Amsterdam. University. Institute econometrical research and operations

R.G. Kreijger

Solving nonlinear Input-Output systems

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Abstract: An algorithm is developed with which a nonlinear generalization of the classical linear interindustry model may be solved. The algorithm is shown to be convergent. It is subsequently applied to a small nonlinear input-output model.

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I. Introduction

In input-output analysis it is usually assumed that there exists a linear relationship between intermediate deliveries and total production. We write $x_{ij} = a_{ij}x_j$, where x_{ij} represents the deliveries of sector i to sector j, x_j represents the total production of sector j, and a_{ij} is a constant, the so called technical coefficient. Let us denote the elasticity of x_{ij} with respect to x_j $E(x_{ij},x_j)$. The assumption of linearity thus implies $E(x_{ij},x_j)=1$ for all i,j.

It is not difficult to think of cases where the assumption that $E(x_{ij},x_j)=1$ seems a bit implausible. In empirical systems, some factors may be fixed; (for example: the total amount of land, the labor force). In that case there may well be diminishing returns for some inputs, implying that $E(x_{ij},x_j) > 1$. On the other hand, inputs may have the character of overhead costs, in which case we would expect $E(x_{ij},x_j) < 1$. An example of diminishing returns could be the delivery of fertilizers to agriculture, an example of increasing returns could be the use of paper for administrative purposes by a steel factory.

It thus seems possible to obtain more realistic models by allowing the $E(x_{ij},x_j)$ to be different from 1. We therefore consider input-output relationship of the form $x_{ij} = \Phi_{ij}(x_j)$, where the $\Phi_{ij}(x_j)$ are assumed to belong to a family of nonlinear functions including the linear specification as a special case. The logarithmic derivatives of the $a_{ij}(x_j)$ may be smaller than or greater than 1.

The following questions now arise:

(a) Theoretical: How do we know whether solutions to the non-linear model exist? How do we obtain these solutions? Can we say something about the properties of these solutions?

(b) Practical: Are the results of non-linear models, in terms of forecasting performance, sufficiently better than the results of the linear model to justify the greater complexity of the non-linear model?

Question (b) seems particularly interesting in view of the fact that in many cases one has no alternative but to use linear models due to the lack of data. We could have more confidence in the linear model if it were possible to give a negative answer to question (b). On the other hand, a clear positive answer to question (b) implies that it would be possible to obtain better sectoral forecasts in those cases where sufficient data exist to implement non-linear models.

The above considerations suggest a research programme, in which non-linear specifications are compared with the standard model. Before embarking on such a programme we must answer question (a). This will be the main purpose of the present paper. We shall develop and analyse an algorithm by which a wide dass of input-output models can be solved. This algorithm will subsequently be applied to a small nonlinear I-O model of the Dutch economy.

II. The model

Let the economy be divided into N sectors. Let x_j be the total output of sector j, x_{ij} the deliveries of sector i to sector j, and f_i the (exogenously given) final demand in sector i. As the total product of a sector can be devided into intermediate deliveries and final deliveries, we have $x_i = \sum_j x_{ij} + f_i$. As the x_{ij} are supposed to be functions of the x_j this becomes $x_i = \sum_j \Phi_{ij}(x_j) + f_i$, which is a system of N non-linear equations, from which we want to solve the x_i.

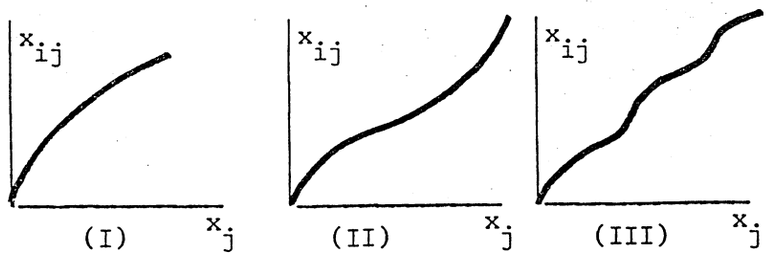
In order to facilitate the discussion we introduce the following notation: $a_{ij}(x_j) = \Phi_{ij}(x_j)/x_j$; x is the vector with x_i in the i'th position; f is the vector with f_i in the i'th position; A(x) is the matrix with $a_{ij}(x_j)$ in the i,j'th position. The system may then be written as $[I-A(x)]\cdot x = f$.

We shall consider systems of the above form, satisfying the following restrictions and assumptions:

1 We consider only $x > 0$ and $f > 0$.

2 The $\Phi_{ij}(x_j)$ are assumed to be either identically zero for all values of x_j, or to be non-decreasing continuous functions of x_j, tending to infinity as x_j tends to infinity. The $a_{ij}(x_j)$ are thus non-negative. $\Phi_{ij}(0) = 0$ for all i, j.

3 There exist values $\bar{x}_1 \ldots \bar{x}_n$ beyond which the $a_{ij}(x_j)$ are either monotonic non-decreasing or monotonic non-increasing (or constant). This condition is mathematically convenient, and it does not cause too much pain on the economic side. One would expect the graphs of the Φ_{ij} to be gently bending curves like diagrams (I) or (II). The exclusion of curves like (III) does not seem to be a great loss from the economic point of view.



4 We consider only a_{ij} such that $\lim_{x_j \to \infty} a_{ij}(x_j) > 0$ for all i and j.

Like condition (3) this is very convenient mathematically, while it does not great harm to the economic content of our model.

5 We assume that A(x) is indecomposable. If this were not the case, we could decompose our system into smaller subsystems, and apply the theory to each subsystem in turn. This would make matters more tedious, without adding anything essential.

The only study of nonlinear I-O models that we know of is I.W. Sandberg's article in Econometrica [1973]. An important difference between Sandberg's model and the present one is that we allow for diminishing returns. Neither a priori reasoning nor visual inspection of time series of technical coefficients suggest that diminishing returns do not occur. It therefore seems too restrictive to exclude them.

III. Solution of the model

We shall first discuss the solution of the model in an intuitive way, laying stress on economic interpretation. Then we shall back up this discussion with mathematical arguments.

The problem is to solve the system $x = \Phi(x) + f$ for given f. Here Φ denotes the vector with $\sum_j \Phi_{ij}(x_j)$ in the i-th position. Imagine the following sequence of events, which resembles the well-known dynamic multiplier process in a simple Keynesian model: In the first round, producers decide to produce f. They therefore need a production of intermediate goods, $\Phi(f)$. Hence, total production must be at least $x^{(1)} = f + \Phi(f)$. In order to be able to produce $x^{(1)}$ producers need an input of intermediate goods $\Phi(x^{(1)})$. Hence production is increased to $x^{(2)} = \Phi(x^{(1)}) + f$. In general, we shall have $x^{(n)} = \Phi(x^{(n-1)}) + f$ in the (n+1)-th round. If this process converges, it will reach a solution (within a prescribed level of accuracy) in a finite number of rounds.

The sequence $f, x^{(1)}, x^{(2)}, \ldots$, which we denote as $\{x^{(n)}\}$, will in case of convergence produce a solution. It will be proved that the existence of a solution is sufficent to ensure convergence of $\{x^{(n)}\}$. We conclude that the sequence of rounds defined above always leads to a solution if there exists one. The question now is: what will happen if no solution exists? How do producers in that case decide that the above process is not going to converge and that it will be impossible to satisfy the final demand f?

The answer is that in case of divergence there exists a point x^* beyond which no positive value added can be realized in at least one sector. The existence of this point depends only on the fact of divergence of $\{x^{(n)}\}$ and not on the price system. We assume that producers will refrain from increasing output at ever increasing losses. Hence if production grows above x^*, production will stop without a solution being reached. The interesting point here is that the physical fact of the nonexistence of a solution has its economic reflection in the fact that it is impossible to realize a positive value-added.

The sequence $\{x^{(n)}\}$ will thus come to an end in a finite number of steps, either by providing a solution (within a prescribed level of accuracy) or with the conclusion that no solution exists.

We now turn to a formal analysis of the solution procedure in the form of a few theorems.

<u>Theorem 1</u>. If there exists a solution $\zeta > 0$ to the system $x = \Phi(x) + f$, then sequence $\{x^{(n)}\}$ defined by $x^{(n)} = \Phi(x^{(n-1)}) + f$; $x^{(0)} = f$ converges to a solution ξ and $\xi \leqq \zeta$.

<u>Proof</u>. As all the Φ_{ij} are monotonic non-decreasing we have $\Phi(x) \geqq \Phi(y)$ if $x \geqq y$. Now $x^{(1)} = \Phi(f) + f$. As $f > 0$, $\Phi(f) \geqq 0$ so $x^{(1)} = \Phi(f) + f \geqq f = x^{(0)}$, so $x^{(1)} \geqq x^{(0)}$. As $x^{(1)} \geqq x^{(0)}$ we have $x^{(2)} = \Phi(x^{(1)}) + f \geqq \Phi(x^{(0)}) + f = x^{(1)}$, so $x^{(2)} \geqq x^{(1)}$. By iteration: $x^{(n)} \geqq x^{(n-1)}$ for all n. The sequence $\{x^{(n)}\}$ is thus monotonic increasing. By assumption, there exists $\zeta > 0$ such that $\zeta = A(\zeta) + f$. As $A(\zeta) \geqq 0$ we have $\zeta \geqq f \rightarrow \zeta = A(\zeta) + f \geqq A(f) + f = x^{(1)}$ so $\zeta \geqq x^{(1)}$. Then $\zeta = A(\zeta) + f \geqq A(x^{(1)}) + f = x^{(2)}$, so $\zeta \geqq x^{(2)}$. Iterating, we have $\zeta \geqq x^{(n)}$ for all n, and $x^{(n)}$ is bounded from above. As $\{x^{(n)}\}$ is monotonic increasing and bounded from above, it must possess an unique limit ξ. Clearly, ξ is a solution and $\xi \leqq \zeta$. Q.E.D.

The process thus always finds a solution if there exists one. The solution that is found will be economically meaningful in the sense that it is the smallest possible production vector that is needed in order to satisfy the given final demand.

The question of existence of a solution is a bit more complicated. In the following we use a number of properties of non-negative matrices. As this materical is nowadays standard in mathematical economics, it is assumed here that the reader is familiar with it. Good references include, on the mathematical side, the books by Gantmacher [1960] and Seneta [1973], and on the economic side the books by Nikaido [1972], Takayama [1974], and Lancaster [1968].

It will turn out that the existence of a solution depends on the behavior of the dominant eigenvalue of $A(x)$. Let us denote this dominant eigenvalue as $\lambda(A(x))$. By the theorem of Frobenius, this is a real number. It is a well-known fact in input-output analysis that, for a meaningful solution to an input-output system to exist, it is a necessary condition that the dominant eigenvalue of the matrix of technical coefficients be smaller than 1. Hence, if ξ is a solution to our system, we must have $\lambda(A(\xi)) < 1$. It is also well-known that the condition that $\lambda(A) < 1$ is equivalent to the so-called Hawkins-Simon conditions which demand, that all principal minors of the matrix (I-A) be positive. See Hawkins and Simon [1949]. Furthermore, as the determinant of a matrix is equal to the product of its eigenvalues, we must have that the absolute value of these minors be smaller than 1. We intend to show that, in case of divergence of $\{x^{(n)}\}$, $\lambda(A(x\ ))$ will eventually become greater than 1, and remains greater than 1. First we need the following lemma:

<u>Lemma</u>. Either the sequence $\{x^{(n)}\}$ converges or all elements of $x^{(n)}$ go to infinity.

<u>Proof</u>. By assumption (4), the matrix $A(x)$ is indecomposable. Suppose all elements of $\{x^{(n)}\}$ are bounded. Then, as $\{x^{(n)}\}$ is monotonic non-decreasing it must have a limit, and the sequence converges. Suppose that $\{x_1^{(n)}\} \to \infty$, and $\{x_2^{(n)}\}$ ... $\{x_N^{(n)}\}$ are all bounded. As $x_i^{(n)} = \sum_j a_{ij}(x_j^{(n-1)}) + f_i$. This implies, in view of assumption (5), that $a_{21}(x_1) \equiv \ldots \equiv a_{N1}(x_1) \equiv 0$. This however contradicts assumption (6). Hence at least one of the elements $x_2^{(n)}$ ... $x_N^{(n)}$ goes to infinity.

Without loss of generality we assume that $\{x_2^{(n)}\} \to \infty$. Suppose the sequences $\{x_3^{(n)}\}$ $\{x_4^{(n)}\}$ are all bounded.

This would imply $a_{31}(x_1) \equiv a_{32}(x_2) \equiv \ldots \equiv a_{N1}(x_1) \equiv a_{N2}(x_2) \equiv 0$, again contradicting assumption (6). By repeating this argument N-2 more times, we prove the lemma.

From lemma, we conclude that divergence of $\{x^{(n)}\}$ implies that all elements of $x^{(n)}$ diverge. We will now show that in case of divergence the Hawkins-Simon conditions on A(x) will be violated if x becomes greater than a certain bound $\bar{x}$.

Let us define the matrix $A^*(x)$ as follows: $a^*_{ij}(x_j) = a_{ij}(x_j)$ if $a_{ij}(x_j)$ is (beyond a certain point) monotonic non-decreasing $a^*_{ij}(x_j) = \varepsilon_{ij}$ (the lower bound of $a_{ij}(x_j)$) if a_{ij} is monotonic decreasing. By assumption, A^* is indecomposable. As $A^* \leqq A$ we have $\lambda(A) \geqq \lambda(A^*)$. We define $B = (\frac{1}{2})^{N-1}(I+A^*)^{N-1}$.

The indecomposability of A^* implies $B > 0$. For a proof of this, see e.g. Gantmacher [1960] vol. II, p. 51.

Clearly, $\lambda(A^*) < 1$ implies $\bar{\lambda}(B) < 1$.

We now consider two cases:

(I) A^* (and hence A) has no limit, i.e. there exist i, j such that $\lim_{x_j \to \infty} a_{ij}(x_j) = \infty$.

(II) A^* has a limit, i.e. there exists a matrix C such that $\lim_{x_j \to \infty} a_{ij} = c_{ij}$.

For case (I) we prove the following

<u>Theorem 2</u>. In case I, divergence of $\{x^{(n)}\}$ implies that, beyond a certain point x, (I-A) will no longer satisfy the Hawkins-Simon conditions.

<u>Proof</u>. Consider B as defined above. For $\lambda(A)$ to be smaller than 1, it is necessary that $\lambda(B) < 1$. Hence for (I-A) to satisfy the Hawkins-Simon conditions, it is necessary that (I-B) satisfies these conditions. As some elements of A go to infinity, some elements of B go to infinity, and as all elements of A^* are monotonic increasing, all elements of B are.

Now let us consider first the diagonal elements of I - B. If one of the elements $b_{11} \ldots b_{NN}$ increases beyond all bounds, say b_{ii} then $1 - b_{ii}$ eventually becomes negative, and the Hawkins-Simon conditions are no longer satisfied.

Let us now suppose that all elements $b_{ii} \leqq 1$, whatever the value of x. Consider then the second principal minor of I - B:

$$\det \begin{pmatrix} 1-b_{11} & -b_{12} \\ -b_{21} & 1-b_{22} \end{pmatrix} = (1-b_{11})(1-b_{22})\, b_{21}b_{12} = \det[I-B]_{22} \,.$$

If one of the elements b_{21} or b_{12} (or both) increase beyond all bounds, this expression will eventually become negative, thus violating the Hawkins-Simon conditions.

Suppose then that all elements of the second principal minor are bounded and that $[I-B]_{22}$ satisfies the Hawkins-Simon conditions. Consider the third principal minor:

$$\det(I-B)_{22} = \det \left(\begin{array}{cc|c} 1-b_{11} & -b_{12} & -b_{13} \\ -b_{21} & 1-b_{22} & -b_{23} \\ \hline -b_{31} & -b_{32} & 1-b_{33} \end{array}\right) .$$

We denote this as $\det \begin{pmatrix} [I-B]_{22} & -q \\ -p' & 1-b_{33} \end{pmatrix}$, where $q = \begin{vmatrix} b_{13} \\ b_{23} \end{vmatrix}$, $p = \begin{vmatrix} b_{31} \\ b_{32} \end{vmatrix}$.

Clearly, this is equal to $\det[I-B]_{22} \,.\, (1-b_{33}-p'[I-B]_{22}^{-1}q)$.

Now $\det(I-B)_{11} > 0$, $1-b_{33} \geqq 0$, $(I-B)_{22}^{-1} > 0$, $p > 0$ and $q > 0$. So if either p or q (or both) contain elements that increase beyond all bounds, the second principal minor will eventually become negative. If all elements of p and q are bounded, we repeat the same discussion for the third principal minor, and so on. As B is supposed to possess at least one element which increases beyond all bounds, we conclude that (I-B) cannot possibly satisfy the Hawkins-Simon conditions $f\{x^{(n)}\}$ diverges. The same conclusion must then hold for (I-A). Q.E.D.

For case (II), we consider two possibilities:

II_a: $\lambda(C) < 1$, II_b: $\lambda(C) > 1$. The possibility $\lambda(C) = 1$ we shall rule out as having probability 0 in any practical application.

For case II_a we prove the following

<u>Theorem 3</u>: In case II_a there is no divergence of $\{x^{(n)}\}$ possible.

Proof. Define $A^*(x)$ in the following way:

$a^*_{ij}(x_j) = c_{ij}$ for those $a_{ij}(x_j)$ that are monotonic increasing

$a^*_{ij}(x_j) = a_{ij}(x_j)$ for those a_{ij} that are monotonic decreasing.

It then follows that $\lambda\left(A^*(x)\right)$ is a monotonic decreasing function of x, and $\lim_{x\to\infty} \lambda\left(A^*(x)\right) = \lambda(C) < 1$. Hence there exists an $\bar{x}$ such that for all $x^{(n)} \geqq \bar{x}$ we have $\lambda\left(A^*(x^n)\right) < 1$.

Suppose now that the sequence $\{x^{(n)}\}$ diverges. It then follows from lemma 2 that there exists a number m such that for $n > m$ we have $x^{(n)} > \bar{x}$. Hence $A^*(\bar{x}) \geqq A^*(x^{(n)}) \geqq A(x^{(n)})$, and $A^*(\bar{x}).\ x^{(n)} \geqq A(x^{(n)}).\ x^{(n)}$, for all $n > m$.

It follows then that $x^{m+k} \leqq \left[A^*(\bar{x})\right]^k x^m + \left[I+A^*(\bar{x}) + \ldots + A^*(x)^{k-1}\right] f$.

As $\lambda\left(A^*(\bar{x})\right) < 1$, we have $\lim_{k\to\infty}\left[A^*(\bar{x})\right]^k = 0$, and

$\lim_{k\to\infty}\left[I+A^*(\bar{x}) + \ldots + A^*(x)^{k-1}\right] = \left[I-A^*(\bar{x})\right]^{-1}$. Hence $\lim_{n\to\infty} x^n \leqq \left[I-A^*(x)\right]^{-1} f$.

We conclude that the sequence is bounded. By theorem 1 it must then converge. The assumption of divergence of the sequence thus leads to a contradiction and hence must be false. Q.E.D.

Theorem 2 holds good in case II_b: divergence of $\{x^{(n)}\}$ in this case implies that from a certain point onwards the Hawkins-Simon conditions will be violated.

Proof: Define A^*_{ij} as $\lim_{x_j\to\infty} a_{ij}(x_j)$ for those a_{ij} that are decreasing, and as $a_{ij}(x_j)$ for those a_{ij} that are increasing.

Then $\lambda\left(A^*(x)\right) \leqq \lambda\left(A(x)\right)$. Lim $\lambda\left(A^*(x)\right) = \lambda(C) > 1$, hence there exists an $\bar{x}$ such that $\lambda\left(A^*(\bar{x})\right) > 1$.

In case of divergence of $\{x^{(n)}\}$, by lemma 2 there exists a number m such that $x^{(n)} > x^{(m)} > \bar{x}$ for all $n > m$. As $\lambda\left(A^*(x^{(n)})\right)$ is monotonic increasing with n, we have that, for all $n > m$, $\lambda\left(A(x^{(n)})\right) > 1$. Q.E.D.

IV. An application

In order to examine the applicebility of the present approach, a small I-O model of the Dutch economy was estimated. The coefficients were estimated on the basis of a time series of input-output tables ranging from 1953 to 1962*). The algorithm was then used to generate "forecasts" for the years 1963-1976 on the basis of known final demands.

The (34 sector) CBS input-output tables were aggregated into the following 5 sectors:

(I) Agriculture, forestry, fishery, food and allied products, beverages and tobacco, corresponding to the sectors 1, 4, 5 and 6 of the CBS tables.

(II) Textile industry, fooryear and clothes, wood-working and furniture, paper industry, printing and publishing, leather and rubber, chemical industry, oil refining, building materials, corresponding to sectos 7 through 14 of the CBS tables.

(III) Metallurgy, machines, electronics, transport equipment, metal processing and diamond-industry (sectors 15 through 19 of the CBS tables).

(IV) Building industry (sector 20 of the CBS tables)

(V) Public utilities, transport, trade and services (sector 21 through 34 of the CBS tables)

Sectors 2 and 3 of the CBS tables, coal mining and other mining and quarrying, were omitted because of the great structural changes that have taken place in these sectors during the sixties: the shutdown of a number of coal mines and the great finds of natural gas.

The following relationships were supposed to hold: $x_{ij}(t)=\alpha_{ij}x_j(t)^{\beta_{ij}}e^{u_{ij}(t)}$. The relation of constant proportionality was thus replaced by a relation of constant elasticity. $\underline{u}_{ij}(t)$ is a stochastic disturbance.

After taking logarithms we have:

$$\log x_{ij}(t) = \log\alpha_{ij} + \beta_{ij}\log x_j(t) + \underline{u}_{ij}(t) \ldots \qquad (1)$$

The $\underline{u}_{ij}(t)$ are assumed to be normally distributed, with $E(\underline{u}_{ij}(t)) = 0$. We will allow them to be both autocorrelated and contemporaneously correlated: we write $\underline{u}_{ij}(t) = \rho_{ij}\underline{u}_{ij}(t-1) + \underline{v}_{ij}(t)$, and $E(v_{ij}(t)v_{kj}(t)) = \sigma_{ikj}$. All other covariances are assumed to be zero.

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*) Data were taken from "de productiestructuur van de Nederlandse Volkshuishouding" (the production structure of the Dutch economy) published by the Central Bureau of Statistics (CBS) vols I - V

For the estimation of the relationships a variant of a three-stage method developed by Parks [1967] was used.
Parhs' method is the following: after applying ordinary least squares to (1) an estimate of ρ_{ij} is obtained from the residuals. This estimate is subsequently used to remove the autocorrelation from 1:

$$\log x_{ij}(t)-\rho_{ij}\log x_{ij}(t-1)=(1-\rho_{ij})\log\alpha_{ij}+\beta_{ij}\left(\log x_j(t)-\rho_{ij}\log x_j(t-1)\right)+v_{ij}(t)\ldots \quad (2)$$

The equations (2) are then subjected to the well-known Zellner procedure for seemingly unrelated equations. See e.g. Theil [1971].
Parks has shown that the resulting estimates have a number of desirable properties (unbiasedness, consisteney, relative efficiency).

We have modified Parks' approach in one aspect: we did not estimate the ρ_{ij} from the residuals of an ordinary least squares regression, but we estimated ρ_{ij} from the regression (2) directly, ignoring the restrictions on the coefficients that the specification of this equation implies. From simulation studies conducted by Rao and Griliches [1969] it appears that this approach leads to better results than estimation from residuals.

The computations produce unbiased estimates of β_{ij}. Taking the exponent of $\log\alpha_{ij}$ does not lead to unbiased estimates of α_{ij}. Furthermore, even if we had unbiased estimates of the α_{ij}, we would not have $E(x_{ij}) = \alpha_{ij}x_j^{\beta_{ij}}$, because $E(e^{u_{ij}}) \neq 1$. We tried several ways of correcting the α_{ij}'s, but finally decided in our experiment to use $\alpha_{ij} = \dfrac{x_{ij}(T)}{x_j(T)^{\beta_{ij}}}$, where T is the time-index of the last observation (T=1962).

The following estimates for β_{ij} were found:

Sectors	1	2	3	4	5
1	1.0359 * (2.667)	0.2942 ** (-11.602)	.0012 ** (-2290.7)	2.4607 — (1.3378)	0.6449 ** (-7.5746)
2	1.92 ** (9.7267)	0.8812 ** (-4.8718)	0.9275 — (-1.5563)	1.091 ** (3.6089)	0.97751 — (-0.5397)
3	1.1327 — (0.9814)	1.4291 ** (2.8834)	1.0976 — (1.6347)	1.1392 ** (2.8635)	1.148 — (1.1677)
4	1.7593 ** (6.5116)	1.9518 ** (6.7106)	1.5515 ** (11.0684)	1.0437 — (0.5816)	1.3419 ** (2.9429
5	1.5724 ** (7.3291)	1.1520 — (1.3610)	1.1559 — (1.9330)	1.0293 — (0.4532)	0.9949 — (-0.0355)

The numbers in brackets are the t-values obtained in the last stage of the estimation procedure, for testing the hypothesis $\beta_{ij} = 1 : t = (\hat{\beta}_{ij}-1)/S(\hat{\beta}_{ij})$, where $S(\hat{\beta}_{ij})$ is the estimated standard deviation of $\hat{\beta}_{ij}$. A * in the upper right hand corner indicates that $\hat{\beta}_{ij}$ differs significantly from 1 at the 95% level, a ** indicates that $\hat{\beta}_{ij}$ differs from 1 at the 98% level. A blank indicates a $\hat{\beta}_{ij}$ not significantly different from 1.

Of course it should be kept in mind that these tests are only approximations as we do not know the exact distribution of the estimators resulting from a three-stage procedure.

For the following experiment, we replaced all the non-significant $\hat{\beta}_{ij}$'s by 1 and retained the significant $\hat{\beta}_{ij}$'s. Then the α_{ij}'s were estimated.

We used data from the C.B.S. tables 1963 to 1967 to generate 'forecasts' by the nonlinear method with β_{ij}'s as above and by the usual method, using the aggregated I-O table for 1962. As a convergence criterion for the algorithm we used the following rule: stop when $|x^n-x^{n-1}| < \frac{1}{1000}|f|$, where $|f| = \sum_i |f_i|$. Using the x-value of 1972 as a starting value, the algorithm converged reasonably fast, the maximum number of iterations (obtained for 1967) being 8.

Let the forecast for the production of sector i be x_i^F, and let the true value of the production of sector i be x_i^T. Let final demand in sect r i be f_i. We computed $\{(x_i^F-f_i)/(x_i^T-f_i)-1\} = e_i$ as a measure of the forecasting error.

The results for the nonlinear method and the usual method are displayed in the following table:

forecasting errors of intermediate demand for 1963-1967 by the non-linear method and by the usual method

	1963		1964		1965		1966		1967	
sectors	nonl.	usual	nonl.	usual	nonl.	usual	nonl.	usual	nonl.	usual
I	-.038	-.013	.020	.047	.016	.044	.002	.031	.020	.048
II	-.023	-.013	-.001	.003	.025	.029	.039	.042	.093	.089
III	.0046	-.002	.031	.011	.048	.019	.057	.020	.102	.055
IV	-.259	.024	-.178	.086	-.234	.030	-.156	.059	-.081	.116
V	-.021	-.023	.054	.0008	.055	-.006	.042	.023	.036	-.039

The results do not indicate that introducing nonlinearity leads to a marked increase in forecasting performance, better results in some sectors being offset by worse results in other sectors. However, we think that more research is needed before we may conclude that no significant improvement is possible by using non-linear specifications. We are working at it.

References

(1) F.R. Gantmacher: Theory of matrices, vol II (1960).

(2) D. Hawkins and H.A. Simon: Note: some conditions on macro economic stability. Econometrica 17 (1949).

(3) K. Lancaster: Mathematical Economics (1968).

(4) H. Nikaido: Introduction to sets and mapping in modern economics (1972).

(5) R.W. Parks: Efficient estimation of a system of regression equations when disturbances are both serially and contemporeaneously correlated J.A.S.A. 62 (1967).

(6) P. Rao and Z. Grilches:'Small-sample properties of several two-stage regression methods in the context of auto-correlated errors' J.A.S.A. 64 (1969).

(7) I.W. Sandberg: A nonlinear input-output model of a multisectored economy Econometrica 41 (1973).

(8) E. Seneta: Non-negative matrices (1973).

(9) A. Takayama: Mathematical economics (1974).

(10) H. Theil: principle of econometrics (1971).