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Natural resources

# FACULTY OF ACTUARIAL SCIENCE & ECONOMETRICS

# A & E REPORT

REPORT AE 1/82

*THE OPTIMAL EXPLOITATION OF A NATURAL RESOURCE WHEN THERE IS FULL COMPLEMENTARITY*

C. Withagen

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*The Optimal Exploitation of a Natural Resource when there is full Complementarity*

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ABSTRACT

In the present paper we analyse optimal economic growth with an exhaustible natural resource when the economy's non-resource output is produced by means of capital and the withdrawal from the resource, whereas these two factors of production are highly complementary.

## 1. Introduction

As Stiglitz (1974) has pointed out " there are at least three economic forces offsetting the limitations imposed by natural resources: technical change, returns to scale and the substitution of man-made factors of production (capital) for natural resources". It can be doubted however whether all these forces exist in reality. For the Netherlands for example there is some evidence that energy and capital are complements rather than substitutes (see Magnus(1979)). Moreover the existence of returns to scale, especially in the case of energy, is not guaranteed.

The aim of the present paper is to characterize optimal time-paths of the economy when substitution possibilities are ruled out. In order to make the results comparable with Stiglitz's conclusions, we shall use the same utility function and a Cobb-Douglas production function. For simplicity labour will be neglected but all results can easily be generalized.

In section 2 Stiglitz's model is briefly reviewed. Section 3 gives the model with complementarity and presents the derivation of necessary conditions for an optimum. It also provides some preliminary results. In section 4 properties of the optimum are derived. Finally section 5 gives the conclusions.

## 2. The Stiglitz model

The model Stiglitz uses can be described as follows. The economy's output (Y) is produced by means of two factors of production: capital (K) and the withdrawal of the natural resource (E), which shall be called energy. Production takes place according to a Cobb-Douglas technology, possibly exhibiting technical progress at a rate $\gamma(\gamma\geq 0)$. The time index t is omitted where there is no danger of confusion.

$$Y = e^{\gamma t} K^{\alpha} E^{1-\alpha}, \; 0 < \alpha < 1. \tag{1}$$

Output is devoted to consumption (C) and investments ($\dot{K} = dK/dt$). Depreciation is neglected. The initial stock of capital is given: $K_0$.

$$Y = C + \dot{K}. \tag{2}$$

The size of the pool of energy is denoted by R and it decreases according to the rate of exploitation. The initial pool is given: $R_0$.

$$\dot{R} = -E. \qquad (3)$$

The economy's objective is to maximize

$$I = \int_0^\infty e^{-\rho t} \cdot C^\nu/\nu \, dt \ , \qquad (4)$$

where $\rho$ is the constant rate of time preference,
$\nu$ is a constant, $\nu < 1$, $\nu \neq 0$ .
The results can be summarized in the following proposition.

Proposition 1

A necessary condition for the existence of an optimum is that $\rho < \gamma\nu/(1-\alpha)$. Along an optimal path the output-capital ratio (Y/K), the savings rate ($\dot{K}/Y$) and the rate of resource utilization ($-\dot{R}/R$) monotonically approach the values $\gamma/\alpha\ (1-\alpha)$, $\alpha(\gamma-(1-\alpha)\rho)\ /\ \gamma(1-\nu)$ and $-(\gamma\nu-(1-\alpha)\rho)\ /\ (1-\alpha)(1-\nu)$ respectively. The asymptotic growth rate of output is $(\gamma-(1-\alpha)\rho)\ /\ (1-\alpha)(1-\nu)$ and in the limit the growth rate of consumption is positive if and only if $\gamma > (1-\alpha)\rho$.

## 3. Complementarity

Here we modify the previous model in one respect. It is now assumed that capital and energy are "complements" by which is meant that there exists a constant elasticity ($\sigma$) of energy demand with respect to capital use. In addition $\sigma \geq 1$. For this many examples can be found in reality: heating of buildings, petrol for transportation purposes etc. The economy's technology is now described by:

$$Y = e^{\gamma t} K^\alpha \ , \qquad (5)$$

$$E = \phi K^\sigma, \qquad (6)$$

where $\gamma \geq 0$, $0 < \alpha < 1$, $\phi > 0$, $\sigma \geq 1$. The model (2), (3), (5) and (6) can be reduced to:

$$\dot{K} = e^{\gamma t} K^\alpha - C, \ K_0 \text{ given}, \qquad (7)$$

$$\dot{R} = -\phi K^\sigma, \ R_0 \text{ given}. \qquad (8)$$

We apply Pontryagin's maximum principle (see e.g. Takayama(1974)). It is necessary for $(\hat{K},\hat{C})$ to constitute an optimum that (7) and (8) are fulfilled and that there exist $p_1(t)$ and $p_2(t)$ such that for all t:

$$-\dot{p}_1 = p_1\, \alpha e^{\gamma t}\, \hat{K}^{\alpha-1} - p_2 \phi\sigma\, \hat{K}^{\sigma-1}, \tag{9}$$

$$-\dot{p}_2 = 0\ , \tag{10}$$

$$e^{-\rho t}\, \hat{C}^{\nu-1} = p_2\ . \tag{11}$$

$p_1$ and $p_2$ are the costate variables of the stock of capital and the pool of energy respectively. Equation (11) follows from the maximization of the Hamiltonian with respect to the rate of consumption. $\hat{C}$ has to be strictly positive. From (10) we see that the shadow price of the pool of energy is constant, a familiar result.

It is easily seen that Stiglitz's results on convergence do not hold in the present model.

- a constant capital-output ratio implies that marginal productivity of capital is constant. But this requires that the stock of capital is non-decreasing which is incompatible with the limited pool of energy.
- a constant savings rate s implies that

$$K(t) = \left\{\frac{(1-\alpha)s}{\gamma}\, e^{\gamma t} - \frac{(1-\alpha)s}{\gamma} + K_0^{\,1-\alpha}\right\}^{\frac{1}{1-\alpha}}\ ,$$

which can be verified by substitution. A necessary condition for the stock of capital to decrease, which is required in view of the limited pool of energy, is that $s < 0$. But then the stock of capital becomes negative eventually.

- concerning the rate of resource utilization (E/R) it is clear that it is feasible to have such a rate at a constant level $\beta$. Then we have:

$$C(t) = a\, e^{(\gamma-\beta\alpha/\sigma)t} + b\, e^{(-\beta/\sigma)t}\ ,$$

where a and b are appropriate constants. Hence if the economy would choose $\beta$ small enough a growing rate of consumption can be realized in the case of technical progress.

However, substitution into (11) and using (9) shows that the necessary conditions are not fulfilled.

A few final remarks are in order. Mirrlees (1967) has studied this type of model for the case of no natural resource or, if one wishes, of an abundant natural resource. He has found that $\rho \geq \gamma\nu$ is a sufficient condition for the existence of an optimum. Obviously this a fortiori holds for the model at hand and consequently we shall assume that $\rho \geq \gamma\nu$ . On the other hand it is easily shown that when $\rho < \gamma\nu$ no optimum exists since it is better to postpone consumption. It is also found that the growth rate of the rate of consumption ($\dot{C}/C$) is bounded above. This should hold in our case as well.

## 4. The optimal path

In this section some properties of the optimal path are derived. Unfortunately the necessary conditions look rather complicated. We can however make some positive statements. In aid of the propositions the following lemma turns out to be useful.

Lemma 1.

For any two paths $(K,C)$ and $(\bar{K},\bar{C})$, satisfying (7) and for any interval $[a,b]$, where $K(a) = \bar{K}(a)$:

$$\int_a^b e^{-\rho t}(u(C) - u(\bar{C}))\, dt > \int_a^b e^{-\rho t} u'(C)(K-\bar{K})(\alpha\, e^{\gamma t}\, K^{\alpha-1} - \rho - (1-\nu)\dot{C}/C) dt - (K(b) - \bar{K}(b))\, e^{-\rho t}\, u'(C(b)) ,$$

where $u(C) = C^{\nu}/\nu$ , $u'(C) = du/dC$ .

Proof.

The proof is straightforward, using strict concavity of the functions involved and integrating by parts. ■

Proposition 2

Be $\tilde{K}(t) = \{(\alpha/(\rho + (1-\nu)\gamma))\, e^{\gamma t}\}^{1/1-\alpha}$. If $(\hat{K},\hat{C})$ is optimal then $\hat{K}(t) > \tilde{K}(t)$ for some t implies that $\hat{K}$ decreases at t.

Proof.

Suppose the proposition is false. However there is a moment at which $\hat{K}(t) = \tilde{K}(t)$ in view of the limited availability of energy. Now construct an alternative path (K,C) in the following way (illustrated below).

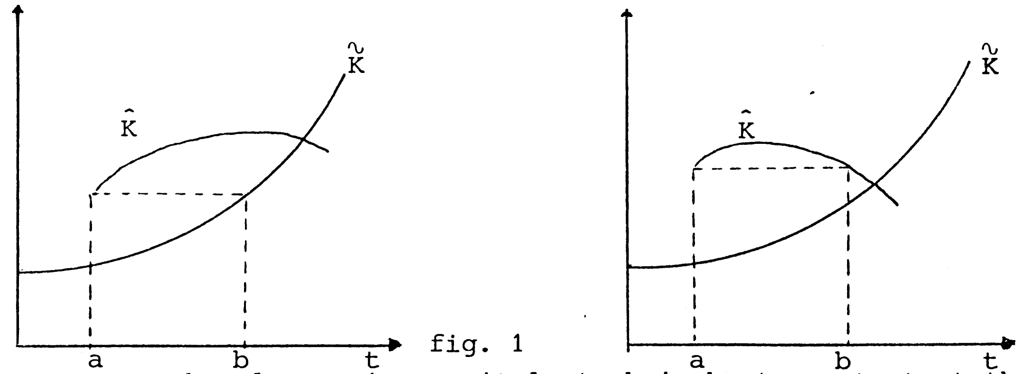

fig. 1

From t = a on the alternative capital stock is kept constant at the $\hat{K}(a)$ level. We consider two cases:

1) $\hat{K}(a) = \tilde{K}(t)$ for some t = b and $\hat{K}$ (b) > $\hat{K}(a)$. In this case follow the constant path $\hat{K}(a)$ until time b and from b on the alternative path is set equal to $\tilde{K}$ until $\tilde{K}(t) = \hat{K}(t)$. As long as K is constant $\alpha e^{\gamma t} K^{\alpha-1} - \rho - (1-\nu)\dot{C}/C$ is negative because $K(t) > \tilde{K}(t)$. If $K(t) = \tilde{K}(t)$ then this expression is also negative as can be verified by straightforward calculations. Now it is easy to apply lemma 1, substituting $\hat{K}(t)$ for $\bar{K}(t)$ and realizing that $\hat{K}(t) > K(t)$. Finally the alternative path uses less energy.

2) $\hat{K}(a) = \hat{K}(b)$ for some b and $\hat{K}(b) > \tilde{K}(b)$. In this case, keep K constant up to time b and follow $\hat{K}$ from b on. The same argument as used in case 1) now applies.

Hence in both cases the optimal path can be overtaken by an alternative path that uses less energy. This contradicts the optimality of $\hat{K}$. ■

The necessary conditions (7) - (11) can be written as:

$$\dot{K} = e^{\gamma t} K^{\alpha} - C , \tag{12}$$

$$(1-\nu)\ \dot{C}/C + \rho = \alpha\ e^{\gamma t} K^{\alpha-1} - p_2 \phi\, \sigma\, e^{\rho t} C^{1-\nu} K^{\sigma-1} . \tag{13}$$

At first sight nothing in the necessary conditions seems to exclude the possibility of having "bulges" in the optimal time-path of the stock of capital.

A solution K(t) of the system of differential equations (12) and (13) shows a bulge if there is an interval [a,b] where K(a) = K(b) and $K(t) \geq K(a)$ for all $a \leq t \leq b$ with K(t) > K(a) for at least one t in the interval. For the case of no technical progress it is easy to show that only one bulge can occur and that if it occurs this only happens in the very beginning of the planning period.

Proposition 2

If $\dot{\gamma} = 0$ a bulge can only occur in the beginning of the planning period.

Proof.

In figure 2I a bulge is sketched.

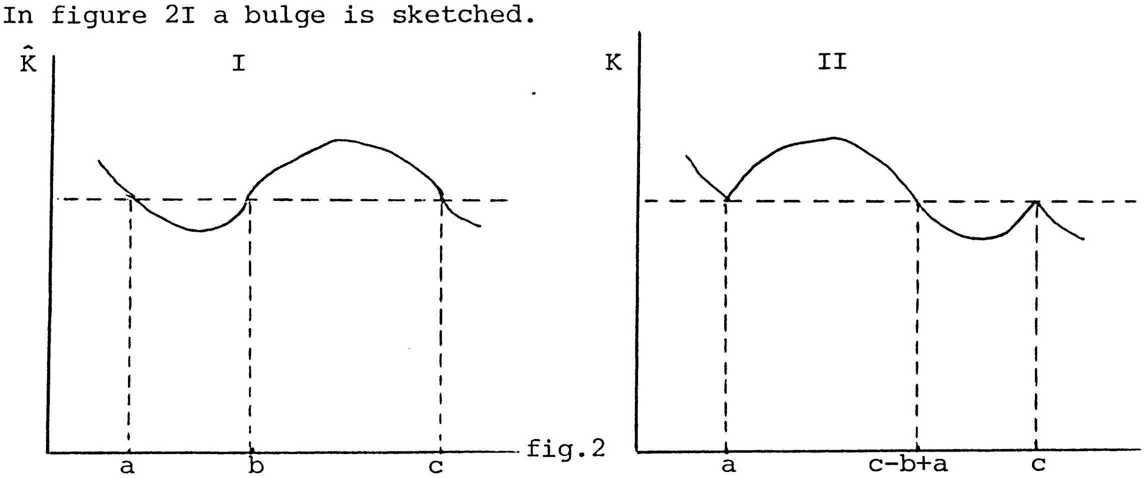


fig.2

Suppose the optimal path can be depicted as in figure 2 I. Then an alternative path (K,C) can be constructed that is better. This is shown in figure 2 II. In figure 2 II the optimal path is copied but on different segments. Both paths are identical up to time a and from time c on. After reaching a we copy the optimal path from [b,c] on the interval [a, c − b + a] and from c − b + a to c we take the optimal path of interval [a,b]. Total utility on both paths is compared. By $\hat{u}$ we denote the utility of the optimal path obtained between a and c and by u the utility of the alternative path. In the subsequent proof energy does not play a part because both paths require the same amount of it.

$$\hat{u} = \int_a^b e^{-\rho t}\, u(\hat{C})\, dt + \int_b^c e^{-\rho t}\, u(\hat{C})\, dt .$$

$$u = \int_a^{c-b+a} e^{-\rho t}\, u(C)\, dt + \int_{c-b+a}^{c} e^{-\rho t}\, u(C)\, dt .$$

For $a \leq t \leq c - b + a$, $K(t) = \hat{K}(t+b-a)$, $\dot{K}(t) = \dot{\hat{K}}(t+b-a)$,
for $c - b + a \leq t \leq c$, $K(t) = \hat{K}(t-c+b)$, $\dot{K}(t) = \dot{\hat{K}}(t-c+b)$.
For $a \leq t \leq c - b + a$ put $t' = t + b - a$. Then $t = a$ implies $t' = b$ and $t = c - b + a$ implies $t' = c$. For $c - b + a \leq t \leq c$ put $t' = t + b - c$. Then $t = c - b + a$ implies $t' = a$ and $t = c$ implies $t' = b$. It follows that:

$$u = \int_a^b e^{-\rho(t+c-b)}\, u(\hat{C}(t+c-b))dt + \int_b^c e^{-\rho(t-b+a)}\, u(\hat{C}(t-b+a))dt \text{ and}$$

$$u - \hat{u} = \int_a^b e^{-\rho t}(1/\nu)\{(e^{(-\rho/\nu)(c-b)}\,\hat{K}^{\alpha} - e^{(-\rho/\nu)(c-b)}\,\dot{\hat{K}})^{\nu} - (\hat{K}^{\alpha} - \dot{\hat{K}})^{\nu}\}dt +$$

$$\int_b^c e^{-\rho t}(1/\nu)\{(e^{(-\rho/\nu)(a-b)}\,\hat{K}^{\alpha} - e^{(-\rho/\nu)(a-b)}\,\dot{\hat{K}})^{\nu} - (\hat{K}^{\alpha} - \dot{\hat{K}})^{\nu}\}dt \,.$$

Hence

$$u - \hat{u} = -(1-e^{-\rho(a-b)})\int_b^c e^{-\rho t}\, u(\hat{C})dt - (1-e^{-\rho(c-b)})\int_a^b e^{-\rho t}\, u(\hat{C})dt \,.$$

From the optimality of $\hat{C}$ we know:

$$\int_b^c e^{-\rho t}\, u(\hat{C})\, dt > \int_b^c e^{-\rho t}\, u(\bar{C})dt = -(1/\rho)\, u(\bar{C})(e^{-\rho c} - e^{-\rho b}) \,,$$

where $\bar{C}$ corresponds with a $\bar{K}$ that is constant from b to c. Secondly we have:

$$\int_a^b e^{-\rho t}\, u(\hat{C})\, dt < \int_a^b e^{-\rho t}\, u(\bar{C})\, dt = -(1/\rho)\, u(\bar{C})(e^{-\rho b} - e^{-\rho a}) \,.$$

It follows that $\hat{u} - u < 0$ by straightforward calculations. This contradicts the optimality of $\hat{C}$ . □

The impossibility of having bulges for the case where there is technical progress cannot be proved in such an easy way.

This is caused by the fact that in the top of the bulges consumption is not only benefitted by the large stock of capital but by the technical progress as well. But we can prove the following.

Proposition 3

There exists a $t_0$ such that $\dot{K} < 0$ for all $t > t_0$.

Proof.

Suppose that the proposition is not correct. Consider the figure below

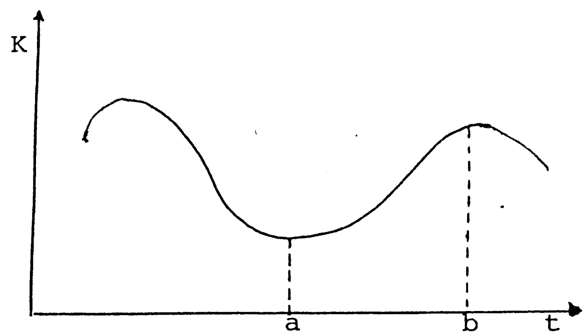


fig. 3

In a and b $\dot{K} = 0$. Hence $C = e^{\gamma t} K^{\alpha}$. $\dot{C} = e^{\gamma t} K^{\alpha}(\gamma + \alpha \dot{K}/K) - \ddot{K}$, from (12). $\ddot{K} > 0$ in a and hence $\dot{C}/C < \gamma$. For b we have $\dot{C}/C > \gamma$. Consider the function

$$f(t) = \alpha e^{\gamma t} K^{\alpha-1} - e^{(\rho+\gamma(1-\nu))t} K^{\sigma-1+\alpha(1-\nu)} p_2 \phi\sigma ,$$

where K(t) fulfils the necessary conditions (12) and (13). For $t = a$ $\dot{K} = 0$, $C = e^{\gamma t} K^{\alpha}$ and $f(t) = (1-\nu) \dot{C}/C + \rho$. This also holds for $t = b$. It follows that $f(a) < (1-\nu)\gamma + \rho$ and $f(b) > (1-\nu)\gamma + \rho$. Hence for at least one t in the interval [a,b], where the stock of capital is increasing, f(t) must have a positive derivative.

$$f'(t) = (\alpha(\alpha-1)\, e^{\gamma t} K^{\alpha-1} - e^{(\rho-\gamma\nu+\gamma)t}\, (\alpha - \alpha\nu+\sigma-1) K^{\alpha-\alpha\nu+\sigma-1} p_2 \phi\sigma)\, \dot{K}/K +$$

$$+ \alpha\, \gamma\, e^{\gamma t} K^{\alpha-1} - e^{(\rho-\gamma\nu+\gamma)t}\, (\rho-\gamma\nu+\gamma)\, K^{\alpha-\alpha\nu+\sigma-1}\, p_2 \phi\sigma .$$

The first term is negative ($\nu < 1$, $\sigma \geq 1$). Hence for at least one t the sum of the last two terms must be positive. This condition is equivalent to having at least one t for which

$$K < (\alpha\gamma/p_2\phi\sigma(\rho-\gamma\nu+\gamma))^{1/\sigma - \alpha\nu}\, e^{-t(\rho-\gamma\nu)/\sigma-\alpha\nu} . \qquad (14)$$

Since in the interval (a,b) the stock of capital is increasing we have $C < e^{\gamma t}K^{\alpha}$.

Using the inequality and (14) in (13) it is found that:

$$(1-\nu)\ \dot{C}/C + \rho > \alpha\ \pi^{\alpha-1}\{1-\frac{1}{\rho-\gamma\nu+\gamma}\}e^{(\rho(1-\alpha)+\gamma(\sigma-\nu))t/\sigma-\alpha\nu}$$

where $\pi = \{\alpha\ \gamma/p_2\phi\ \sigma(\rho+\gamma-\gamma\nu)\}^{1/\sigma-\alpha\nu}$. Since $\pi > 0$, $\rho \geq \gamma\nu$, $\alpha < 1$ and $\sigma \geq 1 > \nu$, $\dot{C}/C$ is unbounded above if there are infinitely many bulges: a contradiction. ■

Given that the stock of capital is eventually decreasing the following propositions can easily be proved

Proposition 4

There is a $t_0$ such that $\dot{C}/C > -\rho/1-\nu$ for all $t > t_0$.

Proof.

Denoting the right hand side of (13) by g(t) and differentiating with respect to time yields:

$$g'(t) = \alpha\ e^{\gamma t}\ K^{\alpha-1}(\gamma+(\alpha-1)\dot{K}/K) - p_2\phi\sigma\ C^{1-\nu}\ K^{\sigma-1}\ e^{\rho t}(\rho+(1-\nu)\dot{C}/C+(\sigma-1)\dot{K}/K).$$

$\dot{K}/K$ is negative eventually (which means for all t larger than some t').

Hence $g'(t) > 0$ if $\dot{C}/C \leq -\rho/1-\nu$. ■

From this proposition it follows that $(1-\nu)\dot{C}/C + \rho > 0$ eventually and hence

$$K \leq (p_2\phi\ \sigma\ /\ \alpha)\ e^{(-\rho+\gamma\nu)t/\sigma-\alpha\nu} \text{ eventually.}$$

Proposition 5

For all $\underline{\gamma} < \gamma$ there is a $t_0$ such that

$$C > e^{(\gamma\sigma-\alpha\rho)t/\sigma-\alpha\nu} \text{ for all } t > t_0.$$

Proof.

Suppose the proposition is not correct. Substitution of the reversed inequality and of the inequality for K, established above, into (13) shows that $\dot{C}/C$ is unbounded above, a contradiction. ■

Proposition 6

For all $\bar{\gamma} > \gamma$ there is some $t_0$ such that if $\gamma\sigma - \alpha\rho > 0$

$$C < e^{(\bar{\gamma}\sigma-\alpha\rho)t/\sigma-\alpha\nu} \text{ for all } t > t_0.$$

Proof.

1) Be $\pi = (\gamma\sigma-\alpha\rho)/\sigma-\alpha\nu$, $\bar{\pi} = (\bar{\gamma}\sigma-\alpha\rho)/\sigma-\alpha\nu$.
2) Suppose that for some $\bar{\gamma}>\gamma$ $C > e^{\bar{\pi} t}$ eventually. Then $C > e^{\bar{\gamma} t} K^{\alpha}$ eventually since

$$e^{\bar{\pi} t} > e^{\bar{\gamma} t}\, e^{\frac{\alpha(-\rho+\gamma\nu)}{\sigma-\alpha\nu}t} = e^{\bar{\pi} t}\, e^{\alpha\nu(\gamma\nu-\bar{\gamma}\nu)t/\sigma-\gamma\nu} .$$

Hence $\dot{K}/K < K^{\alpha-1}(e^{\gamma t} - e^{\bar{\gamma} t})$ eventually and $\rho+(1-\nu)\ \dot{C}/C + (\sigma-1)\ \dot{K}/K < 0$ eventually. Then $g'(t) \geq \varepsilon > 0$ for some $\varepsilon$ and $\dot{C}/C$ is unbounded, a contradiction.

We conclude that if there is some $\bar{\gamma} > \gamma$ such that $C > e^{\bar{\pi} t}$ for an infinite series of times, we must also have that $C = e^{\bar{\pi} t}$ for an infinite series of times.
3) When C gets larger than $e^{\bar{\pi} t}$ C is increasing (see figure 4)

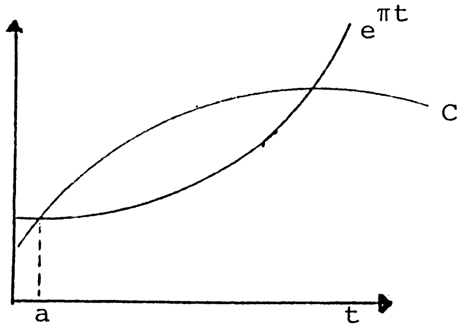


fig. 4

Since there are infinitely many points such as a $\dot{K}/K$ becomes arbitrarily small by taking t sufficiently large. For such t's $\rho+(1-\nu)\dot{C}/C+(\sigma-1)\dot{K}/K < 0$ and $d\ \dot{C}/C\ /dt > 0$. It follows that $\dot{C}$ increases and C becomes more positive. But $e^{\bar{\gamma} t}\ K^{\alpha}$ decreases for sufficiently large t's in the set of points of time where $C = e^{\bar{\pi} t}$. The inequality $C > e^{\bar{\gamma} t}\ K^{\alpha}$ thus persists which is not allowed by the reasoning in 1). This establishes the proposition. ■

Proposition 7

Suppose $\gamma\sigma - \alpha\rho < 0$. Then eventually C monotonically approaches zero.

Proof.

Suppose there is some $\varepsilon > 0$ such that for all $t_0$ there is some $t > t_0$ with $C > \varepsilon$. There does not exist a $\bar{\gamma} > \gamma$ such that $C > e^{\bar{\pi}t}$ eventually. This has been shown in part 2) of the previous proposition. Hence for all $\bar{\gamma} > \gamma$ $C < e^{\bar{\pi}t}$ for an infinite series of times. This holds for $\bar{\pi} < 0$ as well. But if $C > \varepsilon$ then $\dot{C} > 0$ for an infinite series of times. By taking these t's sufficiently large the reasoning followed in part 3) of the previous proposition can be applied. ∎

Finally we prove that the necessary conditions are also sufficient conditions for an optimum.

Proposition 8

If $(\hat{K},\hat{C})$ fulfils (12) and (13) and if $\hat{R}(t) \geq 0$ for all t then $(\hat{K},\hat{C})$ constitutes an optimum.

Proof.

Substitute in lemma 1 : $C = \hat{C}$, $K = \hat{K}$, $a = 0$, $b = T$, $\bar{K} = K$, where K is some feasible-time path of the stock of capital. Then:

$$\int_0^T e^{-\rho t}(u(\hat{C})-u(C))dt > \int_0^T e^{-\rho t}\, u'(\hat{C})(\hat{K}-K)(\alpha e^{\gamma t}\,\hat{K}^{\alpha-1} - \rho - \dot{\hat{C}}/\hat{C})dt - (\hat{K}(T) - K(T))\, e^{-\rho T}\, u'(C(T))$$

$$= \int_0^T e^{-\rho t}\, u'(\hat{C})(\hat{K}-K)\;(e^{\rho t}\,\hat{C}^{1-\nu}\,\hat{K}^{\sigma-1}\; p_2\phi\,\sigma)dt - (\hat{K}(T) - K(T))\; e^{-\rho T}\,\hat{C}(T)^{\nu-1}$$

$$\geq \int_0^T p_2(\phi\,\hat{K}^{\sigma} - \phi K^{\sigma})dt - \hat{K}(T)\; e^{-\rho T}\,\hat{C}(T)^{\nu-1}, \text{ since } \sigma \geq 1.$$

On any path satisfying the necessary conditions the size of the resource will approach zero. Therefore the first term of the right hand side will equal zero for $T = \infty$. The second term approaches zero as can be seen by applying (14) and proposition 5. ∎

Summarizing the results of this section we can say that in the case of no technical progress the stock of capital is monotonically decreasing except possibly at the beginning of the planning period if the initial stock of capital is small. When there is technical progress the stock of capital monotonically decreases eventually at a rate smaller than $(-\rho+\gamma\nu)/\sigma-\alpha\nu$.

If $\gamma\sigma-\alpha\rho > 0$ then eventually

$$e^{(\underline{\gamma}\sigma-\alpha\rho)t/\sigma-\alpha\nu} < C < e^{(\bar{\gamma}\sigma-\alpha\rho)/\sigma-\alpha\nu} \quad \text{for all } \underline{\gamma} < \gamma < \gamma$$

It is easy to see that in this case the rate of consumption is benefitted by the rate of technical progress, the elasticity of marginal utility and the ratio of the elasticity of production with respect to capital over the elasticity of energy demand with respect to capital. The rate of time preference has a negative impact. If $\gamma\sigma-\alpha\rho < 0$ the rate of consumption eventually decreases monotonically and approaches zero.

## 5. Conclusions

In the present paper we have considered the impact of introducing complementarity between capital and energy into the theory of exhaustible resources. The optimal path has been characterized. All capital will be eaten up. In that respect one should not be optimistic when any value is attached to the possession of capital. However when technical progress is sufficiently large a growing rate of consumption will be realized.

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