PRODUCTION FUNCTION, COST OF PRODUCTION, AND ASSOCIATED OPTIMALITY LINKAGES: A TEXTBOOK SUPPLEMENT

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One of the most important concepts of production economics, the linkage of costs of production (expressed in terms of output) to the production function, is poorly treated in most intermediate level microtheory and production economics texts. In several texts the linkage is so sketchy it is of limited pedagogical value (e.g., Gould and Ferguson; Heady; Leftwich). The best available treatments are those of Doll, Rhodes, and West, Buse and Bromley, and Goodwin, all of which are in introductory texts. Certainly most students are not ready for abstract, analytical geometrics in a first course. However, many upper-division students not only do not understand the input side — output side linkages and the resulting consistency of the input profit-maximizing conditions and the output profit-maximizing conditions, but also become confused as to the meaning of and relationship between marginal revenue and marginal value productivity and between marginal cost and marginal factor cost. Treatments like Goodwin's Figure 9.6 (p. 145) must be reinforced and more fully developed at the intermediate level.

The purpose of our note is to lay out in elementary mathematical and geometric terms the connection between these concepts for the single product — single input case. This development is followed by a suggested format for classroom presentation. We make no claim of originality of concept. Rather, our purpose is to contribute a consistent and clearly developed framework that will be useful as a textbook supplement for production economics teachers and students.

THE ALGEBRA

For the single factor case the algebra of deriving the cost function from the production function is simple and direct. Given the production function

(1) $Y = f(X)$

and the cost equation

(2) $C = rX$

where Y, X, C, and r denote output, input, cost, and factor price,[1] respectively, it follows immediately that costs can be expressed as a function of the product by solving the production function for X in terms of Y (taking an inverse)

(3) $X = g(Y)$
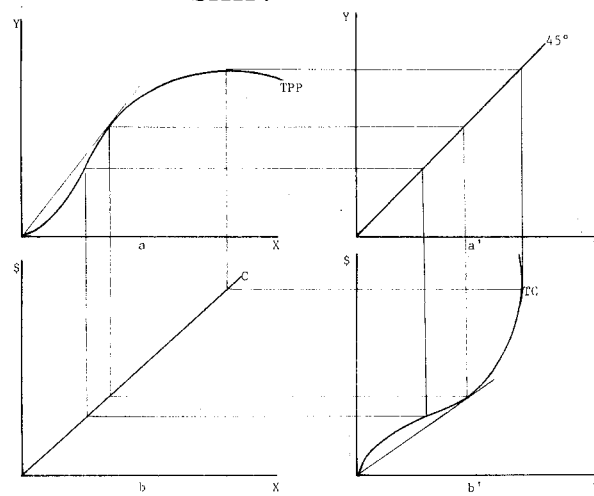
and substituting into the cost equation. Thus, we obtain

(4) $TC = C = rg(Y)$

where TC denotes cost when expressed in terms of output.

THE GEOMETRY

The geometry of this basic linkage is presented in Figure 1. In panel a is the production

FIGURE 1. LINKAGE BETWEEN PRODUCTION FUNCTION AND COST-OUTPUT RELATIONSHIP.

function (corresponding to equation 1); beneath it (panel b) is the cost equation (equation 2). In panel a' is a 45° line permitting the trans-

[1]For simplicity, perfect competition in all product and factor markets and no fixed costs are assumed throughout.

lation of output from the vertical to the horizontal axis; beneath it (panel b') is the resulting cost-product relationship (equation 4) obtained by "closing the circle" and connecting the locus of intersection points. From this kind of visual presentation the student can immediately see the correspondence of inflection, ray-line tangency, and maximum output points between the production function and the total cost curve; these points are noted by the connecting lines in Figure 1. Also the implication of a three-stage production function for the shape of cost curves is made clear, including the often omitted Stage III as noted by Tangri.

By adding total value product (TVP) to panel b and total revenue (TR) to panel b' (Figure 2),

FIGURE 2.   LINKAGE BETWEEN PRODUCTION FUNCTION AND COST-OUTPUT RELATIONSHIP WITH EXTENSIONS TO MARGINAL CONDITIONS.
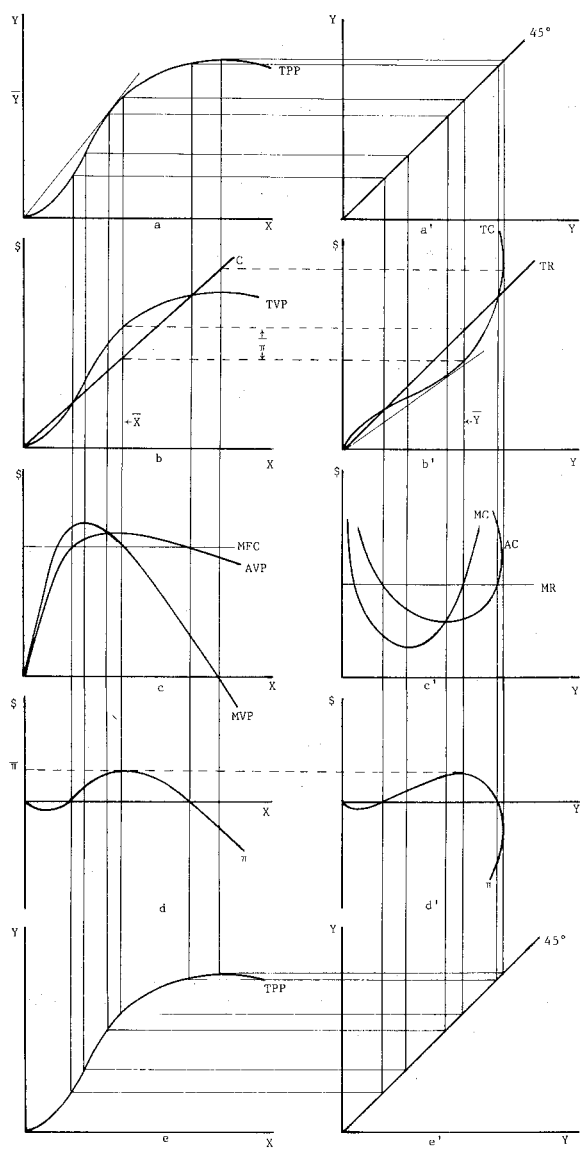
one can see the consistency of optimal levels of Y, $\overline{Y}$, and X, $\overline{X}$, as viewed from either the factor or product side (see dashed lines on panels a, a', b, and b'). The student can verify that the horizontal distance to $\overline{Y}$ in panel b' equals the vertical distance in panel a. The addition of panels c, c', d, d', e, and e' permits the completion of the visual linkage in terms of the factor marginal conditions with the product marginal conditions. Panels a, a', e, and e' are the geometric equivalents of the mathematical inverses which permitted the "completion of the circle" in the algebraic section.

Following the lines in Figure 2, one can see the correspondence of breakeven points (first and fifth vertical lines), inflection and maximum and minimum marginal points (second vertical line), ray-line tangency and maximum and minimum average points (third vertical line), profit maximization point (fourth vertical line), and maximum physical product[2] (sixth vertical line). As in the case of optimal Y, the student can verify that $\overline{\pi}$ in panels b and b' equal $\overline{\pi}$ in panels d and d'.

Figures 1 and 2 are also useful in demonstrating the impact of improved technology (shifting TPP), changing factor price (change in slope of C), and changes in fixed cost (shifting intercept of C) on the firm's supply function (marginal cost curve) and optimum factor and product levels. Exercises of this type, using Figure 1 and 2 as well as equations 1 through 4, are extremely valuable in getting across these fundamental relationships. For this purpose we suggest duplicating and distributing several copies of the figure and asking the students to trace through several postulated changes in the parameters.[3]

SUGGESTED PRESENTATION FORMAT

We have found the following format to be a logical and useful way to present the essence of Figures 1 and 2. First, the theory is fully developed in terms of optimization from the factor side (left side of Figure 2, exclusive of panel e). In this connection we have found that the addition of panel d aids significantly the students' understanding of panels b and c (Doll, Rhodes, and West do this in their text).

The next step is to use Figure 1 to develop the production function—costs of production linkage slowly and in detail. This relationship has to be one of the most important (albeit most often neglected) in the theory of the firm.

We proceed then to product optimization ideas and concepts of panels b', c', and d' of Figure 2 following from panel b' of Figure 1.[4] Finally the additional linkages (beyond those shown in Figure 1) are developed by adding

[2]Space limitations in panel c' preclude showing that MC approaches infinity as output approaches its maximum (last vertical line).

[3]To this end copies of Figures 1 and 2, suitable for reproduction, can be obtained for classroom use by writing the authors.

[4]Concepts of product supply can, of course, be incorporated into the presentation at this point by adding emphasis to (darkening) the relevant portion of MC. Factor demand can be incorporated similarly in panel c.

panels e and e', thereby "closing the circle."

If we would take time in our curricula to thoroughly ingrain our students with the ideas and linkages demonstrated in Figures 1 and 2, we could vastly improve their professionalism and competence. These vertical and horizontal linkages are so fundamental that the scarcity of these developments in intermediate level production economics and microtheory texts is a serious deficiency.

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