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ORDERING OF RISKS AND WEIGHTED COMPOUND DISTRIBUTIONS

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Abstract:

Some invariance properties of net stop loss ordering of risks are examined and proved in the framework of weighted compound distributions.

1. Introduction

Let $\{X_j\}$ be a sequence of non-negative i.i.d. random variables with common distribution function F_X. One is often interested in compound sums

$$S = X_1 + X_2 + \dots + X_N \tag{1.1}$$

where the counting variable N is independent of the terms X_j (see e.g. [2]). In the collective risk model in insurance S represents the total amount of claims in a portfolio, modeled as the sum of a random number of individual claims.

If

$$p_n = \Pr\ (N = n) \tag{1.2}$$

then the distribution of S reads

$$F_S(x) = \sum_{n=0}^{\infty} p_n F_X^{n*}(x) \tag{1.3}$$

where $F_X^{n*}(x)$ is the distribution function of $X_1 + X_2 + \dots + X_n$, so F_X^{n*} is the n-fold convolution of F_X .

The distribution F_S is called a compound distribution. Suppose further that the probabilities p_n depend on a parameter θ . We may interpret $p_n(\theta)$ as a conditional probability of $N = n$ given $\boldsymbol{\theta} = \theta$:

$$p_n(\theta) = \Pr(N = n | \boldsymbol{\theta} = \theta) \tag{1.4}$$

We then obtain

$$F_{S|\boldsymbol{\theta}}(x|\theta) = \Pr(S \le x | \boldsymbol{\theta} = \theta) = \sum_{n=0}^{\infty} p_n(\theta) F_X^{n*}(x) \tag{1.5}$$

Hence

$$F_S(x) = \int F_{S|\boldsymbol{\theta}}(x|\theta) dU(\theta) \tag{1.6}$$

where

$$U(\theta) = \Pr(\boldsymbol{\theta} \le \theta) \tag{1.7}$$

Consequently

$$F_S(x) = \sum_{n=0}^{\infty} \int p_n(\theta)dU(\theta)F_X^{n*}(x) \qquad (1.8)$$

This distribution is called a weighted compound distribution. In the special case where $p_n(\theta) = e^{-\theta}\theta^n/n!$ denotes a Poisson (θ) distribution, and the weighting distribution $U(\theta)$ is a Gamma (γ,c) distribution, the weighted compound Poisson distribution is the compound Negative Binomial $(\gamma, \frac{c}{1+c})$ distribution.

2. Stop loss dominance

A well known ordering of random variables is provided by stochastic dominance (of the secondorder), where X is preferred over Y if for all $x \in \mathbb{R}^+$

$$\int_0^x F_X(y)dy \le \int_0^x F_Y(y)dy \qquad (2.1)$$

One may show that for distributions having the same mean this ordering coincides with the one given in

Definition 1 X precedes Y in the loss stop order, written $X \prec Y$, (or equivalently $F_X \prec F_Y$) if E(X) is finite and for all $X \in \mathbb{R}^+$

$$\int_x^{\infty}(1 - F_X(y))dy \le \int_x^{\infty}(1 - F_Y(y))dy \qquad (2.2)$$

Stop loss ordering uses the tails of the distribution function, and is more appropriate to order losses than stochastic dominance, often used for gains.

Using partial integration an equivalent definition may be derived.

Definition 2 $X \prec Y$ if and only if E(X) is finite and for all $X \ge 0$

$$\int_x^{\infty}(y - x)dF_X(y) \le \int_x^{\infty}(y - x)dF_Y(y) \qquad (2.3)$$

The integral in the left hand member equals the expected value of the random variable X as far as it exceeds x . Using the notation $t_+ = \max\{t,0\}$ we may rewrite (2.3) as

$$E(X - x)_+ \leq E(Y - x)_+ \tag{2.4}$$

In a stop loss contract an insurer pays the loss X over a retained amount x , and $E(X - x)_+$ is the net premium for such a contract, so the term stop loss order is explained.

A sequence $\phi(0),\phi(1),\dots,\phi(n)$ is convex when for $\nu = 0,1,\dots,n-2$ we have

$$\Delta^2\phi(\nu) = \{\phi(\nu + 2) - \phi(\nu + 1)\} - \{\phi(\nu + 1) - \phi(\nu)\} \geq 0 \tag{2.5}$$

In the following section we will use the following lemma's:

<u>Lemma 1</u> The inequality

$$\sum_{\nu=0}^{n} \phi(\nu)a_\nu \geq 0 \tag{2.6}$$

holds for any convex sequence ϕ if and only if the sequence $a_0,a_1,\dots,a_\nu$ satisfies

$$\sum_{\nu=0}^{n} (\nu - k)_+ a_\nu \geq 0 \quad , \quad k = 0,1,\dots,n-1 \tag{2.7}$$

and

$$\sum_{\nu=0}^{n} a_\nu = \sum_{\nu=0}^{n} \nu a_\nu = 0 \tag{2.8}$$

<u>Proof</u> (This lemma is a special case of a much more general result obtained in [4]).

The conditions are necessary since the sequences

$$\phi(\nu) = \pm 1 \ , \quad \phi(\nu) = \pm \nu \quad \text{and} \quad \phi(\nu) = (\nu - k)_+ \ , \quad k = 0,1,\dots,n$$

are convex.

To prove sufficiency, first observe that we may write

$$\phi(\nu) = \phi(0) + \Delta\phi(0) \ . \ \nu + \sum_{j=1}^{n-1} [\Delta^2\phi(j - 1)](\nu - j)_+ \tag{2.9}$$

Using (2.7) and (2.8) we obtain from (2.9)

$$\sum_{\nu=0}^{n} \phi(\nu)a_\nu = \phi(0) \sum_{\nu=0}^{n} a_\nu + \Delta\phi(0) \sum_{\nu=0}^{n} \nu a_\nu$$
$$+ \sum_{j=1}^{n-1} [\Delta^2\phi(j - 1)] \sum_{\nu=0}^{n} (\nu - j)_+ a_\nu \geq 0 \tag{2.10}$$

so the lemma holds.

In fact we will need a less restrictive sufficient condition:

Corollary 1 If $\phi(\nu)$ is a non-decreasing convex sequence, and (2.7) holds together with $\sum_{\nu=0}^{n} a_\nu = 0$, then still (2.6) holds.

Proof We have $\sum_{\nu=0}^{n} \nu a_\nu \geq 0$ by (2.7) with k = 0 , and $\Delta\phi(0) \geq 0$ in (2.10) gives the desired inequality.

Lemma 2 Let for some $\alpha \geq 0$

$$\phi(n) = \int_{\alpha}^{\infty} (x - \alpha) dF^{n*}(x) ,$$

then $\phi(0),\phi(1),\ldots$ denotes a non-decreasing convex sequence.

Proof We must prove $\Delta^2\phi(n) \geq 0$, $n = 0,1,\ldots$. Since $F^{n*}(x)$ is the distribution function of $X_1 + X_2 + \ldots + X_n$, this is equivalent to

$$E(\sum_{i=1}^{n} X_i + X_{n+1} - \alpha)_+ + E(\sum_{i=1}^{n} X_i + X_{n+2} - \alpha)_+ \leq$$

$$\leq E(\sum_{i=1}^{n} X_i - \alpha)_+ + E(\sum_{i=1}^{n+2} X_i - \alpha)_+ \qquad (2.12)$$

For this to hold it is sufficient that for all x, y, $z \geq 0$,

$$(x + y - \alpha)_+ + (x + z - \alpha)_+ \leq (x - \alpha)_+ + (x + y + z - \alpha)_+ \qquad (2.13)$$

For $x \geq \alpha$, equality prevails in (2.13). For $x < \alpha$, let $\beta = \alpha - x$, then (2.13) becomes

$$(y - \beta)_+ + (z - \beta)_+ \leq (y + z - \beta)_+ \qquad (2.14)$$

This inequality is easy to check, so $\phi(0),\phi(1),\ldots$ is a convex sequence. For it to be non-decreasing it is sufficient that $\phi(1) \geq \phi(0)$. But always

$$0 = \phi(0) \leq \phi(1) = E(X_1 - \alpha)_+$$

3. Invariance properties of stop loss dominance

It is easily seen that stop loss ordering is preserved under mixing [1]:

Theorem 1 Let $F_1, F_2, \ldots$ and $H_1, H_2, \ldots$ be distributions and let $p_1, p_2, \ldots$ be a discrete probability distribution.
If $F_n \prec H_n$ for all n and if $\sum_n p_n \int x dF_n(x) < \infty$ then

$$\sum_n p_n F_n \prec \sum_n p_n H_n$$

Proof Trivial.

We will next show that stop loss dominance is preserved under convolution.

Lemma 1 Let X, Y and Z be independent random variables with $E(X) < \infty$ and $E(Y) < \infty$. If $Y \prec Z$, then $X + Y \prec X + Z$.

Proof First we observe that $E(X+Y) < \infty$. Also, for all t

$$E(X + Y - t)_+ = E(E(X + Y - t)_+ | X)$$

$$\leq E(E(X + Z - t)_+ | X) = E(X + Z - t)_+$$

By repeated application of lemma 1 we obtain

Theorem 2 If $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are sequences of independent random variables with for all i $X_i \prec Y_i$, then

$$X_1 + X_2 + \ldots + X_n \prec Y_1 + Y_2 + \ldots + Y_n$$

From theorems 1 and 2 we immediately obtain

Theorem 3 If X_i and Y_i are as in theorem 2 and N is an independent counting variable with $E(N) < \infty$, we have

$$X_1 + X_2 + \ldots + X_N \prec Y_1 + Y_2 + \ldots + Y_N$$

To prove that stop loss order of counting variables is preserved under compounding is slightly more complex.

Theorem 4 If $N \prec N'$, and $X_1, X_2, \ldots$ are independent with common distribution function F , then

$$X_1 + X_2 + \ldots + X_N \prec X_1 + X_2 + \ldots + X_{N'}$$

Proof Let $a_\nu = \Pr(N' = \nu) - \Pr(N = \nu)$, then by $N' \prec N$ the conditions of corollary 2.1 are satisfied. Defining

$$\phi(\nu) = \int_x^\infty (y - x)\,dF^{n*}(y)$$

we have by lemma 2 $\sum_{\nu=0}^\infty \phi(\nu) a_\nu \geq 0$ or equivalently

$$\sum_{\nu=0}^\infty \Pr(N' = \nu) \int_x^\infty (y - x)\,dF^{\nu*}(y) \geq \sum_{n=0}^\infty \Pr(N = \nu) \int_x^\infty (y - x)\,dF^{\nu*}(y)$$

So

$$E(X_1 + \ldots + X_{N'} - x)_+ \geq E(X_1 + \ldots + X_N - x)_+$$

The previous results may be summarized in the following theorem:

Theorem 5 Let $S_1 = X_1 + \ldots + X_{N_1}$, $S_2 = Y_1 + \ldots + Y_{N_2}$, with $N_1 \prec N_2$ and $X \prec Y$. Then $S_1 \prec S_2$.

Next we show that stop loss dominance is preserved for weighted compound Poisson distributions.

Theorem 6 Consider two weighted Poisson distributions S_1 and S_2 with

$$F_{S_j}(x) = \sum_{n=0}^\infty \int_0^\infty e^{-\theta}\theta^n/n!\,dU_j(\theta) F^{n*}(x) \quad , \quad j = 1,2.$$

If $U_1 \prec U_2$, then $S_1 \prec S_2$.

Proof Let N_1 and N_2 be random variables satisfying

$$p_n^{(j)} = \Pr(N^{(j)} = n) = \int_0^\infty e^{-\theta}\theta^n/n!\,dU_j(\theta) \tag{3.1}$$

for j = 1,2 and n = 0,1,... . By theorem 5 it suffices to show $N_1 \prec N_2$. Let

$$a_n = p_n^{(1)} - p_n^{(2)} , \tag{3.2}$$

$$V(\theta) = U_1(\theta) - U_2(\theta) ,$$

$$\phi(\theta) = -\int_\theta^\infty V(w)\,dw$$

Consider

$$a_n = \int_0^\infty e^{-\theta}\theta^n/n!\,dU_1(\theta) - \int_0^\infty e^{-\theta}\theta^n/n!\,dU_2(\theta) = \int_0^\infty e^{-\theta}\theta^n/n!\,dV(\theta) \tag{3.3}$$

Two successive partial integrations yield

$$a_0 = \int_0^\infty dV(\theta) + \int_0^\infty (e^{-\theta} - 1)dV(\theta) =$$

$$= 0 + (e^{-\theta} - 1)V(\theta)\Big|_0^\infty + \int_0^\infty e^{-\theta}V(\theta)d\theta$$

$$= 0 + 0 + e^{-\theta}\phi(\theta)\Big|_0^\infty + \int_0^\infty e^{-\theta}\phi(\theta)d\theta$$

$$= -\phi(0) + \int_0^\infty [\frac{d^2}{d\theta^2} e^{-\theta}]\phi(\theta)d\theta$$

$$a_1 = \int_0^\infty \theta e^{-\theta} dV(\theta)$$

$$= \theta e^{-\theta}V(\theta)\Big|_0^\infty - \int_0^\infty (e^{-\theta} - \theta e^{-\theta})V(\theta)d\theta$$

$$= -(e^{-\theta} - \theta e^{-\theta})\Big|_0^\infty + \int_0^\infty \frac{d}{d\theta}(e^{-\theta} - \theta e^{-\theta})\phi(\theta)d\theta$$

$$= \phi(0) + \int_0^\infty [\frac{d^2}{d\theta^2} \theta e^{-\theta}]\phi(\theta)d\theta$$

$$a_n = \int_0^\infty [\frac{d^2}{d\theta^2} e^{-\theta}\theta^n/n!]\phi(\theta)d\theta \;, \quad n = 2,3,... \tag{3.4}$$

It is easily shown that

$$\sum_{k=0}^{n} \sum_{\nu=0}^{k} [\frac{d^2}{d\theta^2} e^{-\theta}\theta^n/n!] = e^{-\theta}\theta^n/n! \tag{3.5}$$

So

$$\sum_{k=0}^{n} \sum_{\nu=0}^{k} a_\nu = (n + 1)(-\phi(0)) + n(+\phi(0))$$

$$+ \sum_{k=0}^{n} \sum_{\nu=0}^{k} \int_0^\infty [\frac{d^2}{d\theta^2} e^{-\theta}\theta^\nu/\nu!]\phi(\theta)d\theta$$

$$= -\phi(0) + \int_0^\infty e^{-\theta}\theta^n/n!\phi(\theta)d\theta \tag{3.6}$$

On the other hand

$$\sum_{k=0}^{n} \sum_{\nu=0}^{k} a_\nu = \sum_{\nu=0}^{n} \sum_{k=\nu}^{n} a_\nu = \sum_{\nu=0}^{n} (n - \nu + 1)a_\nu =$$

$$= (n+1)\sum_{\nu=0}^{n} a_\nu - \sum_{\nu=0}^{n} \nu a_\nu$$

$$= -(n+1)\sum_{\nu=n+1}^{\infty} a_\nu + \sum_{\nu=n+1}^{\infty} \nu a_\nu - \phi(0) \qquad (3.7)$$

using the fact that

$$\sum_{\nu=0}^{\infty} \nu a_\nu = \int_0^\infty \sum_{\nu=0}^{\infty} \nu e^{-\theta}\theta^\nu/\nu!\,dV(\theta) = \int_0^\infty \theta\,dV(\theta) = \phi(0) \qquad (3.8)$$

From the fact that $U_1 \prec U_2$ we obtain for all $\theta \geq 0$

$$\phi(\theta) = -\int_\theta^\infty V(w)dw = \int_\theta^\infty (1 - U_1(w))dw - \int_\theta^\infty (1 - U_2(w))dw \leq 0 \qquad (3.9)$$

So for all $n = 0,1,\ldots$ by (3.7), (3.6) and (3.9)

$$\sum_{\nu=n+1}^{\infty} (\nu - (n+1))(p_\nu^{(1)} - p_\nu^{(2)}) = \int_0^\infty e^{-\theta}\theta^n/n!\,\phi(\theta)d\theta \leq 0$$

Since by (3.8) $\sum_{\nu=0}^{\infty} \nu(p_\nu^{(1)} - p_\nu^{(2)}) \leq 0$, we have $N_1 \prec N_2$, and the theorem holds.

4. Application: minimal and maximal distributions

In this section we exhibit minima and maxima in the sense of stop loss ordering in restricted classes of compound distributions. First we prove a lemma giving an easy to check sufficient condition for stop loss dominance.

Lemma 1 Suppose a real number $c \geq 0$ exists such that

$$F_1(x) \leq F_2(x) \quad \text{for} \quad 0 \leq x < c \qquad (4.1)$$

$$F_1(x) \geq F_2(x) \quad \text{for} \quad x \geq c$$

and suppose $\int_0^\infty x dF_1(x) \leq \int_0^\infty x dF_2(x)$.

If $\int_0^\infty x dF_1(x) < \infty$, then $F_1 \prec F_2$.

Proof For $t \geq c$ we immediately have

$$\int_t^\infty (1 - F_1(x))dx \leq \int_t^\infty (1 - F_2(x))dx \quad ,$$

and for $t < c$

$$\int_t^\infty \{(1-F_1(x)) - (1-F_2(x))\}dx \leq \int_0^\infty \{(1-F_1(x)) - (1-F_2(x))\}dx \leq 0$$

If F_1 and F_2 are as in lemma 1 we say that F_2 is more dangerous than F_1.
First consider the class of compound Poisson distributions with parameter λ and with terms having mean μ. The minimal distribution in this class has terms equal to μ with probability one. It is easy to see that this distribution is less dangerous than any other with mean μ.
If the terms are restricted to have a bounded range, say $[0,M]$, we can identify the following distribution as the most dangerous one:

$$\Pr(X = 0) = 1 - \frac{\mu}{M}, \quad \Pr(X = M) = \frac{\mu}{M}$$

So the compound Poisson (λ) distribution with terms as X is maximal in the sense of stop loss ordering in this restricted class.
To compute the upper bounds for net stop loss premiums it is easier to consider the compound Poisson ($\lambda\mu/M$) distribution with terms equal to M with probability one. This is the same distribution as the maximal distribution found above, as one sees by comparing moment generating functions.

Now let the distribution of the terms F be fixed, and also the expected number of terms $E(N) = \lambda$. In this class there is a minimal element. Its number of terms N' satisfies

$$\Pr(N' = [\lambda]) = 1 - (\lambda - [\lambda])$$

$$\Pr(N' = [\lambda] + 1) = \lambda - [\lambda]$$

The distribution functions F_N and $F_{N'}$ can have only one point of intersection, either at $[\lambda]$ or at $[\lambda] + 1$ (remember that F_N is constant between $[\lambda]$ and $[\lambda] + 1$, too). So $N \succ N'$, and by theorem 2.5 the minimal element is this class is found.

Finally, consider those compound distributions with some fixed F as distribution function for the terms, and a counting variable N that may be written as

$$N = A_1 + A_2 + \ldots + A_n$$

with the A_j Bernoulli(p_j) distributed and independent.
Let the vector $p = (p_1,\ldots,p_n)^T$, and let N' with terms A_j' with probability vector $p' = (p_1',\ldots,p_n')$ be another such counting variable.
We will show that if $E(N) = \lambda$ is fixed, the Binomial $(n,\frac{\lambda}{n})$ distribution is maximal. For $E(N) = \lambda$ fixed but n arbitrary, we obtain the Poisson (λ) distribution as a supremum.
First let n = 2. If the vectors p and p' are related as

$$p' = \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix}$$

for some $t \in [0,1]$, we have

$$\begin{aligned} p_1'p_2' &= (tp_1 + (1-t)p_2)((1-t)p_1 + tp_2) \\ &= t(1-t)\{p_1^2 + p_2^2 - 2p_1p_2\} + p_1p_2 \geq p_1p_2 \end{aligned}$$

and with the same reasoning

$$(1-p_1')(1-p_2') \geq (1-p_1)(1-p_2)$$

This implies that $N' \succ N$, since N' is more dangerous.
Of course by theorem 2.2 we have for any m

$$A_1'+A_2' \succ A_1+A_2 \Rightarrow A_1'+A_2'+A_3+A_4+\ldots+A_m \succ A_1+A_2+A_3+A_4+\ldots+A_m$$

So if the matrix $T = T(k,\ell,t)$ satisfies

$$\begin{aligned} T_{kk} &= T_{\ell\ell} = t\ , \\ T_{k\ell} &= T_{\ell k} = 1 - t \qquad (4.2) \\ T_{ij} &= \delta_{ij} \quad \text{otherwise} \end{aligned}$$

for some k and ℓ and some $t \in [0,1]$, and $p' = Tp$, then $N' > N$.
Now suppose we have proven that if

$$N = A_1+A_2+\dots+A_{n-1}$$

then

$$N \prec B(n-1, \frac{1}{n-1} EN) \quad ,$$

where B(r,t) denotes a Binomial (r,t) random variable. Note that for $n - 1 = 2$, the assertion is proved by taking $T = \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$.
Let $\bar{p} = \frac{1}{n}\sum_{j=1}^{n} p_j$. If all p_j are equal to $\bar{p}$, N has a Binomial $(n,\bar{p})$ distribution. Now suppose k and ℓ exist such that $p_k > \bar{p} > p_\ell$.
Take $t \in [0,1]$ such that $tp_k + (1-t)p_\ell = \bar{p}$.
Let $p' = T(k,\ell,t)p$, then N' with probability vector p' satisfies $N \prec N'$.
Then $N' - A_k' \prec B(n-1, \frac{1}{n-1} E(N'-A_k')) = B(n-1,\bar{p})$ by induction.
But then

$$(N'-A_k') + A_k' \prec B(n-1,\bar{p}) + A_k'$$

so $N' \prec B(n,\bar{p})$, which was to be proved.

If for some $\lambda > 0$ F_n is the Binomial $(n, \frac{\lambda}{n})$ distribution function, we have $F_1 \prec F_2 \prec \dots$.
Also, for all x we have $F_1(x) \geq F_2(x) \geq \dots$ with

$$\lim_{n\to\infty} F_n(x) = F_\infty(x)$$

where F_∞ is the Poisson (λ) distribution function. Using definition 2.1 we conclude

$$\lim_{k\to\infty} \int_x^\infty (1 - F_k(y))dy = \int_x^\infty (1 - F_\infty(y))dy$$

so F_∞ is a supremum in the sense of stop loss order in the class of distributions N that may be written as a finite sum of Bernoulli experiments.

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