INVENTORY AND PRICE EQUILIBRIUM MODELS APPLIED TO THE STORAGE PROBLEM

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Currently developments in two bodies of literature address the determination of optimum levels of storage. One approach, the use of inventory models, has developed as a generalization of the familiar lot size problem in management science models [9, 17]. The decision unit in these models is the firm or frequently a decentralized management unit or authority [4, 12, 15].

A second approach to the storage problem has developed in economics and applies the idea of price equilibrium in an intertemporal context [10, 13]. The focus of these models is the market, with representations for demand and supply at different periods of time and costs of holding inventories or transferring the good from one period to the next. The objective of these models is the representation of market behavior under competitive conditions.

The purpose of this article is to establish linkages between the two different approaches to the storage problem. Aside from the analytical advantages that follow from combining the results obtained through the two streams of inquiry, the connection of the two approaches is motivated by the growing similiarity of market and management problems [7]. Work on stabilization policies and trade increasingly treats markets, storage, and equilibrium prices in a manipulated contest [1, 5, 8, 11]. The separation between the management-oriented inventory models and the price equilibrium market models is becoming increasingly artificial as the two approaches continue to be applied to more complex problems.

INVENTORY MODEL

The inventory model used to develop the connection to the price equilibrium framework is a standard dynamic version of the economic lot size problem [2, 16]. For simplicity, the problem is taken to be deterministic with no back orders or lost sales allowed. The planning period is assumed finite and of length T, with decisions occurring at discrete intervals t = 1, 2, . . . , T. The following notational conventions are used in formulating the dynamic version of the economic lot size model.

D_t = the quantity demanded in period t, assumed known for all t
X_t = the quantity produced or ordered at time t
Y_t = the inventory at the beginning of period t, with Y_1 assumed known
$C_t(X_t)$ = the cost of producing or ordering X_t at time t, with $C_t(0) = 0$
$G_t(Y_t)$ = the inventory holding cost in period t
α_t = a discount factor which, when applied to costs in period t, yields their present value.

The discounted cost function for period t can be written

(1) $r_t(X_t, Y_t = \alpha_t[C_t(X_t) + G_t(Y_t)]$

for t = 1, 2, . . . , T. Similarly, the sum of the discounted costs over the planning horizon is

(2) $Z = \sum_{t=1}^{T} r_t(X_t, Y_t).$

This summation represents the cost of satisfying the known quantities demanded over the finite time horizon T. In this case Y_T is assumed to be zero because there is no carryover to satisfy the demand in period T+1, which is beyond the scope of the model.

The inventory problem is one of minimizing the discounted sum of the costs in equation 2 subject to a constraint ensuring the satisfaction of demands and a basic accounting relationship

(3) $Y_t + X_t - D_t = Y_{t+1}$

for t = 1, 2, . . . , T. Observe that the constraint is in the form of a difference equation. Equivalently written, the constraint in equation 3 is

(4) $Y_{t+1} = Y_1 + \sum_{i=1}^{t} (X_i - D_i).$

equation 4 states that the carryover to period t+1 is equal to the initial inventory, or carrying for the first period, plus the sum of the excesses of production or orders over demand in periods 1 to t.

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In this formulation of the problem the quantities produced or ordered, X_t, are decision variables and the beginning inventories, Y_t, which describe the position of the firm or management unit at the beginning of the period, are state variables. The objective is to determine the sequence of decision variables X_t which minimize equation 2 subject to equation 3. A solution to the inventory problem gives optimizing values for production and, correspondingly, carryover for each of the included time periods.

PRICE EQUILIBRIUM MODEL

Intertemporal price equilibrium models are another method of determining optimal levels of storage over time [10, 14]. For simplicity, the demand and supply functions are assumed to be known with certainty.[1] In formulation of the intertemporal price equilibrium model for t = 1, 2, . . . , T time periods the following notation is adopted.

$P_D(t) =$ $e(t) - f(t)D(t)$: demand function for period t with $P_D(t)$ as the demand price, D(t) the quantity demanded, and f(t) positive.

$P_x(t) =$ $g(t) + h(t)X(t)$: supply function for period t with $P_x(t)$ as the supply price, X(t) the quantity supplied, and h(t) positive.

$G(t, t+1) =$ the cost of storing one unit between periods t and t+1.[2]

$Y(t, t+1) =$ the quantity carried over between periods t and t+1.

$\alpha_t =$ a discount factor which, when applied to costs and benefits in period t, yields their present value.

Intertemporal price equilibrium models determine optimal levels of storage, prices, and consumption by maximizing the discounted net social payoff, a quasi welfare function derived from the demand and supply equations. The social payoff function for any time period is defined as the algebraic area under the excess demand curve for that period [10, 14].[3] The discounted net social payoff function for the intertemporal model is the sum of the social payoff functions over the T periods minus storage costs. The model is designed to represent the intertemporal price determination process in competitive markets.

The algebraic expression for the discounted net social payoff is

$$(5)\quad NSP = \sum_{t=1}^{T} \alpha_t \Big[\int_{D'(t)}^{D(t)} (e(t) - f(t)D(t))\,dD(t) - \int_{X'(t)}^{X(t)} (g(t) + h(t)X(t))\,dX(t)\Big] - \sum_{t=0}^{T-1} \alpha_{t+1} G(t, t+1)Y(t, t+1)$$

where $D'(t) = X'(t)$ is the equilibrium demand or supply for each period without storage. The upper limits of integration, D(t) and X(t), are variables representing quantity demanded and quantity supplied, respectively. Equation 5 can be written alternatively as

$$(6)\quad NSP = \sum_{t=1}^{T} \alpha_t \Big[\int_{O}^{D(t)} (e(t) - f(t)D(t))\,dD(t) - \int_{O}^{X(t)} (g(t) + h(t)X(t))\,dX(t) - \int_{O}^{D'(t)} (e(t) - f(t)D(t))\,dD(t) + \int_{O}^{X'(t)} (g(t) + h(t)X(t))\,dX(t)\Big] - \sum_{T=O}^{T-1} \alpha_{t+1} G(t, t+1)Y(t, t+1)$$

Equation 6 shows that the net social payoff is the sum of consumers' and producers' surpluses after storage, minus the sum of consumers' and producers' surpluses before storage, minus storage costs, summed over all periods. This increase in consumers' and producers' surpluses due to storage is a measure of derived net benefit. Clearly, the same solution set could also be obtained by maximizing.

$$(7)\quad NSP = \sum_{t=1}^{T} \alpha_t \Big[\int_{O}^{D(t)} (e(t) - f(t)D(t))\,dD(t) - \int_{O}^{x(t)} (g(t) + h(t)X(t))\,dX(t)\Big] - \sum_{t=0}^{T-1} \alpha_{t+1} G(t, t+1)Y(t, t+1).$$

[1] This condition currently is being relaxed in some advanced applications to trade and stabilization problems. However, for expository purposes, the simplified form of the price equilibrium model is used in this analysis.

[2] Constant storage cost is assumed in this analysis for simplicity. A more realistic alternative to this assumption can be found in the price of storage literature.

[3] Social payoff also can be defined as the sum of producers' and consumers' surplus in period t. The traditional formulation of the model found in [10] is used in this analysis.

In equation 7, the total of producers' and consumers' surpluses with storage minus storage costs is maximized. For a given set of demand and supply functions, the sum of producers' and consumers' surpluses without storage would be constant. Thus, maximizing equation 6 is the same as maximizing equation 7 minus a constant. In the present analysis, equation 7 is used as the objective function for the intertemporal price equilibrium model.

This objective function is maximized subject to the constraint

$$(8) \quad X(t) + Y(t-1, t) - D(t) - Y(t, t+1) = 0$$

for all t.[4] This constraint states that production in period t and the carryover from $t-1$ to t must be equal to consumption in period t plus carryover from t to $t+1$. This problem can be solved for $\overline{D}(t)$, the optimal consumption in t; $\overline{X}(t)$, the optimal production in t; $\overline{Y}(t, t+1)$, the optimal carryover from t to $t+1$; and $\overline{P}(t)$, the optimal price in t, for all t. Quadratic programming is normally used as the solution algorithm when the demand and supply functions are linear [14].

COMPARISON OF THE INVENTORY AND PRICE EQUILIBRIUM MODELS

The inventory and intertemporal price equilibrium models are compared on the basis of their respective constraints and objective functions. First, observe that the constraints for the two models are identical. In equations 3 and 8, Y_t and $Y(t-1,t)$ both represent inventory levels at the beginning of period t, Y_{t+1} and $Y(t, t+1)$ are inventory levels at the beginning of period $t+1$, $X(t)$ and X_t are quantities produced or ordered in period t, and $D(t)$ and D_t are quantities demanded in period t. Thus, the notation is the only basis for differentiating the accounting or balance conditions in the two models.

The similarity of the objective functions is demonstrated with equations 2 and 7. The integral of the supply curve,

$$\int_{0}^{X(t)} (g(t) + h(t)X(t))dX(t),$$

from the price equilibrium model, and $C_t(X_t)$, from the inventory model, can both be regarded as total costs of producing the commodity. For the inventory model this is interpreted in a straightforward context, whereas in the price equilibrium model the cost is rationalized on the basis of competitive assumptions for firms operating on the supply side. Also, $G(t, t+1)\ Y(t, t+1)$, which is the per-unit storage cost multiplied by the number of units of inventory held, and $G_t(Y_t)$ can both be regarded as the inventory holding cost or the storage cost. Finally, α_t represents the discount factor in each equation. Thus, the difference between the objective functions for the inventory and price equilibrium models is that in equation 7 a term which accounts for the price-related demand for the commodity is included. For the case in which $f_t = 0$, the models are easily seen to be identical in structure, although rationalized for the solution of different problems.

The absence of a demand term in the objective function for the inventory model can be explained in the following way. In general, quantities demanded in each period are assumed to be known constants in the inventory model, whereas in the intertemporal price equilibrium model only the demand function in each period is assumed known. Thus, in the inventory model, consumers' surplus can be viewed as a constant corresponding to D_t units consumed in any period t. Because D_t is prespecified for each period, there is no possibility of changing consumers' surplus. Thus, in the inventory model, storage is determined by minimizing the total cost of meeting these known requirements. Minimizing the sum of the total cost of production and storage over the planning horizon has the effect of maximizing producers' surplus minus storage costs over the planning horizon. Thus, the inventory model maximizes the sum of consumers' and producers' surpluses, as does the intertemporal price equilibrium model, but for a special case in which consumers' surplus is a constant.

To illustrate the similarity of the two objective functions, let TC_t represent the total cost of production and SC_t represent the storage cost. By elimination of the term representing demand in equation 7, the objective function for the intertemporal model becomes

$$(9) \quad \max \sum_{t=1}^{T} (-TC_t - SC_t).$$

Similarly, the objective function for the inventory model is

$$(10) \quad \min \sum_{t=1}^{T} (TC_t + SC_t).$$

Thus, it follows that

$$(11) \quad \max \sum_{t=1}^{T} (-TC_t - SC_t) = \min \sum_{t=1}^{T} (TC_t + SC_t)$$

and the two problems are algebraically identical under this simplifying assumption.

Let DC_t represent the area under the demand curve. Introducing a demand term into the in-

[4]Other constraints necessary for a meaningful solution to both the inventory and the price equilibrium models are non-negativity conditions on inventory, quantity demanded, quantity supplied, and price. In this analysis the comparison of the models is confined to interior solutions and boundary conditions which are not examined.

ventory model in equation 2 gives the objective function

$$\text{(12)} \quad \min \sum_{t=1}^{T} (-DC_t + TC_t + SC_t).$$

Because this objective function is being minimized, the area under the demand curve is introduced with a negative sign. This is done because minimizing the negative of total benefits, measures by the area under the demand curve, is the same as maximizing positive total benefits.

The choice of objective functions reduces to the nature of the assumptions on demand. It is not surprising that the inventory problems which originated in firm management contexts treat demand as a constant. The usual situation is one in which the firm does not know or wish to represent market demand. The firms are not concerned with consumers' surplus or welfare in the markets in which they operate.

The advantage of observing the similarity between the two models follows from the increasing use of price equilibrium formulations in decision contexts. For example, beginning with the work of Gustafson [2] and continuing through more recent analyses such as those of Johnson and Sumner [6], the problem of optimizing storage has come to be an important area of inquiry. One approach to this problem has been the use of intertemporal price equilibrium models. Another approach addresses this buffer stock problem on an inventory basis rather than within the context of competitive markets. The present analysis shows that if such inventory models are appropriately specified, competitive arbitrage between the time periods will accomplish an optimizing inventory policy. The broadened basis for interpreting the results of inventory models and the policy implications for their solutions are obvious and important. The government or planning authority should pursue storage rules that emulate competitive behavior if the quasi welfare function is to be maximized. Other examples relating to intertemporal allocation can be interpreted similarly if a market and competitive equilibrium is to be simulated with an inventory model.

Observing this linkage may also offer computational advantages. The dynamic rules for optimizing in inventory problems are comparatively well developed. For more realistic and thus complex and/or stochastic formulations of the storage problem, the connection between the two models is similar in principle, relating to the simplification in characterizing demand and the interpretations of the supply and/or cost functions. The fact that inventory problems have been solved in complex stochastic contexts should make this analogy useful.

CONCLUSION

Because the main focus of inventory models is typically a firm or decentralized management unit and the focus of price equilibrium models is the market, different behavioral interpretations are attached to the solutions of each of these models. However, in certain cases, these two models can be shown to be algebraically identical. This relationship provides a linkage between the two models which proves beneficial in two ways. First, the linkage provides a broadened basis for interpreting the results of inventory models and the policy implications for their solutions. The present analysis shows how solutions obtained in these contexts can be interpreted in a competitive equilibrium framework. This approach is particularly useful where inventory formulations are used in a sequential decision-making context to analyze such important policy issues as grain reserves and buffer stocks. Second, through this linkage, many of the powerful computational techniques developed in the inventory literature can be brought to bear on price equilibrium models.

The inventory and price equilibrium models are shown to be algebraically identical for the storage problem in which quantities demanded are constants. The most familiar and simplest versions of these two models are used in deriving this result. It is hoped that the results of this analysis will lead researchers to try to generalize these models to more realistic situations.

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