A CONTROL THEORY APPROACH TO OPTIMAL IRRIGATION SCHEDULING IN THE OKLAHOMA PANHANDLE

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Climatic conditions in semiarid regions like the Oklahoma Panhandle result in wide fluctuations in rainfall, dryland crop yields, and returns to agricultural producers in the area. Irrigated crop production increases per-acre yields and significantly reduces fluctuations in yields and net returns.

Irrigated production of food and fiber in the Oklahoma Panhandle has developed rapidly during the past three decades, increasing from 11,500 to 385,900 acres since 1950 (Schwab). The primary source of irrigation water in the area is the Ogallala Formation, an aquifer underlying much of the Great Plains region. Until the past couple of years, the presence of relatively low cost natural gas led producers to expand irrigated production and apply high levels of water to crops irrigated in the area. Water withdrawals for irrigated production are considerably greater than natural recharge to the aquifer. Declines in the groundwater table reduce well yields and increase pumping costs, thus (other things equal) lowering net returns to the farmer. Continued overdraft of the aquifer is expected to lead to the eventual economic exhaustion of the aquifer. These factors combined with recent rapid increases in the price of natural gas have greatly increased producers' interest in irrigation strategies which will permit them to maintain net returns while simultaneously reducing water and energy use within the growing season.[1]

Static microeconomic analysis provides a useful theoretical framework for describing optimum resource use in a timeless environment. However, decisions to apply alternative quantities of irrigation water under uncertain weather conditions are dynamic and complex. The irrigation decision requires knowledge of the relationship between soil and water content, growth stress of the plant, and the stage of plant development for the crop. In our study optimal control theory and systems analysis are used to evaluate the potential impact of alternative irrigation strategies within the growing season and to derive optimal time path strategies which reduce water use while maintaining net returns to the producer.

MODEL DEVELOPMENT

Optimal control has gained acceptance by economists as a tool for deriving optimal time path strategies in solving dynamic economic problems. Optimal control theory has been used in numerous studies in general economics (Arrow, Chow, Dorfman, and others). The agricultural economics literature also contains several applications of optimal control theory (Richardson et al., Trapp, Zaveleta et al., and others).

Time path strategies from optimal control theory maximize a given performance criterion while simultaneously satisfying model constraints. In our analysis, optimal control is used to derive an irrigation strategy for the growing season which maximizes returns to the grain sorghum producer in the Oklahoma Panhandle, subject to a constraint on water use. A grain sorghum plant growth model developed by Arkin et al. is modified and used to derive results for alternative irrigation strategies. The model simulates the growth of a single grain sorghum plant through time by linking climatological factors and plant growth equations. In development of the grain sorghum model, the physical and physiological processes of light interception, photosynthesis, respiration, and water use were modeled independently and incorporated into the model. Equations describing seedling emergence, leaf area development, canopy light interception, and potential net photosynthesis were derived empirically from field measurements (Arkin et al., pp. 622-4). A series of efficiency functions relating the effects of non-optimum plant temperature and available soil water on net photosynthesis and plant growth are included in the model. Development of the grain sorghum plant is modeled on a daily basis with input data on such climatic variables as temperature, rainfall, and solar radiation. The yield per acre is determined by multiplying the head weight of the modeled "average" plant times the plant population per acre.

Five stages of growth for the grain sorghum plant are simulated by the model: emergence

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[1]Questions of the optimal temporal allocation of the underground stock water resource, from either the individual or the social viewpoint, are not addressed in our study. Readers interested in these problems may refer to the articles by Bekure and Eidman and by Burt.

to differentiation (Stage 1), differentiation to end of leaf growth (Stage 2), end of leaf growth to anthesis (Stage 3), anthesis to physiological maturity (Stage 4), and physiological maturity and beyond (Stage 5). Inadequate soil water at different stages of plant development results in different grain sorghum yields at the end of the growing season. However, soil water stress early in the season may not reduce final yield appreciably.

OPTIMAL CONTROL

The objective of optimal control theory is to determine the control signals that cause a process to satisfy the physical constraints and either minimize or maximize some performance criterion (Kirk). The formulation of an optimal control problem requires a mathematical description of the process to be controlled, such as a simulation model of an agricultural production system; a statement of the physical constraints, such as minimum and maximum supplies of groundwater; specification of the control variable, such as a scheduled irrigation; and specification of a performance criterion, such as net returns to the producer.

The input to the system for optimal control is the vector $\underset{\sim}{U}(t)$, variables to be controlled (irrigation application), and the vector $\underset{\sim}{X}(t)$, the output or state variable which is measured (net returns). The controller of an optimal control problem determines optimal levels of the input signals (groundwater applications). Kuhn-Tucker conditions are applied to difference equations in order to derive the appropriate discrete maximum principle.

The optimal allocation of groundwater during a single growing season can be described as:

$$(1)\quad S[\underset{\sim}{X}(t), \underset{\sim}{U}(t), t] = \sum_{t=0}^{T=1} F[\underset{\sim}{X}(t), \underset{\sim}{U}(t), t] + F(X_T)$$

where $S[\underset{\sim}{X}(t), \underset{\sim}{U}(t), t]$ is the objective function, defined as the summation of net returns that will be earned over the single crop season subject to N constraints and to any boundry conditions that may apply. The $F[\underset{\sim}{X}(t), \underset{\sim}{U}(t), t]$ is the intermediate function and shows dependence of the function on time paths of the state variables, control variables, and time within the relevant time range. For our study, the past climatological factors, quantities of irrigation water pumped, and growth stage of the grain sorghum plant interact to determine the value of the objective function.

$$(2)\quad \underset{\sim}{X}(t+1) - \underset{\sim}{X}(t) = f(\underset{\sim}{X}(t), \underset{\sim}{U}(t), t)$$

where the change in the value of the state variable (net returns to the producer) from one period to another is a function of the current state (current net returns), the decision taken (quantity of irrigation water applied), and the time period (stage of plant growth).

The values of the control variable $\underset{\sim}{U}(t)$ are restricted as follows:

$$(3)\quad g_i(\underset{\sim}{U}(t) \leq b_i(t) \qquad i = 1, 2, ..., m \quad t = 0, 1, ..., T-1$$

where $b_i(t)$ is a constant indicating the maximum quantity of irrigation water pumped for a single application to be 4.5 acre-inches. Any control variable $\underset{\sim}{U}(t)$ or irrigation application that satisfies the constraints is an admissible or feasible control. The control problem becomes one of deriving the values of the control variable or quantity of water applied, $\underset{\sim}{U}(t)$, through a crop season such that:

$$(4)\quad \text{Maximize:}\quad S[\underset{\sim}{X}(t), \underset{\sim}{U}(t), t] = \sum_{t=0}^{T-1} F[\underset{\sim}{X}(t), \underset{\sim}{U}(t), t] + F(X_T)$$

subject to:

$$(5)\quad X(t+1) - X(t) = f[X(t), U(t), t] \quad t = 1, 2, ..., M$$

$$(6)\quad g_i(U(t)) \leq bi(t) \qquad t = 1, 2, ..., T-1$$

and where X_o is given (Benavie).

The objective function of the control problem is expressed in terms of the output of the simulation model which, for our analysis, is the yield of the grain sorghum plant. Because the objective function is not explicitly expressed in terms of the decision variables, optimization techniques which rely on derivatives cannot be directly applied to the problem (Pedgen and Gately).

THE BOX COMPLEX SOLUTION PROCEDURE

Two general approaches often used for optimizing simulation models are response surface methodology and search methodology. The response surface methodology involves fitting first or second order equations to the simulation response surface using a series of simulation replications based on an appropriate experimental design. Search techniques, however, do not require derivative information. For this problem, the Box complex nonlinear programming procedure is used. The Box complex algorithm is a sequential search technique which can be used for solving problems with nonlinear objective functions subject to linear or nonlinear inequality constraints.[2] The sequential technique is incorporated into the simulation sequences of the plant growth model to derive an optimal irrigation schedule for a growing season. The procedure finds the maximum of a multivariable, nonlinear function subject to linear or nonlinear inequality constraints such as:

$$(7) \quad \text{Maximum:} \quad F(X_1, X_2, ..., X_n)$$

subject to:

$$(8) \quad G_i \leq X_i \leq H_i \qquad i = 1, 2, ..., m.$$

The implicit variables $X_{n+1}, X_{n+2}, ..., X_m$ are dependent functions of the explicit independent variables $X_1, X_2, ..., X_n$. The upper and lower constraint levels are designated by H_i and G_i, respectively, and can be either constant or functions of the independent variables $X_1, X_2, ..., X_n$. For our analysis the explicit variables are the quantities of groundwater applied in each irrigation period and the implicit variable is net returns to the producer.

ANALYSIS AND RESULTS

Three different production scenarios for grain sorghum were analyzed by using the grain sorghum plant growth model. The three different scenarios were run for 23 replications with actual weather data for the period 1953 to 1975. The first scenario derived annual production and net returns to the producer from dryland grain sorghum production. This scenario provided a partial test of the sensitivity and validity of the model. The second scenario simulated the traditional irrigation practice in the area of applying about 15 inches of groundwater each year during five different irrigation periods. The five irrigations included a three-inch preplant application and three-inch applications beginning on the first day of each of the first four growth stages.

The final scenario incorporated optimal control procedures to derive the irrigation sequence (using from zero to nine irrigation applications) that would maximize returns to the producer for each of the 23 replications. Preplant irrigations were assumed to be initiated on Julian Day 145 (May 25), 10 days before planting, if needed. In addition, up to eight postplant irrigations were possible. When Stage 1 was reached, three irrigation applications could occur (early Stage 1, mid Stage 1, and late Stage 1), each irrigation being 10 days apart. With the occurrence of Stage 2 and Stage 3, one irrigation application could be scheduled in each respective growth stage. With the attainment of Stage 4, three irrigation applications (early Stage 4, mid Stage 4, and late Stage 4) 10 days apart could be initiated.

Producers of irrigated grain sorghum in the Oklahoma Panhandle often apply between one and three inches of groundwater during each scheduled irrigation. The constraints for the model at each of the nine irrigation periods were set between zero and three inches. The model was designed so that either zero or from one to three inches of water was applied. That is, if any water was applied at one of the nine irrigation periods, at least a one-inch application was used. By incorporating the grain sorghum plant growth model with the desired performance criteria and appropriate physical constraints, an optimal irrigation schedule was developed from the nine proposed irrigation periods.

The Box complex derives the net returns to the producer from different values of the explicit variables. The convergence criterion requires the objective function values at each point to be within β units of each other for ξ consecutive iterations. For our analysis, the Box complex derived quantities of water pumped at each of the eight irrigation periods, and these values were used in the grain sorghum plant model to derive field yields. Field yield and quantities of groundwater pumped were used by the objective function to derive net returns. The complex continued to derive irrigation values and to incorporate these values into the plant growth model until five consecutive objective functions were 0.10 units or 10 cents apart. For each replication, the Box complex was initiated three times with different pseudo random numbers which enabled different initial configurations to be derived. From these three different initiations, a maximum value for the objective function was derived which could be considered a global maximum for each replication.

The representative quarter section of land used for the analysis contains 155 acres of grain sorghum irrigated from a 900 gallon per minute well. For each replication the well was 350 feet deep with a water level of 250 feet and a surface distribution system. The price of grain sorghum for each replication was stated at \$3.98/cwt. Nonirrigation costs were determined from the Oklahoma State University budgets for the Oklahoma Panhandle area. The Oklahoma State University irrigation cost generator was used to derive fixed and variable irrigation costs. Natural gas was the fuel used to deliver the irrigation water and was priced \$1.50/MCF.

[2]The Box complex is a direct search technique; however, unlike the Hooke and Jeeves pattern search, Rosenbrock's method of rotating coordinates, and the simplex method of Nelder and Mead, the Box complex can be used for problems that incorporate constraints.

Variations in annual results are functions of the input values of the plant growth model. Quantities of groundwater applied along with the climatic factors such as rainfall, solar radiation, and maximum and minimum temperatures affect the output of the grain sorghum plant growth model.

Dryland Scenario

The results of the dryland run are given in Table 1. In three of the 23 years simulated, grain sorghum did not make a stand. For the years in which a stand was achieved, the yields range from a minimum of 7.86 cwt/acre to a maximum of 59.33 cwt/acre. From Table 1, variations in yields between years are characteristic of dryland yields in the area. Returns to dryland producers also vary because of the fluctuations in yields and prices.

TABLE 1. SIMULATED YIELDS, REVENUES, COSTS, AND RETURNS FOR DRYLAND GRAIN SORGHUM USING 1953-75 CLIMATIC DATA

Replications	Field Yield	Revenues	Costs	Net Returns
	(cwt./ac.)	($/ac.)	($/ac.)	($/ac.)
1	9.01	35.86	37.48	-1.62
2	7.86	31.28	37.48	-6.20
3	9.14	36.38	37.48	-1.10
4	0.00	0.00	37.48	-37.48
5	20.86	83.02	37.48	45.54
6	30.05	119.60	37.48	82.12
7	10.23	40.72	37.48	3.24
8	15.90	63.28	37.48	25.80
9	43.63	173.65	37.48	136.17
10	54.17	215.60	37.48	178.12
11	8.99	35.78	37.48	-1.70
12	34.67	137.99	37.48	100.51
13	17.70	70.45	37.48	32.97
14	19.96	79.44	37.48	41.96
15	21.70	86.37	37.48	48.89
16	18.12	72.12	37.48	34.64
17	0.00	0.00	37.48	-37.48
18	0.00	0.00	37.48	-37.48
19	28.60	113.83	37.48	76.35
20	59.33	236.13	37.48	198.65
21	19.12	76.10	37.48	38.62
22	41.04	163.34	37.48	125.86
23	12.79	50.90	37.48	13.42
Average	20.99	83.56	37.48	46.08

Typical Irrigation Scenario

In this scenario, a three-inch preplant irrigation was applied and three-inch applications were made when Stages 1-4 were attained. For this scenario, 15 acre-inches was received by the plant, requiring gross irrigation pumping of 22.5 inches. From Table 2, the average yield for this scenario is 58.87 cwt/acre with maximum production of 68.78 cwt/acre and minimum production of 49.76 cwt/acre. Returns to the producer average $79.95/acre with a maximum of $119.38/acre and a minimum of $43.68/acre. Variations in annual yields, even with intensive irrigation, are due to other climatological variables. That is, temperature and solar radiation in a particular year may be insufficient for maximum production.

TABLE 2. SIMULATED GRAIN SORGHUM YIELDS, REVENUES, COSTS, AND RETURNS FROM CONSTANT 15 INCH IRRIGATION WATER APPLICATION USING 1953-75 CLIMATIC DATA

Replications	Field Yield	Revenues	Variable Irrigation Cost	Total Irrigation Cost	Total Costs	Net Returns
	(cwt/ac)	($/ac)	($/ac)	($/ac)	($/ac)	($/ac)
1	60.19	239.56	35.77	58.82	154.36	85.20
2	55.13	219.42	35.77	58.82	154.36	65.06
3	68.78	273.74	35.77	58.82	154.36	119.38
4	60.23	239.72	35.77	58.82	154.36	85.36
5	61.53	244.89	35.77	58.82	154.36	90.53
6	65.70	261.49	35.77	58.82	154.36	107.13
7	64.34	256.07	35.77	58.82	154.36	101.71
8	66.96	266.50	35.77	58.82	154.36	112.14
9	56.90	226.46	35.77	58.82	154.36	72.10
10	61.53	244.89	35.77	58.82	154.36	90.53
11	60.27	239.87	35.77	58.82	154.36	85.51
12	49.76	198.04	35.77	58.82	154.36	43.68
13	52.80	210.14	35.77	58.82	154.36	55.78
14	51.46	204.81	35.77	58.82	154.36	50.45
15	53.42	212.61	35.77	58.82	154.36	58.25
16	55.26	219.93	35.77	58.82	154.36	65.57
17	55.26	219.93	35.77	58.82	154.36	65.57
18	58.48	232.75	35.77	58.82	154.36	78.39
19	54.83	218.30	35.77	58.82	154.36	63.94
20	60.23	239.72	35.77	58.82	154.36	85.36
21	64.54	256.86	35.77	58.82	154.36	102.51
22	51.48	204.89	35.77	58.82	154.36	50.53
23	64.99	258.66	35.77	58.82	154.36	104.30
Average	58.87	234.32	35.77	58.82	154.36	79.96

Optimal Control Scenario

For this scenario, the objective was to develop an irrigation strategy from the nine proposed irrigation stages which would maximize returns to the producer. If an irrigation were initiated, the irrigator would apply from one to three inches of water. The results from this scenario are presented in Tables 3 and 4.

From Table 3, the mean field yield for this scenario is 58.51 cwt/acre with maximum production of 68.46 cwt/acre and minimum production of 49.47 cwt/acre. Returns range from $130.79/acre to $62.19/acre with a mean return for the 23-year study period at $93.76/acre.

In Table 4, the first column depicts the mean application of irrigation water for the 23-year study period. The results of this scenario show that a preplant irrigation was required in only seven years of the 23-year study period. The most intensive irrigation occurred during late Stage 1, Stage 2, Stage 3, and early Stage 4. During one wet year, only one irrigation of 1.5 inches in Stage 3 was applied whereas in a dry year 24 acre-inches was required.

CONCLUDING COMMENTS

Comparable grain sorghum yields are achieved under the optimal control and 15 acre-inch irrigation scenarios. The average production for the optimal control scenario is 58.51 cwt/acre, whereas for the 15-inch scenario the yield averages 58.87 cwt/acre. However, the amount of irrigation water applied and net returns are substantially different for the two scenarios. Irrigation pumping for the 15-inch scenario averages 22.5 inches, whereas in the optimal control scenario the irrigation applications average 13.0 inches/acre. Thus, approximately 9.5 additional acre-inches of water is pumped in the 15-acre-inch scenario with no appreciable increase in grain sorghum production. Net returns to the producer average $93.76/acre in the optimal control scenario and $79.95/acre in the 15-inch scenario.

The results of the preliminary analysis indicate the potential for irrigation producers in the Oklahoma Panhandle to reduce irrigation water applications on grain sorghum while maintaining yields and increasing net returns. The control theory approach developed to test this hypothesis derives optimal irrigation strategies *ex post* using historical weather data. As such, it is not appropriate for field-level usage for scheduling irrigation applications.

Efforts are currently underway to develop an irrigation scheduling model based on the grain sorghum plant growth relationships. Short-term weather predictions, soil water monitoring devices, and feedback loops to update soil water and plant growth data on a daily basis will be components of the irrigation scheduling model.

TABLE 3. SIMULATED GRAIN SORGHUM YIELDS, REVENUES, COSTS, AND RETURNS FOR THE OPTIMAL CONTROL SCENARIO USING 1953-75 CLIMATIC DATA

Replications	Field Yield	Revenues	Variable Irrigation Cost	Total Irrigation Cost	Total Costs	Net Returns
	(cwt/ac)	($/ac)	($/ac)	($/ac)	($/ac)	($/ac)
1	60.49	240.75	29.43	52.48	148.02	92.73
2	53.96	214.76	26.19	49.24	144.78	69.98
3	68.46	272.47	23.09	46.14	141.68	130.79
4	60.57	241.07	37.92	60.97	156.51	84.56
5	61.17	243.46	20.84	43.89	139.43	104.02
6	64.68	257.43	14.31	37.36	132.90	124.53
7	64.25	255.71	26.83	49.88	145.42	110.29
8	66.50	264.67	26.66	49.71	145.25	119.42
9	57.21	227.70	13.90	36.95	132.49	95.20
10	60.94	242.54	5.82	28.87	124.41	118.13
11	59.14	235.38	21.46	44.51	140.05	95.32
12	49.47	196.89	14.76	37.81	133.35	63.54
13	52.45	208.75	15.62	38.67	134.21	74.54
14	50.96	202.82	22.04	45.09	140.63	62.19
15	52.85	210.34	11.76	34.81	130.35	79.99
16	55.04	219.06	19.34	42.39	137.93	81.13
17	55.15	219.50	33.34	56.39	151.93	67.56
18	58.39	232.39	31.98	55.03	150.57	81.82
19	54.68	217.63	19.46	42.51	138.05	79.57
20	60.00	238.80	2.39	25.43	120.97	117.82
21	64.08	255.04	17.51	40.56	136.10	118.94
22	50.96	202.82	17.77	40.82	136.36	66.46
23	64.39	256.27	19.80	42.85	138.39	117.89
Average	58.51	232.88	20.53	43.58	139.12	93.76

TABLE 4. MEAN, VARIANCE, STANDARD DEVIATION, MAXIMUM AND MINIMUM VALUES FOR SELECTED CATEGORIES IN THE OPTIMAL CONTROL SCENARIO

Categories	Units	Mean	Variance	Standard Deviation	Coefficient of Variation	Maximum	Minimum
Irrigations:							
Preplant	acre inch	0.91	2.24	1.50	1.64	4.42	0.00
Early Stage 1	acre inch	0.57	1.69	1.30	2.28	4.50	0.00
Mid Stage 1	acre inch	0.86	2.47	1.57	1.83	4.50	0.00
Late Stage 1	acre inch	2.21	4.14	2.03	0.92	4.50	0.00
Stage 2	acre inch	3.10	2.65	1.63	0.52	4.50	0.00
Stage 3	acre inch	2.53	3.04	1.74	0.69	4.50	0.00
Early Stage 4	acre inch	1.87	2.78	1.67	0.89	4.50	0.00
Mid Stage 4	acre inch	0.73	1.04	1.02	1.40	3.94	0.00
Late Stage 4	acre inch	0.13	0.18	0.42	3.24	1.52	0.00
Total	acre inch	12.91	27.15	5.21	0.40	23.85	1.50
Field Yield	cwt/ac	58.51	28.18	5.31	0.09	68.46	49.47
Returns	$/ac	93.76	469.71	21.67	0.23	130.79	62.19

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