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"A Simple Alternative Development of Expected Utility"

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A SIMPLE ALTERNATIVE DEVELOPMENT OF EXPECTED UTILITY

I.

INTRODUCTION

"Functional structure," introduced into economics by Wassily Leontief (2) in the 1940's, has become very familiar to economists. We routinely discuss separability, homotheticity, subadditivity, etc., and their relationships to demand functions, elasticities of substitution, and other economic notions. Indeed, economists have found that theoretical and empirical efforts are greatly facilitated when one can impose these structural restrictions on underlying preferences or technologies. As a consequence, we have begun to develop comfortable intuitions about the meaning of separability and related concepts.

In this note, I use concepts of functional structure to approach expected utility. This perspective allows one to see quite clearly that modeling consumer preferences under uncertainty is essentially the same as modeling these preferences under certainty. Further, the perspective is a useful expositional tool in graduate teaching. Students who have become accustomed to making assumptions about utility functions' attributes find little mystery in expected utility models when those models are cast simply as another instance of common structural restrictions on preferences.

II.

THE CLASSIC FRAMEWORK

We begin in the traditional fashion by assuming that the objects of choice over which preferences are defined are sets of m commodity bundles and their associated probabilities of being received. Each bundle is a set of n commodities that will be realized with certainty if some particular event occurs; the events are mutually exclusive and exhaustive. All uncertainty in this framework is captured in the probabilities of the events, $P_1,\ldots,P_m$.

We then assume that a consumer can say about any two ordered mx(n+1)-tuples (called lotteries $\bar{L}$ and L^*)

$$\bar{L} = \{(\bar{P}_1,\bar{x}_{11},\ldots,\bar{x}_{1n}),\ldots,(\bar{P}_m,\bar{x}_{m1},\ldots,\bar{x}_{mn})\}$$

and

$$L^* = \{(P^*_1,x^*_{11},\ldots,x^*_{1n}),\ldots,(P^*_m,x^*_{m1},\ldots,x^*_{mn})\},$$

(where the x_{ij} are commodities) that one lottery is preferred to the other or that they are equally preferred. Furthermore, we make the usual assumptions of transitivity, reflexivity, completeness, and continuity of preferences, thereby ensuring that some continuous utility function exists which orders all lotteries:

$$U = f(P_1, x_{11}, \ldots, x_{1n}, \ldots, P_m, x_{m1}, \ldots, x_{mn})$$

where $\sum_{i=1}^{m} P_i = 1$ and $x_{ij} \geq 0$ for all i,j.

So far, this development is the typical formulation of the consumer's choices under uncertainty; see, eg., Varian's (4, pp. 155-57) textbook treatment. However, at this point most writers follow the seminal approach of von Neumann and Morgenstern (3) and rely on a purely axiomatic treatment of consumer preferences to arrive at the expected utility formulation. The axioms are generally stated either (a) in terms of which of two lotteries the consumer would choose in given circumstances, or (b) as conditions under which the consumer would be indifferent among lotteries. The utility function itself receives little or no attention in this traditional approach except for the ultimate condition that utility must take the expected utility form

$$U = \sum_{i=1}^{m} p_i v(x_{i1},\ldots,x_{im}).$$

While this traditional approach seems quite natural (what could be more natural than deriving preferences from hypotheses about choices?), it is contrary to what we do in the rest of microeconomics. Rather, our habit is to first specify the utility function, or at least some of its traits, and then infer the behavioral implications of that specification for the economic agent involved. By extending this more familiar approach to preferences under uncertainty, we here emphasize the commonality of modeling consumer behavior with and without uncertainty.

An additional advantage of this alternative is that it may facilitate analysis of preferences under uncertainty when one or more of the behavioral postulates fail, for it gives a direct linkage between functional form and each of the behavioral assumptions.

III

THE NEW DERIVATION

As a first restriction we assume that the utility function is weakly separable in the partition $\{A\}$ where

$$\{A\} = \{(P_1, x_{11}, \ldots, x_{1n}), \ldots, (P_m, x_{m1}, \ldots, x_{mn})\}$$

so that

(1) $U = f(v_1(P_1, x_{11}, \ldots, x_{1n}), \ldots, v_m(P_m, x_{m1}, \ldots, x_{mn}))$.

This restriction implies that if the consumer is presented two lotteries that differ only in the i^{th} outcome, $(P_i, x_{i1}, \ldots, x_{in})$, she can state which lottery is preferred without knowing anything about the other (common) elements of the two lotteries. (Since $\sum_{i=1}^{m} P_i = 1$, the P_i must be the same for the two lotteries if all their other elements are the same; only the x_{ij} can differ.) This condition seems quite plausible.

We next strengthen the separability restriction and require that preferences be <u>strongly</u> separable in the partition $\{A\}$ or equivalently, we require f be representable as additive in the v_i, i.e.,

$$U = \sum_{i=1}^{m} v_i(P_i, x_{i1}, \ldots, x_{in})$$

This restriction implies that if two lotteries differ only in a subset of outcomes and their associated probabilities, the consumer can state which lottery is preferred knowing only the outcomes that differ and their associated probabilities.

The third restriction we impose on functional form is that the x_{ij} are separable from the P_i inside v_i. Thus

$$(2) \quad U = \sum_{i=1}^{m} v_i(P_i, h_i(x_{i1}, \ldots, x_{in})).$$

It is certainly debatable whether one would not care about P_i in judging two bundles of x_{ij}'s (when all other bundles are the same). For instance, I might be uneasy about getting a trophy when it's a sure thing, but like it better when it is an unlikely prospect. Nonetheless, this is a restriction we need to impose if there is more than one commodity.

Next we assume that the form of all the v_i's is the same. This restriction implies that the probablistic events which determine which bundle of goods we receive are of no interest to the consumer except as reflected in the contents of the bundle. Thus

$$(3) \quad U = \sum_{i=1}^{m} v(P_i, h(x_{i1}, \ldots, x_{in})).$$

We are now but a short step from the expected utility representation of preferences.

The final functional restriction we impose is that

(4) $v(P_i, h(x_i)) = P_i g(h(x_i))$.

This seemingly strong restiction actually amounts to the weak requirement that if all outcomes are identical, i.e., if $x_{ik} = x_{jk}$ for each k for every i,j, then the consumer is indifferent among all P_i combinations in the unit simplex. The proof of this is straightforward; we only need to show this latter restriction implies that the partial derivative of v with respect to P_i is a constant, albeit a constant that varies with x_i.

Given identical bundles, $\bar{x}$, in all events, indifference among all feasible probability combinations implies

(5) $$dU = \sum_{i=1}^{m} \frac{\partial v(p_i, h(\bar{x})}{\partial P_i} dP_i = 0$$

for P_i such that $\Sigma P_i = 1$. Consider $dP_3 = dP_4 = \ldots = dP_m = 0$, so $dP_1 = -dP_2$. Equation (5) becomes

$$\frac{\partial v(P_1, h(\bar{x}))}{\partial P_1} dP_1 - \frac{\partial v(P_2, h(\bar{x})}{\partial P_2} dP_1 = 0$$

so that

$$\frac{\partial v(P_1, h(\bar{x}))}{\partial P_1} = \frac{\partial v(P_2, h(\bar{x}))}{\partial P_2} = g(h(\bar{x}))$$

for any P_1 and P_2. Hence v/ P_i is a function, depending only on $h(\bar{x})$. This implies that

$$v(P_i, h(x_i)) = P_i g(h(x_i)) + c$$

thus

$$U = \sum_{i=1}^{m} (P_i g(h(x_i)) + c$$

$$= mc + \sum_{i=1}^{m} P_i g(h(x_i))$$

Since U is an ordinal measure ordering function, mc can be dropped without loss of generality. Hence, if we define $k(x) = g(h(x))$ we arrive at

$$(5) \quad U^* = \sum_{i=1}^{m} P_i k(x_i)$$

which is the expected utility formulation of the utility function.

It is informative to note that the utility function in equation (5) <u>is</u> arbitrary up to any monotonic transformation, but the transformation must be of the entire function and not, generally, of just the $k(x_i)$ function. Also, nonlinear monotonic transformations of $k(x_i)$ (which orders certain prospects) do not upset the ordering property of $k(x_i)$ over certain prospects, but affect only the link between $k(x_i)$ and the P_i in ordering uncertain prospects. These two points deserve emphasis because they ratify the underlying analytical sameness of modeling preferences under certainty and under uncertainty.

However, if one wishes to adhere to the convenient functional form that defines "*ex ante*" utility, (U^*), as the expected value of "*ex post*" utility, $k(x_i)$, then one must restrict oneself to linear transformations of U^* or, equivalently, to linear transformations of $k(x_i)$, as is well known.[2]

IV

NEW RESTRICTION AND OLD AXIOMS

Five restrictions on a utility function defined over lotteries have brought us to the expected utility specification. It is of some interest to briefly discuss how these restrictions relate to the axioms of choice assumed by von Neumann and Morgenstern.

To facilitate this comparison, consider three key axioms as presented by Varian (4, pp. 155-156). Let [pox + (1-p)oy] be a lottery that we have heretofore denoted (p, x, (1-p), y), and

(C2) If $x \sim y$ then $[pox + (1-p)oz] \sim [Poy + (1-p)oz]$

(L3) $[qo[pox + (1-p)oy] + (1-q)oy] = [(qp)ox + (1-qp)oy]$

(L2) $[pox + (1-p)oy] = [(1-p)oy + pox]$

In the terminology of Henderson and Quandt (1, pp. 52-56) axiom (C2) is the "axiom of independence" and (L3) is the "axiom of complexity". We refer to (L2) as the "axiom of symmetry".

The weak separability of preferences in the partition $\{A\}$ is a close analog to the axiom of independence. That axiom says lotteries with indifferent prizes are indifferent.

Strong separability in the partition $\{A\}$ and the requirment that the v_i be separable with respect to probability vis-à-vis commodities serve a role akin to that of the axiom of complexity in the classical treatment. The classical axioms explicitly treat only lotteries with two outcomes, but those outcomes may themselves be lotteries. (In our formulation, the integer m limits the extent of nesting that can be treated.) On its face, the axiom of complexity says that in compound lotteries, the consumer only cares about net probabilities. But more to the point, the axiom of complexity extends the sway of the independence axiom to lotteries with (implicitly) many events. This is just what is accomplished by the two stronger separability restrictions we introduce.

The fourth requirement, that $v_i(\bar{P}, h(\bar{x})) = v_j(\bar{P}, h(\bar{x}))$ for all i and j, corresponds to the classical axiom of symmetry. As long as the x_i are suitably defined, this requirement is nearly tautological.

Finally, the requirement that $v_i(P_i, h(x_i)) = P_i k(x_i)$ corresponds to a special case of the axiom of complexity in which all the prizes of arbitrary, nested lotteries are equal to $\bar{x}$.

With these correspondences in mind, we suggest that the functional structure approach to expected utility affords us a clearer insight than does the classical approach into what may be the least intuitively appealing feature of the expected utility specification. As noted above, there is no pressing intuition underlying the restriction that the x_i are separable from p_i in v_i. However, this restriction is hidden in the von Neumann-Morgenstern axioms; it arises only implicitly, from the recursive application of the axiom of independence by way of the axiom of complexity. Hence these two axioms which, on their faces, seem quite plausible hide in their unstated interaction a deeper lack of intuitive support. There is no such obfuscation in our alternative approach.[3]

V

CONCLUSION

This concludes our effort to underline the inherent similarities in modeling preferences with and without uncertainty. We have shown that the conventional expected utility formuation can be derived from an "ordinary" utility function providing this ordinary utility function (a) has event probabilities as arguments, and (b) satisfies a set of familiar functional structural restrictions. We have also used this alternative formulation to shed fresh light on the intuitive underpinning of the classical von Neumann-Morgenstern axioms of choice under certainty.

FOOTNOTES

[1]In am grateful to Audrey Curtiss, Arlene Liebowitz, Jim Dertouzes, Rodney Smith and Thomas Willett for their comments on earlier drafts, and to the Lincoln Foundation for financial support. All remaining errors are, of course, my responsibility.

[2]It is also well known that the further structural restriction that h* be additively separable and linearly homogeneous, i.e., linear, yields a risk neutral expected utility function.

[3]The functional structure approach also permits us to see clearly that the expected utility formulation may be more intuitively plausible when only a single good is considered than when two or more goods are considered. If there is only one good, there is no need for assuming the P_i separable from a set of goods within each v_i. Hence, while it is not obvious from inspection of the axioms themselves, stronger assumptions are needed to reach the expected utility formulation in multiple good settings than in single good models.

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(4) Varian, H.R., Microeconomic Analysis, 2nd edition, W. W. Norton and Co., Inc., N.Y., N.Y., 1984.