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On the inverse of the autocovariance matrix for a general mixed autoregressive moving average process

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Abstract: Extending an approach given by Tiao & Ali (1971) the exact inverse of the autocovariance matrix of a general mixed autoregressive moving average process is obtained. Next the existence and form of this matrix is established for a general non-stationary process. An explicit expression for the inverse autocovariance matrix is given for the second order mixed autoregressive moving average process.

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1. Introduction

An autoregressive moving average process $\{z_t\}$ of orders p and q is defined by

$$\sum_{j=0}^{p} \phi_j z_{t-j} = \sum_{j=0}^{q} \theta_j a_{t-j} \tag{1.1}$$

where $\{a_t\}$ is a set of uncorrelated, identically distributed, variables with mean zero and unit variance and where the parameters ϕ_j and θ_j are real and $\phi_0 = \theta_0 = 1$. We assume that the stationarity and invertibility conditions for this process are satisfied and we define the autocovariance, at lag k, of $\{z_t\}$ by

$$\gamma_k = E\left[z_t z_{t-k}\right] . \tag{1.2}$$

Given an observed $z_1,\ldots,z_n$ the autocovariance matrix, of order n, is given by

$$P_n = (p_{ij}) \tag{1.3}$$

where $p_{ij} = \gamma_{|i-j|}$. For the process (1.1) we rewrite P_n as $P_n(p,q)$ which reduces to $P_n(p,0)$ and $P_n(0,q)$ respectively for an autoregressive process of order p and a moving average process of order q.

When the z_t are normally distributed the inversion of the autocovariance matrix and its determinant are important, in order to obtain the likelihood function of the process. Therefore several authors have been motivated by this to obtain $P_n^{-1}(p,q)$ and the determinant $|P_n(p,q)|$. Shaman (1969), Uppuluri & Carpenter (1969) and Prabhakar Murthy (1974) have given exact expressions for the inverse for a first-order moving average process (p=0, q=1). Shaman (1973) has extended his results to the general autoregressive moving average case and gives an exact expression for $P_k^{-1}(0,2)$. Tiao & Ali (1971) have given an exact expression for $P_n^{-1}(1,1)$ and the determinant $|P_n(1,1)|$. Newbold (1974) has obtained the same results as Tiao & Ali (1971) by means of an other approach, although his method is in fact the same as that given by Galbraith & Galbraith (1974). Recently Anderson (1976 a,b) showed how general expressions for $P_n^{-1}(0,q)$, $P_n^{-1}(p,0)$ and $P_n^{-1}(p,q)$ could be obtained by means of a generalization of an algorithm given by Sherman & Morrison (1950), though he does not given any exact expression for low order models.*

In this paper, making use of the notation given in Anderson (1976b), we extend the approach given by Tiao & Ali (1971) to obtain a general expression for $P_n^{-1}(p,q)$. Results for a strictly autoregressive and a strictly moving average process follow quickly from it. In section 3 we discuss the existence and form of the inverse autocovariance matrix for a general autoregressive integrated moving average process. Finally, in section 4, we give an explicit form of $P_n^{-1}(p,q)$ for $p = q = 2$.

2. The inverse in the general case

Let $U_{n,r}(\alpha)$ denote the nxn upper triangular matrix whose elements in the i-th row and j-th column ($i \leq j$) are 1 for i=j, α_{j-i} for $i<j\leq r+i$ and 0 for $j>r+i$. Let further $L_r(\alpha)$ denote the rxr lower triangular matrix, whose element in the i-th row and j-th column is α_{r-i+j} ($i \geq j$) and let $Z_{v,w}$ and $A_{v,w}$ stand for the column vectors

$$Z'_{v,w} = (z_v, z_{v-1}, \ldots, z_{v-w+1})$$
$$A'_{v,w} = (a_v, a_{v-1}, \ldots, a_{v-w+1})$$

where ' denotes the transpose of the vectors.

It is easy to show that equation (1.1), for n>p and q, may now be written

$$U_{n,p}(\phi)Z_{t,n} = U_{n,q}(\theta)A_{t,n} + \begin{bmatrix} 0 \\ L_q(\theta) \end{bmatrix} A_{t-n,q} - \begin{bmatrix} 0 \\ L_p(\phi) \end{bmatrix} Z_{t-n,p}. \qquad (2.1)$$

Since $A_{t,n}$ is uncorrelated with any combination of the elements of $A_{t-n,q}$ and $Z_{t-n,p}$, it follows from (2.1) that the n-th autocovariance matrix for an

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* A list of other references to authors who gave exact expressions for the exact inverse in various cases of the autoregressive moving average model can be found in Shaman (1975).

autoregressive moving average process of orders p and q is given by

$$P_n(p,q) = U^{-1}_{n,p}(\phi)\, U_{n,q}(\theta)\left(I+(U^{-1}_{n,q}(\theta)\Delta)V_m(U^{-1}_{n,q}(\theta)\Delta)'\right)\, U'_{n,q}(\theta)U'^{-1}_{n,p}(\phi) \qquad (2.2)$$

where Δ is an nxm matrix, with m = max(p,q), of the form

$$\Delta = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \qquad (2.3)$$

and where V_m is an mxm variance-covariance matrix of the vector

$$(y_{t-n+1}, y_{t-n+2}, \dots, y_{t-n+m})' = \begin{bmatrix} 0 \\ L_q(\theta) \end{bmatrix} A_{t-n,q} - \begin{bmatrix} 0 \\ L_p(\theta) \end{bmatrix} Z_{t-n,p} \;. \qquad (2.4)$$

The inverse autocovariance matrix follows from (2.2)

$$P^{-1}_n(p,q) = \left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right)'\left[I+(U^{-1}_{n,q}(\theta)\Delta)V_m(U^{-1}_{n,q}(\theta)\Delta)'\right]^{-1}\left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right). \qquad (2.5)$$

Now using the following well-known relation between matrices

$$(A+BDB')^{-1} = A^{-1} - A^{-1}B(B'A^{-1}B+D^{-1})^{-1}B'A^{-1} \qquad (2.6)$$

where A and D are nonsingular matrices of orders m and n and B an mxn matrix, (2.5) can be written

$$P^{-1}_n(p,q) = \left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right)'\Big[I-(U^{-1}_{n,q}(\theta)\Delta)\big((U^{-1}_{n,q}(\theta)\Delta)'.$$

$$.(U^{-1}_{n,q}(\theta)\Delta)+V^{-1}_m\big)^{-1}(U^{-1}_{n,q}(\theta)\Delta)'\Big]\left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right)$$

$$= \left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right)'\left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right) +$$

$$- \left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right)'(U^{-1}_{n,q}(\theta)\Delta)\big((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta) +$$

$$+ V^{-1}_m\big)^{-1}(U^{-1}_{n,q}(\theta)\Delta)'\left(U^{-1}_{n,q}(\theta)U_{n,p}(\phi)\right) \;. \qquad (2.7)$$

For a strictly autoregressive process $U_{n,q}(\theta) = I$ and equation (2.7) reduces to

$$P^{-1}_n(p,0) = U'_{n,p}(\phi)\, U_{n,p}(\phi) - (U'_{n,p}(\phi)\Delta)(I+V^{-1}_m)^{-1}\left(\Delta' U'_{n,p}(\phi)\right). \qquad (2.8)$$

Since the first n-p rows of $U'_{n,p}(\phi)\Delta$ consists of zeros the second term on the right-hand side of (2.8), whatever the composition of $(I+V_m^{-1})^{-1}$, is a matrix of zeros except for the pxp submatrix in the lower right corner. The elements of $P_n^{-1}(p,0)$ are, appart from the pxp lower right submatrix, equal to the corresponding elements of $U'_{n,p}(\phi)U_{n,p}(\phi)$. When $n \geq 2p$ the elements of $P_n^{-1}(p,0)$ can be deduced by computing $U'_{n,p}(\phi)U_{n,p}(\phi)$ and taking advantage of the double symmetric property of the matrix $P_n^{-1}(p,q)$, i.e. $p^{ij} = p^{ji} = p^{n+1-j,n+1-i} = p^{n+1-i,n+1-j}$ where p^{ij} is the (i,j)-th element of $P_n^{-1}(p,q)$, in the same way as was done in Box & Jenkins (1970, appendix A 7.5) and in Siddiqui (1958).

For a strictly moving average process $U_{n,p}(\phi) = I$ and the matrix $P_n^{-1}(p,q)$ can be written in the form

$$P_n^{-1}(0,q) = \left(U_{n,q}^{-1}(\theta)\right)'\left(U_{n,q}(\theta)\right) - \left(U_{n,q}^{-1}(\theta)\right)'(U_{n,q}^{-1}(\theta)\Delta)\Big((U_{n,q}^{-1}(\theta)\Delta)'(U_{n,q}^{-1}(\theta)\Delta) + V_m^{-1}\Big)^{-1}(U_{n,q}^{-1}(\theta)\Delta)'\left(U_{n,q}^{-1}(\theta)\right) . \qquad (2.9)$$

The matrix $U'_{n,q}(\theta)$ is an nxm lower triangular matrix which can be partitioned as follows

$$\begin{pmatrix} U'_{n-1,q}(\theta) & \vdots\, 0 \\ \hline (0,\ldots,0,\theta_q,\theta_{q-1},\ldots,\theta_1) & \vdots\, 1 \end{pmatrix} . \qquad (2.10)$$

The inverse of this matrix is given by

$$\left(U'_{n,q}(\theta)\right)^{-1} = \begin{pmatrix} \left(U'_{n-1,q}(\theta)\right)^{-1} & \vdots\; 0 \\ \hline -d'_n\left(U'_{n-1,q}(\theta)\right)^{-1} & \vdots\; 1 \end{pmatrix} \qquad (2.11)$$

where $d'_n = (0,\ldots,0,\theta_q,\theta_{q-1},\ldots,\theta_1)$.In this way we have reduced the inversion of an nxn lower triangular matrix to the inversion of the same matrix with order n-1. We can apply the same process to the inversion of $\left(U'_{n-1,q}(\theta)\right)$ and on repeating this procedure we finally arive on the n-th step at the completely inverted matrix $\left(U'_{n,q}(\theta)\right)^{-1}$. It is easy to see that $\left(U'_{n,q}(\theta)\right)^{-1}$ is of the same form as $U'_{n,q}(\theta)$ with the i+1-th element of the first column given by the recursive relations

$$u_{i+1} = \begin{cases} -(\theta_i u_1 + \theta_{i-1} u_2 + \dots + \theta_1 u_i) & \text{for } i=1,\dots,q \\ \\ -(\theta_q u_{i+q+1} + \theta_{q+1} u_{i-q+2} + \dots \theta_1 u_i) & \text{for } i=q+1,\dots,n-1 \end{cases} \quad (2.12)$$

with $u_1 = 1$.

Using the above relations the matrix product $\left(U^{-1}_{n,q}(\theta)\right)'\left(U^{-1}_{n,q}(\theta)\right)$ can be obtained and since V_m has the same basic form as $U'_{n,q}(\theta)$ the inverse V_m^{-1} is straightforward determined. The only remaining problem is the inversion of the qxq matrix $\left((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta)+V_m^{-1}\right)$.

The calculations for obtaining $P_n^{-1}(0,q)$ can be slightly modified by applying relation (2.6) a second time to $\left((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta)+V_m^{-1}\right)^{-1}$ which gives the following expression

$$\left((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta)\right)^{-1} - \left((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta)\right)^{-1} \Big(\left((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta)\right)^{-1} + V_m\Big)^{-1} \left((U^{-1}_{n,q}(\theta)\Delta)'(U^{-1}_{n,q}(\theta)\Delta)\right)^{-1}. \quad (2.13)$$

Here no inverse of V_m has to be obtained.

3. The inverse for a nonstationary process

We now consider the general autoregressive integrated moving average process

$$\sum_{j=0}^{p} \phi_j \sum_{i=0}^{d} \binom{d}{i}(-1)^i z_{t-i-j} = \sum_{j=0}^{q} \theta_j a_{t-j} \quad (3.1)$$

with $\phi_0 = \theta_0 = 1$.

If we replace $\sum_{i=0}^{d} \binom{d}{i}(-1)^i z_{t-i}$ by w_t, the autoregressive integrated moving average process for $\{z_t\}$ reduces to an autoregressive moving average process

$$\sum_{j=0}^{p} \phi_j w_{t-j} = \sum_{j=0}^{q} \theta_j a_{t-j} . \quad (3.2)$$

The integrated process given by equation (3.1) can be taken up as the limit, for ε approaches one, of the process

$$\sum_{j=0}^{p} \phi_j \sum_{i=0}^{d} \binom{d}{i}(-\varepsilon)^i z_{t-i-j} = \sum_{j=0}^{q} \theta_j a_{t-j}. \qquad (3.3)$$

The equations (3.2) and (3.3) are related with each other in the following way

$$(z_1,z_2,\ldots,z_d,w_{d+1},w_{d+2},\ldots,w_n)' = M_n(\varepsilon)(z_1,\ldots,z_n)' \qquad (3.4)$$

where $M_n(\varepsilon)$ is an nxn lower triangular matrix whose elements in the i-th row and j-th column are 1 for i=j, $\binom{d}{j+d-i}(-\varepsilon)^{i-j}$ for $j<i\leq d+j$ with i=d+1,d+2,...,n and zero for i>d+j or $i\leq j$ with i=d+1,d+2,...,n.

The autocovariance matrix for (3.4) is given by the expression

$$P_n(p,q) = M_n^{-1}(\varepsilon)\begin{bmatrix} V(z'z) & V(z'w) \\ V'(z'w) & V(w'w) \end{bmatrix} \left(M_n'(\varepsilon)\right)^{-1} \qquad (3.5)$$

where V(z'z) is the dxd variance-covariance matrix of $(z_1,z_2,\ldots,z_d)'$, V(w'w) is the n-dxn-d variance-covariance matrix of $(w_{d+1},w_{d+2},\ldots,w_n)'$ and V(z'w) is the n-dxd cross-covariance matrix between the vectors $(z_1,z_2,\ldots,z_d)'$ and $(w_{d+1},w_{d+2},\ldots,w_n)$.

Using the well-known expression for the partitioned inversion of a nonsingular matrix on (3.5) yields

$$P_n^{-1}(p,q) = M_n'(\varepsilon)\begin{bmatrix} V^{-1}(z'z) + V^{-1}(z'z)V(z'w)\,R\,V'(z'w)V^{-1}(z'z) & -V^{-1}(z'z)\,V(z'w)\,R \\ -R\,V'(z'w)V^{-1}(z'z) & R \end{bmatrix} M_n(\varepsilon) \qquad (3.6)$$

with $R = \left[V(w'w) - V'(z'w)\,V^{-1}(z'z)\,V(z'w)\right]^{-1}$.

When ε approaches one several things happen. First of all it is obvious from (3.3) that the dxd matrix V(z'z) tends to infinity. Second, the matrix V(z'w) remains a finite matrix as ε goes to one. This can be easily seen from writing (3.3) in the random shock form

$$z_t = \psi_0 a_t + (d\varepsilon\psi_0 + \psi_1)a_{t-1} + \left(\frac{d(d+1)}{2!}\varepsilon^2\psi_0 + d\varepsilon\psi_1 + \psi_2\right)a_{t-2} +$$

$$+\left(\frac{d(d+1)(d+2)}{3!}\varepsilon^3\psi_0 + \frac{d(d+1)}{2!}\varepsilon^2\psi_1 + d\varepsilon\psi_2 + \psi_3\right)a_{t-3} + \dots$$

$$= \sum_{k=0}^{\infty} \gamma_{za}(k)a_{t-k}$$

where $(1-\varepsilon B)^{-d}\,\theta(B)\,/\,\phi(B) = \sum_{j=0}^{\infty} \binom{d+j+1}{j}(\varepsilon B)^j \sum_{\ell=0}^{\infty} \psi_\ell B^\ell$ with B the backward shiftoperator and where $\gamma_{za}(k) \doteq E\left[z_t a_{t-k}\right]$. Then

$$E\left[w_t\ z_\ell\right] = \sum_{j=t-\ell}^{\infty} \psi_j \gamma_{za}\,(j-t-\ell) \qquad (3.7)$$

for $\ell=1,2,\dots,d$ is finite, since $\gamma_{za}(k)$ has a bound independent of ε for $|\varepsilon|\leq 1$.

Because $V(z'z)$ tends to infinity, as ε goes to one, it follows from

$$V^{-1}(w'w)R^{-1} = V^{-1}(w'w)\left[V(w'w) - V'(z'w)V^{-1}(z'z)V(z'w)\right]$$

$$= I - V^{-1}(w'w)V'(z'w)V^{-1}(z'z)V(z'w)$$

that R tends to $V^{-1}(w'w)$. Hence, combining the above results, it follows that the matrix $P_n^{-1}(p,q)$ goes to zero for ε goes to one except for the matrix R and the expression on the right of (3.6) becomes

$$M_n' \begin{bmatrix} 0 & 0 \\ 0 & V^{-1}(w'w) \end{bmatrix} M_n \qquad (3.8)$$

with $M_n = \lim_{\varepsilon\to 1} M_n(\varepsilon)$. So for $\varepsilon=1$ we have $z'P_n^{-1}(p,q)z = w'V^{-1}(w'w)w$. This means that the likelihood function of n observations from (3.1) has the same exponential form as n-d observations from (3.2).

4. A special case

To illustrate the results of section 2 we consider the case where the series $\{z_t\}$ is generated by the process

$$z_t + \phi_1 z_{t-1} + \phi_2 z_{t-2} = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}\ .$$

The matrices $U_{n,p}(\phi)$, $U_{n,q}(\theta)$ and Δ are

$$U_{n,2}(\phi) = \begin{bmatrix} 1 & \phi_1 & \phi_2 & 0 & \dots & 0 \\ & 1 & \phi_1 & \phi_2 & \dots & 0 \\ & & & & & \vdots \\ & & & 1 & \phi_1 & \phi_2 \\ & & & & 1 & \phi_1 \\ & & & & & 1 \end{bmatrix}, \quad U_{n,2}(\theta) = \begin{bmatrix} 1 & \theta_1 & \theta_2 & 0 & \dots & 0 \\ & 1 & \theta_1 & \theta_2 & \dots & 0 \\ & & & & & \vdots \\ & & & 1 & \theta_1 & \theta_2 \\ & & & & 1 & \theta_1 \\ & & & & & 1 \end{bmatrix},$$

$$\Delta' = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

The vector $(y_{t-n+1}, y_{t-n+2})'$ is given by

$$\begin{pmatrix} y_{t-n+1} \\ y_{t-n+2} \end{pmatrix} = \begin{pmatrix} -\phi_2 & 0 & \theta_2 & 0 \\ -\phi_1 & -\phi_2 & \theta_1 & \theta_2 \end{pmatrix} \begin{bmatrix} z_{t-n} \\ z_{t-n-1} \\ a_{t-n} \\ a_{t-n-1} \end{bmatrix}.$$

The variance-covariance matrix of this vector follows from the relations

$$\begin{bmatrix} 1 & \phi_1 & \phi_2 \\ \phi_1 & 1+\phi_2 & 0 \\ \phi_2 & \phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 & \theta_1 & \theta_2 \\ \theta_1 & \theta_2 & 0 \\ \theta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{za}(0) \\ \gamma_{za}(1) \\ \gamma_{za}(2) \end{bmatrix}$$

where $\gamma_{za}(k) = E\left[z_t \; a_{t-k}\right]$ and is given by

$$V_m = \begin{bmatrix} -\phi_2 & 0 & \theta_2 & 0 \\ -\phi_1 & -\phi_2 & \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_{za}(0) & \gamma_{za}(1) \\ \gamma_1 & \gamma_0 & 0 & \gamma_{za}(0) \\ \gamma_{za}(0) & 0 & 1 & 0 \\ \gamma_{za}(1) & \gamma_{za}(0) & 0 & 1 \end{bmatrix} \begin{bmatrix} -\phi_2 & -\phi_1 \\ 0 & -\phi_2 \\ \theta_2 & \theta_1 \\ 0 & \theta_2 \end{bmatrix}$$

with $\gamma_{za}(0) = 1$, $\gamma_{za}(1) = -\phi_1 + \theta_1$, $\gamma_0 = (\det)^{-1}\{(1+\phi_2)(1+\theta_1^2+\theta_2^2-2\phi_2\theta_2) + 2\phi_1\phi_2(\phi_1-\theta_1)\}$, $\gamma_1 = (\det)^{-1}\{-\phi_1(1+\theta_1^2+\theta_2^2) + (\theta_1-\phi_1\theta_2)(1+\phi_1^2-\phi_2^2) + \theta_2(\phi_2^2+\theta_1+\phi_1^2\theta_1+\phi_2^2\theta_2)\}$ and $\det = (1-\phi_2)\left((1+\phi_2)^2-\phi_1^2\right)$.

The elements of V_m are

$$v_{11} = \phi_2^2\gamma_0 - 2\phi_2\theta_2\gamma_{za}(0) + \theta_2^2$$

$$v_{12} = v_{21} = \phi_1\phi_2\gamma_0 + \phi_2^2\gamma_1 - (\phi_2\theta_1 + \phi_1\theta_2)\gamma_{za}(0) - \phi_2\theta_2\gamma_{za}(1) + \theta_1\theta_2$$

$$v_{22} = (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 - 2(\phi_1\theta_1 + \phi_2\theta_2)\gamma_{za}(0) - 2\phi_2\theta_1\gamma_{za}(1) + \theta_1^2 + \theta_2^2 \; .$$

Using the recursive relations (2.12) the elements of $\left(U'_{n,2}(\theta)\right)^{-1}$ are

$$u_1 = 1,$$

$$u_{i+1} = -(\theta_2 u_{i-1} + \theta_1 u_i) \quad \text{for } i=1,\dots,n-1$$

and the matrix product $(U_{n,2}^{-1}(\theta)\Delta)'(U_{n,2}^{-1}(\theta)\Delta)$ becomes

$$\begin{bmatrix} \sum_{i=0}^{n-1} u_i^2 & \sum_{i=0}^{n-2} u_i u_{i+1} \\ \sum_{i=0}^{n-2} u_i u_{i+1} & \sum_{i=0}^{n-2} u_i^2 \end{bmatrix} .$$

If we apply relation (2.6) a second time to $\left((U_{n,q}^{-1}(\theta)\Delta)'(U_{n,q}^{-1}(\theta)\Delta) + V_m^{-1}\right)^{-1}$ no inverse of V_m is needed and from here the matrix P_n^{-1} (2,2) can be obtained by straitforward matrix multiplication.

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