbf{D}\backslash\mathbf{D}^T} \succ_i \bar{x}_T^i \,\}$$

$$\hat{\mathcal{B}}_T^i = \{\, \hat{x}^i \in \mathbb{R}_+^{\mathbf{D}^T\times\mathbf{L}} \mid \hat{x}^i\chi_{\mathbf{D}^T} + \omega^i\chi_{\mathbf{D}\backslash\mathbf{D}^T} \in \mathcal{B}_T(\bar{p}_T, \bar{q}_T, \omega^i, A) \,\}$$

satisfy $\mathcal{U}_T^i \cap \hat{\mathcal{B}}_T^i = \emptyset$, by the standard separation theorem there exists $\hat{\bar{P}}_T^i \in \mathbb{R}^{\mathbf{D}^T\times\mathbf{L}}, \hat{\bar{P}}_T^i \neq 0$ such that

$$\sup_{\hat{x}^i\in\hat{\mathcal{B}}_T^i} \hat{\bar{P}}_T^i\hat{x}^i \leqq \inf_{\hat{x}^i\in\mathcal{U}_T^i} \hat{\bar{P}}_T^i\hat{x}^i$$

Since $\hat{\bar{x}}_T^i \in \bar{\mathcal{U}}_T^i \cap \hat{\mathcal{B}}_T^i$ (where $\bar{\mathcal{U}}_T^i$ denotes the closure of $\mathcal{U}_T^i$)

$$\hat{\bar{P}}_T^i\hat{x}^i \geqq \hat{\bar{P}}_T^i\hat{\bar{x}}_T^i, \quad \forall\, \hat{x}^i \in \bar{\mathcal{U}}_T^i \tag{a1}$$

$$\hat{\bar{P}}_T^i\hat{x}^i \leqq \hat{\bar{P}}_T^i\hat{\bar{x}}_T^i, \quad \forall\, \hat{x}^i \in \hat{\mathcal{B}}_T^i \tag{a2}$$

Let $\bar{P}_T^i \in \mathbb{R}^{\mathbf{D}\times\mathbf{L}}$ denote the vector which coincides with $\hat{\bar{P}}_T^i$ on $\mathbf{D}^T \times \mathbf{L}$ and is zero on $(\mathbf{D}\backslash\mathbf{D}^T)\times\mathbf{L}$. By (a2) the system of linear inequalities in the variables $(x^i(\xi), \xi \in \mathbf{D}^T, z^i(\xi), \xi \in \mathbf{D}^{T-1})$

$$\bar{p}_T(\xi)(x^i(\xi) - \omega^i(\xi)) - (\bar{p}_T(\xi)A(\xi) + \bar{q}_T(\xi))z^i(\xi^-) + \bar{q}_T(\xi)z^i(\xi) \leqq 0, \quad \forall\, \xi \in \mathbf{D}^T \tag{a3}$$

(with $z^i(\xi_0^-) = 0$ and $\bar{q}_T(\xi) = 0$ if $t(\xi) = T$) imply the inequality

$$\sum_{\xi\in\mathbf{D}^T} \bar{P}_T^i(\xi)(x^i(\xi) - \bar{x}_T^i(\xi)) \leqq 0$$

It follows from Theorem 22.3 in Rockafellar (1970) that there exists a non-negative vector $(\bar{\pi}_T^i(\xi), \xi \in \mathbf{D}^T) \in \mathbb{R}^{\mathbf{D}^T}$ such that

$$\bar{P}_T^i(\xi) = \bar{\pi}_T^i(\xi)\bar{p}_T(\xi), \quad \forall\, \xi \in \mathbf{D}^T \tag{a4}$$

$$\bar{\pi}_T^i(\xi)\bar{q}_T(\xi) - \sum_{\xi'\in\xi^+} \bar{\pi}_T^i(\xi')(\bar{p}_T(\xi')A(\xi') + \bar{q}_T(\xi')) = 0, \quad \forall\, \xi \in \mathbf{D}^{T-1} \tag{a5}$$

$$\sum_{\xi\in\mathbf{D}^T} \bar{\pi}_T^i(\xi)\bar{p}_T(\xi)\omega^i(\xi) \leqq \sum_{\xi\in\mathbf{D}^T} \bar{P}_T^i(\xi)\bar{x}_T^i(\xi) \tag{a6}$$

Since $\bar{P}_T^i \neq 0$ there is a node $\bar{\xi} \in \mathbf{D}^T$ such that $\bar{\pi}_T^i(\bar{\xi}) \neq 0$. Since $\succsim_i$ is strictly monotonic in good 1, $\bar{p}_T(\xi, 1) > 0, \ \forall\, \xi \in \mathbf{D}^T$. By A2 and (a6), $\bar{P}_T^i\bar{x}_T^i > 0$. If there exists a node ξ' with $\bar{\pi}_T^i(\xi') = 0$ then by decreasing a positive component of $\hat{\bar{x}}_T^i$, say good (ξ,ℓ) for which $\bar{p}_T^i(\xi,\ell) > 0$ and by increasing the consumption of good 1 at node ξ' the agent can attain a vector in $\mathcal{U}_T^i$ which costs less than $\hat{\bar{x}}_T^i$, contradicting (a1). Thus $\bar{\pi}_T^i(\xi) > 0, \quad \forall\, \xi \in \mathbf{D}^T$ so that (a) and (c) in Proposition 5.2 hold.

By (a6), $\bar{P}_T^i\omega^i \leqq \bar{P}_T^i\bar{x}_T^i$ and by (a3)-(a5), $\hat{\bar{P}}_T^i(\hat{x}^i - \omega^i) \leqq 0$, $\forall\, \hat{x}^i \in \hat{\mathcal{B}}_T^i$. Since $\hat{\bar{x}}_T^i \in \hat{\mathcal{B}}_T^i$, $\bar{P}_T^i\bar{x}_T^i = \bar{P}_T^i\omega^i$. Since $\bar{P}_T^i\omega^i > 0$, (b) follows from (a1). △

Proof of Lemma 6.2: Let $\alpha_\xi(x^i)$ be defined by

$$\alpha_\xi(x^i) = \inf \{\ \alpha \in \mathbb{R} \ \mid\ 0 \ \leq \alpha \leq 1, \quad \alpha x^i + e_1^\xi \succsim_i x^i \}$$

By A.2 and A.3, $\alpha_\xi(x^i) < 1$. $F \subset \ell_\infty^+(\mathbf{D}\times\mathbf{L})$ is compact in the product topology. Let us show that $x^i \longrightarrow \alpha_\xi(x^i)$ is upper semi-continuous on F in the product topology. Let (x_ν^i) be a sequence of F converging to $\bar{x}^i \in F$ componentwise and suppose that for $\epsilon > 0$, there exists a subsequence (which without loss of generality we call (x_ν^i)) such that

$$\alpha_\xi(x_\nu^i) > \alpha_\xi(\bar{x}^i) + \epsilon$$

Then, by definition of $\alpha_\xi(x_\nu^i)$

$$x_\nu^i \succ_i (\alpha_\xi(\bar{x}^i) + \epsilon)x_\nu^i + e_1^\xi$$

Since on bounded sets the product topology and the Mackey topology coincide, x_ν^i converges to $\bar{x}^i$ in the Mackey topology and by A2

$$\bar{x}^i \succ_i (\alpha_\xi(\bar{x}^i) + \epsilon)\bar{x}^i + e_1^\xi$$

But this contradicts the definition of $\alpha_\xi(\bar{x}^i)$ since by monotonicity of the preferences if $\alpha \in (\alpha_\xi(\bar{x}^i),\ 1]$ then $\alpha\bar{x}^i + e_1^\xi \succ_i \bar{x}^i$.

Thus α_ξ is upper semicontinuous for the product topology on F and attains its maximum on the compact set F. Then $\max_{i\in\mathbf{I}} \max_{x^i\in F} \{\alpha_\xi(x^i)\} < 1$ and there exists $\alpha_\xi < 1$ as in Lemma 6.2.△

Proof of Theorem 7.4: Define $\mathcal{A}^* \subset \mathcal{A}$ as the subset of commodity payoff processes such that the economy $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ has a GEI equilibrium. Consider a payoff process $\bar{A} \in \mathcal{A}$ and let $((\bar{x}, \bar{q}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i\in\mathbf{I}}))$ be a pseudoequilibrium of $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, \bar{A}))$ generated by an artificial asset structure $(\mathbf{K}, \eta, \Gamma)$ with

$$\text{rank}\ \left[\Gamma(\xi', 1, k)\right]_{\substack{\xi'\in\xi^+\\ k\in K(\xi)}} = \min\left(b(\xi), j(\xi)\right) = a(\xi), \quad \forall\, \xi \in \mathbf{D}$$

where $j(\xi) = \# J(\xi)$. Let $\bar{q}$ be the vector of prices of the original securities defined by (7.1). There exists a vector of portfolios $\bar{z} \in Z$ such that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$ is a GEI equilibrium of $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, A))$ if and only if

$$\operatorname{rank}\ [\bar{p}(\xi')\bar{A}(\xi', j) + \bar{q}(\xi', j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = a(\xi), \quad \forall\, \xi \in \mathbf{D} \tag{a6}$$

To prove the theorem we show that if (a6) is not satisfied for $\bar{A}$ then for every $\epsilon > 0$ there exists a commodity payoff process $A \in \mathcal{A}$ with $\| A - \bar{A} \|_\infty < \epsilon$ such that (a6) is satisfied for A, so that $A \in \mathcal{A}^*$. We show that this can be done without changing the underlying pseudoequilibrium $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$.

Let $H \subset \mathcal{A}$ denote the subset of commodity payoff processes A such that $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in \mathbf{I}}))$ is a pseudoequilibrium of $\mathcal{E}_\infty(\mathbf{D}, \succsim, \omega, (\mathbf{J}, \zeta, \bar{A}))$. The payoffs in H must satisfy (7.1) i.e. for all $\xi \in \mathbf{D}$, $\frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi' \in \mathbf{D}^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi')$ must be independent of i and if $q(\xi)$ denotes this common value then (7.2) must be satisfied with $\bar{q}$ replaced by q. Thus $A \in H$ if $A \in \mathcal{A}$ and if for all $\xi \in \mathbf{D}$,

$$\sum_{\xi' \in \mathbf{D}^+(\xi)} \frac{\bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi')}{\bar{\pi}^i(\xi)} = \sum_{\xi' \in \mathbf{D}^+(\xi)} \frac{\bar{\pi}^1(\xi')\bar{p}(\xi')A(\xi')}{\bar{\pi}^1(\xi)}, \quad \forall\, i \in \mathbf{I}$$

$$\langle \Big[\bar{p}(\xi')A(\xi', j) + \sum_{\xi'' \in \mathbf{D}^+(\xi')} \frac{\bar{\pi}^1(\xi'')\bar{p}(\xi'')A(\xi'')}{\bar{\pi}^1(\xi')}\Big]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \rangle \subset \langle [\Gamma(\xi', 1, k)]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \rangle$$

These two equations can be written with date 0 discounted prices as: for all $\xi \in \mathbf{D}$

$$\sum_{\xi' \in \mathbf{D}^+(\xi)} \frac{\bar{P}^i(\xi')A(\xi')}{\bar{P}^i(\xi, 1)} = \sum_{\xi' \in \mathbf{D}^+(\xi)} \frac{\bar{P}^1(\xi')A(\xi')}{\bar{P}^1(\xi', 1)}, \quad \forall\, i \in \mathbf{I}$$

$$\langle \Big[\sum_{\xi'' \in \mathbf{D}(\xi')} \bar{P}^1(\xi'')A(\xi'', j)\Big]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \rangle \subset \langle [\bar{P}^1(\xi')\Gamma(\xi', k)]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \rangle$$

To simplify notation, if $x \in \ell_\infty(\mathbf{D} \times \mathbf{L})$ and $P \in \ell_1(\mathbf{D} \times \mathbf{L})$ define $P \underset{\xi}{\cdot} x = \sum_{\xi' \in \mathbf{D}(\xi)} P(\xi')x(\xi')$ and let $P \underset{\xi^+}{\square} x = (P \underset{\xi'}{\cdot} x)_{\xi' \in \xi^+}$ denote the $b(\xi)$ vector of date 0 discounted values of x from all the successors of ξ onwards. For $A \in \mathcal{A}$ define

$$P \underset{\xi}{\cdot} A = (P \underset{\xi}{\cdot} A(\cdot, j), j \in J(\xi)) \quad P \underset{\xi^+}{\square} A = [P \underset{\xi'}{\cdot} A(\cdot, j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}}$$

Then H can be written as

$$H = \left\{ A \in \mathcal{A} \;\middle|\; \begin{array}{c} \dfrac{I}{\bar{P}^i(\xi,1)} \displaystyle\sum_{\xi' \in \xi^+} \bar{P}^i \underset{\xi'}{\cdot} A = \dfrac{1}{\bar{P}^1(\xi,1)} \displaystyle\sum_{\xi' \in \xi^+} \bar{P}^1 \underset{\xi'}{\cdot} A, \; \forall \xi \in \mathbf{D}, \; \forall \, i \in \mathbf{I} \\ \\ \langle \bar{P}^1 \underset{\xi^+}{\square} A \rangle \subset \bar{\mathcal{L}}(\xi), \; \forall \, \xi \in \mathbf{D} \end{array} \right\}$$

where $\bar{\mathcal{L}}(\xi) = \langle \, [\, \bar{P}^1(\xi')\Gamma(\xi',k) \,]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \rangle$.

Note that H is not empty since $\bar{A} \in H$ and since for each $\xi \in \mathbf{D}$ the second condition $< \cdot > \subset \bar{\mathcal{L}}(\xi)$ can be expressed by a system of linear equations, H is the intersection of a countable collection of closed hyperplanes. Thus H is a closed subset of $\mathcal{A}$. Since $\mathcal{A}$ is a closed subset of the Banach space $\ell_\infty(\mathbf{D} \times \mathbf{L} \times \mathbf{J})$, it follows that H is a Baire space (Rudin (1973), p.42 Theorem 2.2).

For each node ξ let $\hat{J}(\xi)$ be a subset of $J(\xi)$ consisting of $a(\xi)$ securities including the short-lived numeraire bond. For a payoff process $A \in \mathcal{A}$ let $\hat{A}$ denote the payoffs of the securities in $\hat{J}(\xi)$. Let $\hat{\xi}^+$ be a subset of $a(\xi)$ nodes of ξ^+ (to be chosen below). Consider the subset of H

$$H_\xi = \{ \, A \in H \mid \det \, [\, \bar{P}^1 \underset{\hat{\xi}^+}{\square} \hat{A} \,] = 0 \, \}$$

We show that H_ξ has an empty interior in H. It suffices to show that we can perturb any $A \in H_\xi$, $A \longrightarrow A + \Delta A$ with $A + \Delta A \in H$ in such a way that

$$\det \, [\, \bar{P}^1 \underset{\hat{\xi}^+}{\square} (\hat{A} + \Delta \hat{A}) \,] \neq 0 \tag{a7}$$

Consider changes ΔA in the commodity payoff process consisting solely of changes in the amounts of commodity 1 which can be decomposed as follows:

$$\Delta A(\cdot, j) \;=\; \begin{cases} \displaystyle\sum_{\xi' \in \hat{\xi}^+} \alpha^j_{\xi'} e_1^{\xi'} + \sum_{\xi' \in \xi^+ \setminus \hat{\xi}^+} \beta^j_{\xi'} e_1^{\xi'} + \gamma^j e_1^{\xi}, & \text{if } j \in \hat{J}'(\xi) \\ 0, \quad \text{if } j \notin \hat{J}'(\xi) \end{cases} \tag{a8}$$

where $\hat{J}'(\xi)$ is the subset of $\hat{J}(\xi)$ which excludes the short-lived numeraire bond. Note that the commodity payoffs are perturbed only at node ξ and its immediate successors ξ^+ and only securities in $\hat{J}'(\xi)$ have their payoffs perturbed. For security $j \in \hat{J}'(\xi)$ its payoff in good 1 is perturbed by $\alpha^j_{\xi'}$ at node $\xi' \in \hat{\xi}^+$, $\beta^j_{\xi'}$ at node $\xi' \in \xi^+ \setminus \hat{\xi}^+$ and by γ^j at node ξ. For brevity write

$$\alpha = (\alpha^j_{\xi'}, j \in \hat{J}'(\xi), \; \xi' \in \hat{\xi}^+) \in \mathbb{R}^{(a(\xi)-1)a(\xi)}$$

$$\beta = (\beta^j_{\xi'}, j \in \hat{J}'(\xi), \; \xi' \in \xi^+ \setminus \hat{\xi}^+) \in \mathbb{R}^{(a(\xi)-1)(b(\xi)-a(\xi))}$$

$$\gamma = (\gamma^j, j \in \hat{J}'(\xi)) \;\in\; \mathbb{R}^{a(\xi)-1}$$

α can be chosen so that (a7) is satisfied. To see this let $g : \mathbb{R}^{(a(\xi)-1)a(\xi)} \longrightarrow \mathbb{R}$ be defined by

$$g(\alpha) \;=\; \det\;[\bar{P}^1 \underset{\hat{\xi}^+}{\square} (\hat{A} + \Delta\hat{A})] \tag{a9}$$

where ΔA is defined by (a8). While ΔA is a function of (α, β, γ), the determinant in (a9) depends only on α. A straightforward but tedious calculation shows that g has partial derivatives of order $a(\xi) - 1$ evaluated at $\alpha = 0$ which are not zero since $\bar{P}^1(\xi', 1) > 0 \;\forall\; \xi' \in \hat{\xi}^+$. Thus g is not locally constant in a neighborhood of $\alpha = 0$. It follows that there exists α arbitrarily small such that $g(\alpha) \neq 0$.

If $a(\xi) < b(\xi)$ then β is chosen to ensure that

$$\langle \bar{P}^1 \underset{\xi^+}{\square} (A + \Delta A) \rangle \;\subset\; \bar{\mathcal{L}}(\xi) \tag{a10}$$

$\bar{\mathcal{L}}(\xi)$ is a non-trivial subspace of $\mathbb{R}^{b(\xi)}$. There is a choice of $a(\xi)$ nodes $\hat{\xi}^+ \subset \xi^+$ and an appropriate ordering of the nodes such that $\bar{\mathcal{L}}(\xi)$ can be represented by a system of $b(\xi) - a(\xi)$ equations of the form

$$\bar{\mathcal{L}}(\xi) = \{\, v \in \mathbb{R}^{b(\xi)} \mid [I \mid E]v = 0 \,\}$$

The nodes are ordered so that the subset $\xi^+ \backslash \hat{\xi}^+$ constitutes the first $b(\xi) - a(\xi)$ nodes. I is the $(b(\xi) - a(\xi)) \times (b(\xi) - a(\xi))$ identity matrix and E is a $(b(\xi) - a(\xi)) \times a(\xi)$ matrix (see Magill-Shafer (1991), p.1544).

For each $j \in \hat{J}'(\xi)$, once α has been chosen, there is a unique vector $\beta^j = (\beta^j_{\xi'}, \xi' \in \xi^+ \backslash \hat{\xi}^+) \in \mathbb{R}^{b(\xi)-a(\xi)}$ such that the vector

$$v^j = ((\bar{P}^1(\xi', 1)\beta^j_{\xi'}, \xi' \in \xi^+ \backslash \hat{\xi}^+), (\bar{P}^1(\xi', 1)\alpha^j_{\xi'}, \xi' \in \hat{\xi}^+))$$

satisfies the equation $[I \mid E]v^j = 0$.

We have to ensure that $\langle \bar{P}^1 \underset{\hat{\xi}^+}{\square} (A + \Delta A) \rangle \subset \bar{\mathcal{L}}(\tilde{\xi})$ is satisfied for all nodes $\tilde{\xi} \neq \xi$. By construction this inclusion holds for all nodes $\tilde{\xi}$ which are not on the path from ξ_0 to ξ. If γ^j is chosen so that

$$\bar{P}^1(\xi, 1)\gamma^j + \sum_{\xi' \in \xi^+ \backslash \hat{\xi}^+} \bar{P}^1(\xi', 1)\beta^j_{\xi'} + \sum_{\xi' \in \hat{\xi}^+} \bar{P}^1(\xi', 1)\alpha^j_{\xi'} = 0$$

for $j \in \hat{J}'(\xi) \cap J(\xi^-)$ and if $\gamma^j = 0$ for $j \notin \hat{J}'(\xi) \cap J(\xi^-)$ then (a10) is satisfied for all nodes $\tilde{\xi} \in \mathbf{D}$. Thus $A + \Delta A$ satisfies the subspace requirements (the second condition) in the definition of H. To show that the first condition in the definition of H is satisfied it only remains to show that if Δq is defined by

$$\Delta q(\tilde{\xi}, j) = \frac{1}{\bar{\pi}^1(\tilde{\xi})} \sum_{\xi' \in \mathbf{D}^+(\tilde{\xi})} \bar{\pi}^1(\xi')\bar{p}(\xi')\Delta A(\xi', j), \ \forall\, j \in J(\tilde{\xi}), \ \forall\, \tilde{\xi} \in \mathbf{D}$$

then for all $i \in \mathbf{I}$

$$\frac{1}{\bar{\pi}^i(\tilde{\xi})} \sum_{\xi' \in \mathbf{D}^+(\tilde{\xi})} \bar{\pi}^i(\xi')\bar{p}(\xi')\Delta A(\xi', j) = \Delta q(\tilde{\xi}, j), \ \forall\, j \in J(\tilde{\xi}), \ \forall\, \tilde{\xi} \in \mathbf{D}$$

so that all agents agree on the induced changes in the security prices. Since ΔA has only a finite number of non-zero terms this follows from $\langle \bar{P}^1 \underset{\tilde{\xi}^+}{\square} \Delta A \rangle \subset \mathcal{L}(\tilde{\xi}), \ \forall\, \tilde{\xi} \in \mathbf{D}$.

Since α can be chosen to be arbitrarily small and since (β, γ) are deduced from α by linear relations with bounded coefficients, the perturbation ΔA can be made arbitrarily small. Thus for all $\xi \in \mathbf{D}$, H_ξ is closed and has empty interior in H. Since H is a Baire space, the countable union $\bigcup_{\xi \in \mathbf{D}} H_\xi$ has empty interior in H. Thus for all $\epsilon > 0$ there exists an ϵ-perturbation A of $\bar{A}$ such that

$$\langle \bar{P} \underset{\xi^+}{\square} A \rangle = \bar{\mathcal{L}}(\xi), \quad \forall\, \xi \in \mathbf{D}$$

and the proof is complete. $\triangle$

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