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COMPUTING MOMENTS OF COMPOUND DISTRIBUTIONS

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Title: Computing moments of compound distributions

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Abstract: The first few moments of compound distributions may be obtained by conditioning on the number of terms. It is shown how this method can be adapted to construct a recursive scheme for computing higher order moments of compound distributions.

1. Introduction

Suppose that the random variable S, representing e.g. the total amount of claims on an insurance portfolio in a certain year, may be written as

$$S = \sum_{i=1}^{N} X_i \qquad (1)$$

where X_1, X_2, ... are i.i.d. random variables (claimsizes), independent of the random variable N (the number of claims). We know of two algorithms to compute the moments of S when N is Poisson distributed. When λ is the Poisson parameter en $p_j = EX^j$, according to [4] we have

$$E(S-\lambda p_1)^k = k!\{\frac{\lambda p_k}{k!} + \frac{\lambda^2}{2!} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 2}} \frac{p_{k_1} p_{k_2}}{k_1! k_2!} + \frac{\lambda^3}{3!} \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2,k_3\geq 2}} \frac{p_{k_1} p_{k_2} p_{k_3}}{k_1! k_2! k_3!} + \ldots\}$$

and [2], page 12, gives a useful recursion formula

$$E(S-\lambda p_1)^{k+1} = \lambda \sum_{t=0}^{k-1} \binom{k}{t} E(S-\lambda p_1)^t p_{k+1-t}$$

We will present in section 2 a recursive scheme to compute moments $E(S^k)$ when the distribution of N is arbitrary.
The number of arithmetic operations required for computing $E(S^k)$ increases with k^3, the storage needed is proportional to k^2.

In section 3 we show how by the same algorithm the moments of the ruin probability function ψ can be computed.

2. Algorithm

First we will compute conditional expectations of S given N = n. Observe that by symmetry, Newton's Binomial Theorem and independence, for all n = 0, 1, ...

$$E\left(\sum_{i=1}^{n} X_i\right)^k = \sum_{i=1}^{n} EX_i\left(\sum_{j=1}^{n} X_j\right)^{k-1}$$

$$= nEX_n\left(\sum_{j=1}^{n} X_j\right)^{k-1}$$

$$= nEX_n \sum_{t=0}^{k-1} \binom{k-1}{t} X_n^t \left(\sum_{j=1}^{n-1} X_j\right)^{k-1-t}$$

$$= n \sum_{t=0}^{k-1} \binom{k-1}{t} p_{t+1} E\left(\sum_{j=1}^{n-1} X_j\right)^{k-1-t} \qquad (3)$$

Letting $n^{!k} = n(n-1) \ldots (n-k+1)$, we will show that coefficients a_{jk}, $j = 1, 2, \ldots, k$; $k = 1, 2, \ldots$ exist, such that for all n = 1, 2, ...

$$E\left(\sum_{i=1}^{n} X_i\right)^k = \sum_{j=1}^{k} a_{jk} n^{!j} \qquad (4)$$

Indeed, suppose that such $a_{j\ell}$ have been computed for $\ell < k$, taking of course $a_{11} = p_1$, then by (3)

$$E\left(\sum_{i=1}^{n} X_i\right)^k = n\left\{p_k + \sum_{t=0}^{k-2} \binom{k-1}{t} p_{t+1} \sum_{j=1}^{k-1-t} a_{j,k-1-t} (n-1)^{!j}\right\}$$

$$= np_k + \sum_{t=0}^{k-2} \binom{k-1}{t} p_{t+1} \sum_{j=1}^{k-1-t} a_{j,k-1-t} n^{!(j+1)}$$

$$= np_k + \sum_{j=1}^{k-1} n^{!(j+1)} \sum_{t=0}^{k-1-j} \binom{k-1}{t} p_{t+1} a_{j,k-1-t}$$

$$= np_k + \sum_{j=2}^{k} n^{!j} \sum_{t=0}^{k-j} \binom{k-1}{t} p_{t+1} a_{j-1,k-1-t}$$

$$= \sum_{j=1}^{k} a_{jk} n^{!j} \qquad (5)$$

if we take $a_{1k} = p_k$, and for $j = 2, 3, \ldots, k$

$$a_{jk} = \sum_{t=0}^{k-j} \binom{k-1}{t} p_{t+1} a_{j-1,k-1-t} \tag{6}$$

Using (4) we directly obtain

$$\begin{aligned} E(S^k) &= \sum_{n=0}^{\infty} P(N=n) E(S^k | N=n) \\ &= \sum_{n=0}^{\infty} P(N=n) \sum_{j=1}^{k} a_{jk} n^{!j} \\ &= \sum_{j=1}^{k} a_{jk} E(N^{!j}) \end{aligned} \tag{7}$$

The coefficients a_{jk} in (7) are computed using (6); the factorial moments of N can be computed from the ordinary moments, but in fact often are more easily calculated themselves. In [3] one finds expressions for factorial and ordinary moments of many counting distributions, including those used in actuarial work.

3. Application

Consider the compound Poisson process (cf. [1])

$$\{Y(t) = \sum_{j=1}^{N(t)} X_j \quad , \quad t \geq 0\}$$

where

$$\{N(t) \quad , \quad t \geq 0\}$$

is a Poisson stochastic process with $E(N(t)) = t$ and $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with distribution function P and moments $p_1, p_2, \ldots$

Let for some $\lambda > 0$,

$$Z = \sup_{0 \leq t} \left[\sum_{j=1}^{N(t)} X_j - t(p_1+\lambda) \right]$$

If λ is the safety loading included in the premium, and u is the intitial reserve, ruin occurs in case of the event $Z > u$. So if ψ^* is the distribution function of Z, we have for the ruin probability function ψ:

$$\psi(u) = 1 - \psi^*(u)$$

Now by the theorem on p. 67/68 of [1] we have

$$\psi^*(u) = \frac{\lambda}{p_1+\lambda} \sum_{n=0}^{\infty} \left(\frac{p_1}{p_1+\lambda}\right)^n H^{*n}(u) \tag{8}$$

with H^{*n} the n-fold convolution of the following distribution

$$H(x) = \int_0^x \frac{1-P(t)}{p_1}\, dt \qquad \text{(for } x > 0 \text{ , 0 elsewhere)} \tag{9}$$

We may apply the algorithm of the preceding section to compute the moments of ψ^* , as ψ^* is by (8) a compound Geometric $\left(\frac{\lambda}{p_1+\lambda}\right)$ distribution. The moments of H can be obtained by partial integration:

$$\int_0^\infty x^j dH(x) = \frac{1}{p_1} \int_0^\infty x^j (1 - P(x)dx$$

$$= \frac{1}{p_1} \left(\frac{x^{j+1}}{j+1} (1 - P(x)) \Big|_0^\infty + \int_0^\infty \frac{x^{j+1}}{j+1} dP(x) \right)$$

$$= \frac{p_{j+1}}{p_1(j+1)} \tag{10}$$

References

[1] Beekman, J.A., "Two stochastic processes", Almqvist and Wiksell, Stockholm (1974)

[2] Goovaerts, M.J., F. de Vylder and J. Haezendonck, "Insurance Premiums", North-Holland (1984)

[3] Janardan, K.G., "Moments of certain series distributions and their applications", SIAM J. Appl. Math. 1934: 854-368

[4] Shiu, E.S.W., "Moments of two distributions in collective risk theory", Scandinavian Acturial Journal 1977: 185-187