FACULTY OF ACTUARIAL SCIENCE & ECONOMETRICS

A & E REPORT

REPORT AE 19/85

GENERAL BOUNDS ON RUIN PROBABILITIES

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Date: November 1985

Series and Number: Report AE 19/85

Pages: 5

Price: No charge

JEL Subject classification: 213

IAR Subject classification: M10, M13

Keywords: Ruin probabilities, Chebyshev inequality

Abstract:
In this paper we consider general bounds on ultimate ruin probabilities in a Poisson process when the claim severity distribution is not exponentially bounded. The bounds are derived using a variant of the Chebyshev inequality. Ruin probabilities are bounded using the claims distribution function as well as some of its partial moments, and the Poisson parameter.

1. Introduction

Recently several authors (Brockett (1985), Teugels (1985), Waters and Papatriandafylou (1985), and Runnenburg and Goovaerts (1985)), have applied a Chebyshev-like inequality to deduce bounds on compound distributions and ruin probabilities. The inequality used reads

$$F(\infty) - F(x) \le \frac{1}{f(x)} \int_0^\infty f(s)dF(s) \tag{1.1}$$

where F(x) is a defective distribution function, f is any non-decreasing function with f(x) > 0, and x > 0.

We assume in the sequel that claims occur according to a Poisson process with intensity α, that the individual claims are i.i.d. random variables with common distribution function F_X having mean μ, that the gross premium intensity equals some constant c, and that the initial wealth is u. In von Bahr (1975) the asymptotic behaviour of the ruin probability is studied assuming that the tail of the claim distribution varies regularly at infinity, that is

$$1 - F_X(x) \sim x^{-r}L(x), \qquad \text{as } x \to \infty$$

for some function L varying slowly at infinity:

$$L(tx) \sim L(x) \qquad \text{as } x \to \infty$$

for any t > 0. The interpretation of the symbol ~ is that the ratio of the expressions on the left hand side and on the right hand side asymptotically tends to 1. Von Bahr shows that the ruin probability $\psi(u)$ satisfies

$$\psi(u) \sim \frac{qu^{-r+1}L(u)}{p\mu(r-1)} \tag{1.2}$$

where the constants p and q = 1-p are defined by

$$p = 1 - \frac{\alpha\mu}{c} \tag{1.3}$$

Both p and q are assumed to be positive.

In Runnenburg and Goovaerts (1985) bounds are derived for the tail probabilities of compound distributions. Here we use a similar method to give bounds for ruin probabilities when the tail of F_X is not exponentially bounded.

2. Upper bound for ultimate ruin probabilities

It is well known, see e.g., Bowers et al. (1985), that the probability of non-ruin 1 - $\psi(u)$ can be written as a compound Geometric(q) distribution function

$$1 - \psi(u) = F_L(u) = p\sum_{n=0}^{\infty} q^n H^{n*}(u) \qquad (2.1)$$

where p = 1-q is defined in (1.3) and

$$H(x) = \frac{1}{\mu}\int_0^x [1 - F_X(t)]dt \qquad (2.2)$$

Notice that for $x \in [0,u]$, $F_L(x)$ is equal to the following defective distribution

$$\tilde{F}(x) = p\sum_{n=0}^{\infty} q^n H^n(u)\tilde{H}^{n*}(x), \qquad -\infty < x < \infty \qquad (2.3)$$

where $\tilde{H}(x)$ is defined as

$$\begin{aligned}\tilde{H}(x) &= \frac{H(x)}{H(u)}, \quad 0 \le x \le u \\ &= 1, \quad u < x\end{aligned} \qquad (2.4)$$

Now compute the Laplace transform with $\tilde{F}(x)$:

$$\int_0^{\infty} e^{tx} d\tilde{F}(x) = \frac{p}{1-qH(u)\int_0^{\infty} e^{tx} d\tilde{H}(x)} \qquad (2.5)$$

$$= \frac{p}{1-q\int_0^u e^{tx} dH(x)}$$

Using (1.1) one obtains

$$\tilde{F}(\infty) - \tilde{F}(u) \le e^{-tu}\int_0^{\infty} e^{tx} d\tilde{F}(x) = \frac{pe^{-tu}}{1-q\int_0^u e^{tx} dH(x)} \qquad (2.6)$$

As observed earlier we have $\tilde{F}(u) = F_L(u)$, and as

$$\tilde{F}(\infty) = p\sum_{n=0}^{\infty} q^n H^n(u) = \frac{p}{1-qH(u)}, \qquad (2.7)$$

we have

$$1 - F_L(u) \le \frac{q-qH(u)}{1-qH(u)} + \frac{pe^{-tu}}{1 - q\int_0^u e^{tx} dH(x)} \qquad (2.8)$$

Now suppose that H is not exponentially bounded, but that the moments up to some r do exist:

$$\int_0^\infty x^r dH(x) < \infty, \quad \int_0^\infty x^s dH(x) = \infty \quad \text{for all } s > r \tag{2.9}$$

Further, let n be the smallest integer with $r \le n$, and let $\delta = n-r$. Then for all y, $0 \le y \le x$, we have

$$\frac{e^{ty} - \sum_{j=0}^{n-1} (ty)^j/j!}{y^{n-\delta}} \le \frac{e^{tx} - \sum_{j=0}^{n-1} (tx)^j/j!}{x^{n-\delta}} \tag{2.10}$$

Rearranging terms and integrating we obtain the following inequality:

$$\int_0^x e^{ty} dH(y) \le \sum_{j=0}^{n-1} \frac{t^j \nu_j(x)}{j!} + x^{-(n-\delta)} \nu_{n-\delta}(x) \{e^{tx} - \sum_{j=0}^{n-1} (tx)^j/j!\} \tag{2.11}$$

where $\nu_k(x)$ denotes the following partial moment:

$$\nu_k(x) = \int_0^x y^k dH(y) \tag{2.12}$$

Using (2.2) and partial integration, it is easy to express $\nu_k(x)$ in terms of partial moments of $F_X(x)$.
By (2.11) and (2.8), for any $t \ge 0$ an upper bound for the ruin probability is given by

$$\psi(u) \le \frac{q-qH(u)}{1-qH(u)} + pe^{-tu} \Big/ \left[1 - \frac{qe^{tu}}{u^{n-\delta}} \nu_{n-\delta}(u) - \sum_{j=0}^{n-1} q \frac{t^j}{j!} \{\nu_j(u) - \frac{\nu_{n-\delta}(u)}{u^{n-j-\delta}}\}\right] \tag{2.13}$$

Now choose

$$t = [(r-1)\ln(u) + \ln(r-1)] / u \tag{2.14}$$

such that

$$e^{tu} = (r-1)\, u^{r-1} \tag{2.15}$$

then (2.13) may be written as

$$\psi(u) \le \frac{q-qH(u)}{1-qH(u)} + \frac{p}{(r-1)u^{r-1}} \Big/ \tag{2.17}$$

$$\left[1 - \frac{q}{u}(r-1)\nu_r(u) - \sum_{j=0}^{n-1} q\frac{u^{-j}}{j!}\{(r-1)\ln(u)+\ln(r-1)\}^j\{\nu_j(u)-\frac{\nu_r(u)}{u^{r-j}}\}\right]$$

This bound holds for general u. For large u it behaves as

$$\frac{q-qH(u)}{1-qH(u)} + \frac{p}{(r-1)u^{r-1}}\frac{1}{1-qH(u)} \tag{2.18}$$

To compare this with von Bahr's asymptotical result we choose a Pareto claims distribution

$$F_X(x) = 1 - x^{-(r+1)}, \quad x \ge 1 \tag{2.19}$$

such that by (2.2)

$$H(x) = 1 - x^{-r}, \quad x \ge 1 \tag{2.20}$$

and obtain as asymptotical bound for the ruin probability

$$\frac{qu^{-r}}{1-q(1-u^{-r})} + \frac{p}{[(r-1)u^{r-1}]\,[1 - qu^{-r}]} \tag{2.21}$$

which can be rewritten into

$$\psi(u) \le \frac{q+\frac{p}{u(r-1)}}{pu^r + q} \tag{2.22}$$

whereas von Bahr's asymptotical value for the ruin probability is

$$\psi(u) \sim \frac{q}{pu^r} \tag{2.23}$$

using (1.2) with $L(x) \equiv 1$, $r+1$ replacing r, and $\mu = \frac{1}{r}$.

References

B. von Bahr (1975). Asymptotic ruin probabilities when exponential moments do not exist. Scandinavian Actuarial Journal, 6-10.

P.L. Brockett (1985). Probability bounds on downtimes. Naval Research Logistics Quarterly 32, no. 2, 329-335.

N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, and C.J. Nesbitt (1985). Actuarial Mathematics. Society of Actuaries, to appear.

J.Th. Runnenburg, and M.J. Goovaerts (1985). Bounds on compound distributions and stop-loss premiums. Insurance: Mathematics and Economics, 4, no. 4, to appear.

J. Teugels (1985). Approximation and estimation of some compound distributions. Insurance: Mathematics and Economics, 4, no. 3, 143-154.

H.R. Waters, and A. Papatriandafylou (1985). Ruin probabilities allowing for delays in claims settlement. Insurance: Mathematics and Economics, 4, no. 3, 113-122.