Amsterdam. University. Institute of actuarial science and econometrics

H. Neudecker

Bounds for the bias of the LS-estimator of σ^2 in the case of a first-order autoregressive process (positive autocorrelation). AE2/76

Classification

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Abstract: The LS-estimator $\frac{e'e}{n-k}$ of σ^2 and its bias are considered. Bounds are then established for $\frac{1}{\sigma^2} E\frac{e'e}{n-k}$ by means of a well-known theorem by Anderson. The same bounds can be derived by a direct applicant of the Courant-Fischer min-max theorem. Following Anderson and Grenander-Szegö one can find approximations to these bounds.

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Bounds for the bias of the LS estimator of σ^2 in the case of a first-order autoregressive process (positive autocorrelation).

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Introduction

Consider the model

$$y = X\beta + \varepsilon,$$

where ε is the disturbance vector. X is of order (n,k) and rank k. The disturbance elements ε_i are generated by the process

$$\varepsilon_i = \rho\varepsilon_{i-1} + \xi_i \qquad 0 < \rho < 1$$

where the ξ_i are uncorrelated random variables with zero mean and variance σ_0^2.

For ε we have the variance

$$E\,\varepsilon\varepsilon' = \sigma^2 V, \text{ where } \sigma^2 = \frac{\sigma_0^2}{1-\rho^2}$$

and $V = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & & \rho^{n-2} \\ \rho^2 & \rho & 1 & & \rho^{n-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$.

The LS estimator of σ^2 is $\frac{e'e}{n-k}$, where $e = \{I-X(X'X)^{-1}X'\}\varepsilon$.

$$E\,\frac{e'e}{n-k} = \frac{\sigma^2}{n-k}\,\mathrm{tr}MV = \frac{\sigma^2}{n-k}\,\mathrm{tr}\{I-X(X'X)^{-1}X'\}V = \frac{\sigma^2}{n-k}\,\{n-\mathrm{tr}X(X'X)^{-1}X'V\} \neq \sigma^2,$$

as $\mathrm{tr}X(X'X)^{-1}X'V$ is in general not equal to k. The bias is $E\,\frac{e'e}{n-k} - \sigma^2 =$

$$= \frac{\sigma^2}{n-k}\left[k-\mathrm{tr}X(X'X)^{-1}X'V\right].$$

Hence, $\frac{e'e}{n-k}$ is a biased estimator of σ^2.

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* Instituut voor Actuariaat en Econometrie, Universiteit van Amsterdam.
I am grateful to an unknown referee for drawing my attention to the work of Anderson [1] and Grenander and Szegö [4], and commenting on an earlier draft of this paper.

Theil [7, p. 256 et seq.] evaluated $\mathrm{tr}X(X'X)^{-1}X'V$ in the one-variable regression with constant term, where k=2.

For not too small n and ρ such that ρ^2 and higher powers can be neglected he found the result

$$\mathrm{tr}X(X'X)^{-1}X'V \approx \frac{2}{1-\rho} + 2\,r\rho,$$

where $r = \dfrac{\sum_{i=1}^{n-1} X_i X_{i+1}}{\sum_{i=1}^{n} X_i^2}$ and X_i is the i^{th} observation measured from its mean.

The expression $\frac{2}{1-\rho} + 2\,r\rho$ exceeds 2 when both r and ρ are positive.

Similar results for $k > 2$ are not available, it seems.
In this paper we shall establish bounds for $\frac{1}{\sigma^2} E \frac{e'e}{n-k} = \frac{1}{n-k}\mathrm{tr}MV$, by applying a well-known theorem by Anderson [2].
We shall then show that the same bounds can be derived by a direct application of the Courant-Fischer min-max theorem [3], as is being demonstrated in an appendix.
Some remarks about possible approximations to these bounds, where reference is made to Anderson [2] and Grenander and Szegö [4] conclude the paper.

<u>Bounds for trMV (1)</u>

It is obvious that V is positive definite. We shall denote its eigenvalues by λ_i where $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$.
MV (or MVM) will have n-k non zero eigenvalues, μ_i say, where $\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_{n-k}$.

By theorem 10.4.3. [1)] in Anderson [2, p. 611] we then have

$$\lambda_{i+k} \leqslant \mu_i \leqslant \lambda_i \qquad i=1\dots n-k$$

Hence $$\sum_{i=1}^{n-k} \lambda_{i+k} \leqslant \mathrm{tr}MV \leqslant \sum_{i=1}^{n-k} \lambda_i \qquad (1)$$

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1) Durbin and Watson [4] relied on the same result for their bounds test.

From [1] we derive an interval for the expected value of the LS estimator over σ^2:

$$\frac{\sum_{i=1}^{n-k} \lambda_{i+k}}{n-k} \leq \frac{1}{\sigma^2} E\frac{e'e}{n-k} \leq \frac{\sum_{i=1}^{n-k} \lambda_i}{n-k} \tag{2}$$

Explicit results for these λ's are not available. According to Grenander and Szegö [4, p. 69 et. seq.] evaluation in explicit terms does not seem to be possible.

The λ's can, however, be expressed in the form

$$\lambda_i^{(n)} = \frac{1-\rho^2}{1-2\rho\cos\theta_{n+1-i}^{(n)}+\rho^2} ,$$

where the $\cos\theta$'s are the roots of a certain n^{th} degree polynomial in $\cos\theta$, namely

$$\frac{\sin\,(n+1)\theta}{\sin\theta} - 2\rho\,\frac{\sin\,n\theta}{\sin\theta} + \rho^2\,\frac{\sin\,(n-1)\theta}{\sin\theta} = 0 .$$

Grenander and Szegö also show that the θ's are bounded by

$$\frac{i-1}{n+1}\,\pi < \theta_i^{(n)} < \frac{i}{n+1}\,\pi \qquad (i=1...n) \tag{3}$$

For sufficiently large n these bounds are quite tight.

If, as in Anderson [2, formula (39)] one approximates V by $\tilde{V} = (V^{-1} - \frac{\rho}{1+\rho}\,C)^{-1}$, where $C = [c_{ij}]$ is such that $c_{11} = c_{nn} = 1$ and $c_{ij}=0$ otherwise, then $\tilde{V}$ has eigenvalues which can be expressed explicitly as

$$\tilde{\lambda}_i^{(n)} = \frac{1-\rho^2}{1-2\rho\cos\frac{(i-1)\pi}{n}+\rho^2} . \tag{4}$$

This follows from a theorem by Von Neumann [6].
Clearly the angles fall within the same bounds (3):

$$\frac{i-1}{n+1}\,\pi \leq \frac{i-1}{n}\,\pi < \frac{i}{n+1}\,\pi \qquad (i=1...n) \tag{5}$$

We have computed bounds for $\frac{1}{\sigma^2}\,E\,\frac{e'e}{n-k}$ according to formula (2) for various values of n,k and ρ 2). The results are shown in Table 1. (The results for k=2 and k=4 have been represented in the form of two diagrams).

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2) Thanks are due to Messrs J. Broekhuis en J. Kiviet for performing the computations.

The intervals are asymmetrical around 1. They clearly suggest a negative bias. For higher values of ρ the interval grows rapidly given n. Here the practical use of the method becomes limited.

We have used formula (4) to approximate the eigenvalues. The results are shown in Table 2. If can be concluded that this approximation can be used to approximate the limits of the inequality (2), for small values of ρ. Computationally, however, there is no gain from using Anderson's approximation (4); an equal computational effort is required.

Bounds for trMV (2)

Theorem 10.4.3. as stated by Anderson [2, p. 611] is rather difficult to prove.

We can, however, derive result (1) in a much simpler way. We rewrite

$$\begin{aligned} \mathrm{tr}MV &= \mathrm{tr}V - \mathrm{tr}X(X'X)^{-1}X'V \\ &= n - \mathrm{tr}(X'X)^{-\frac{1}{2}}X'VX(X'X)^{-\frac{1}{2}} \end{aligned}$$

and consider the problem of maximizing (minimizing) $\mathrm{tr}Y'VY$ subject to $Y'Y=I$, where Y is (n,k).

It is well-known that

$$\max_{Y'Y=I} \mathrm{tr}Y'VY = \sum_{i=1}^{k} \lambda_i$$

and $$\min_{Y'Y=I} \mathrm{tr}Y'VY = \sum_{i=n-k+1}^{n} \lambda_i \quad .$$

We refer to Courant and Hilbert [3, ch. I, §4]. (See also the appendix to the present paper)

$X(X'X)^{-\frac{1}{2}}$, obviously, satisfies the constraint: $\{X(X'X)^{-\frac{1}{2}}\}'X(X'X)^{-\frac{1}{2}} = I$,

hence $$\sum_{i=n-k+1}^{n} \lambda_i \leq \mathrm{tr}X(X'X)^{-1}X'V \leq \sum_{i=1}^{k} \lambda_i$$

and $$\sum_{i=1}^{n-k} \lambda_{i+k} \leq \mathrm{tr}MV \leq \sum_{i=1}^{n-k} \lambda_i \qquad (1)$$

because $\sum_{i=1}^{n} \lambda_i = n$.

Appendix

Maximization (minimization) of tr $Y'VY$ with respect to Y.

Define the Lagrangean function

$\phi = \frac{1}{2}\text{tr } Y'VY - \text{tr}M(Y'Y-I)$, where M is a Lagrange multiplier matrix .

Necessary for an extremum is

$$0 = d\phi = \text{tr}Y'VdY - \text{tr}\tilde{M}Y'dY - \text{tr}(Y'Y-I)\, dM,$$

where $\tilde{M} = M + M'$.

$$\text{Hence} \quad Y'V = \tilde{M}Y' \qquad (6)$$

$$Y'Y = I \qquad (7)$$

$\tilde{M} = Y'VY$ is positive definite of order k. It can therefore be diagonalized into $T'\tilde{M}T = \Lambda$, with T orthogonal. Clearly T and Λ are functions of Y.

Define: $Z= YT$.

We can then rewrite (6) and (7) as:

$$Z'V = \Lambda Z' \qquad (6^*)$$

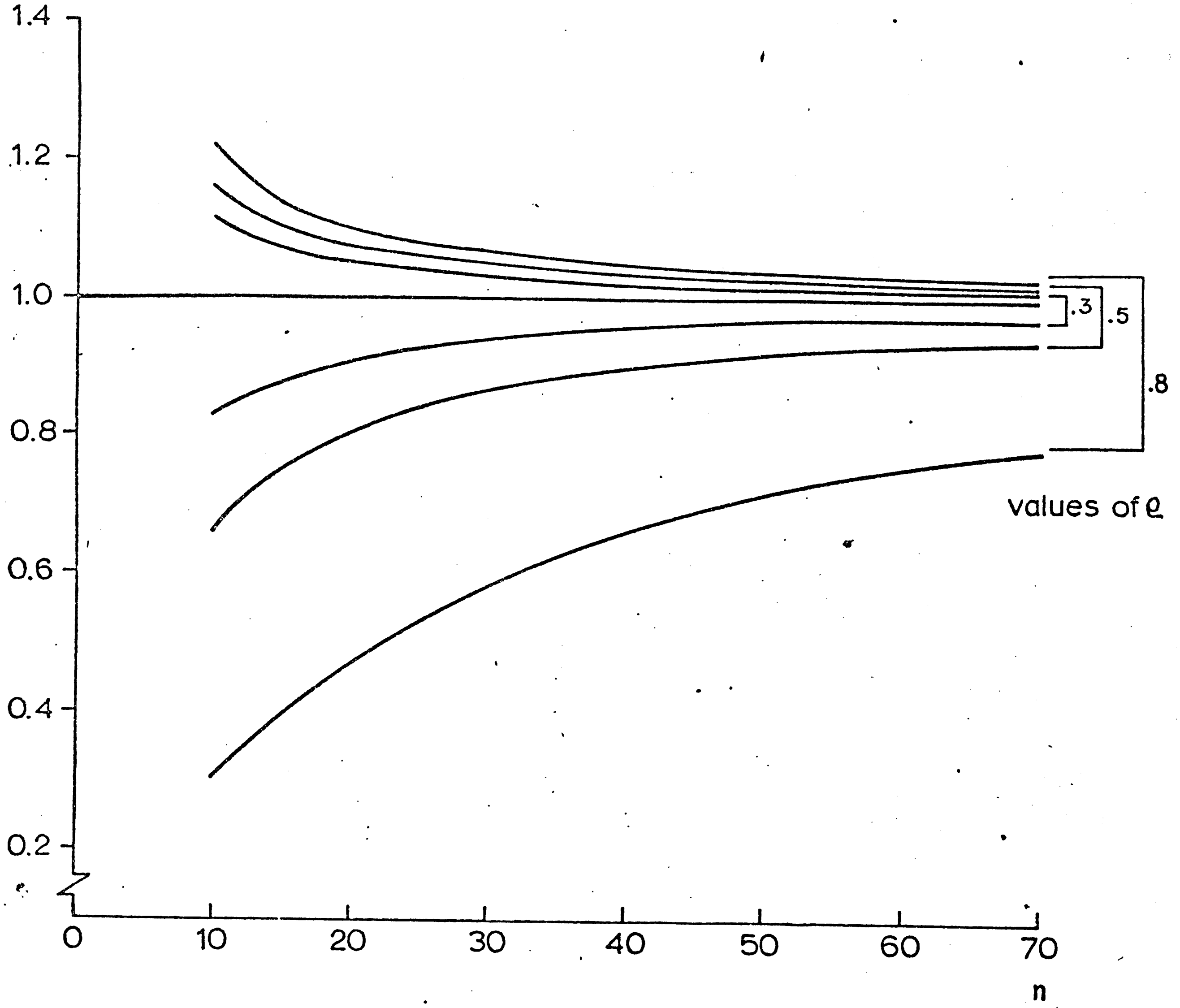
$$Z'Z = I \qquad (7^*)$$

Z is seen to be an (n,k) matrix of k orthonormal eigenvectors of V, Λ is the (k,k) matrix of associated eigenvalues. $\text{tr}Y'VY = \text{tr } \Lambda$.

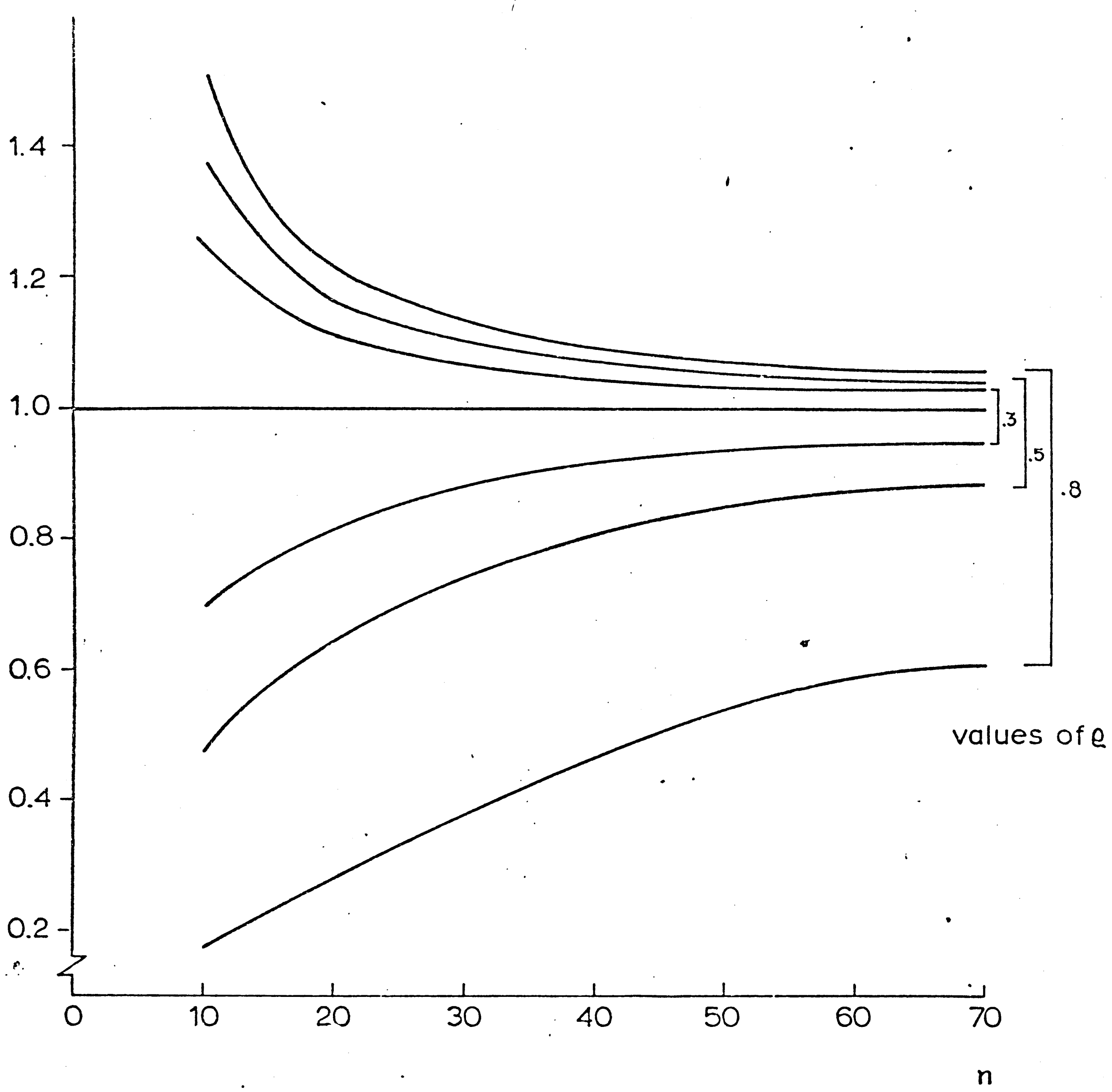
Hence $\max\limits_{Y'Y=I} \text{tr } Y'VY = \lambda_1 + \ldots + \lambda_k$,

and $\min\limits_{Y'Y=I} \text{tr } Y'VY = \lambda_{n-k+1} + \ldots + \lambda_n$, where

$\lambda_1 , \ldots , \lambda_n$ are the eigenvalues of V in descending order of magnitude.



Bounds for $E\,\frac{e'e}{n-k}/\sigma^2$ for $k=2$ and $\varrho = .3, .5, .8$.



Bounds for $E\frac{e'e}{n-k}/\sigma^2$ for $k=4$ and $\varrho = .3, .5, .8$.

Table 1

Bounds of $\frac{1}{\sigma^2} E \frac{e'e}{n-k}$ for various n, k and ρ

		ρ = 0.3		ρ = 0.5		ρ = 0.8	
	k	lower bound	upper bound	lower bound	upper bound	lower bound	upper bound
n=10	2	0.82911	1.10993	0.66266	1.16220	0.30340	1.22047
	3	0.75511	1.17986	0.55458	1.27074	0.21957	1.37501
	4	0.69405	1.26114	0.48034	1.40449	0.17651	1.57638
	5	0.64562	1.35438	0.42905	1.57095	0.15134	1.84866
n=15	2	0.88224	1.06948	0.75133	1.10134	0.39470	1.13628
	3	0.82557	1.11071	0.65387	1.16289	0.28995	1.22077
	4	0.77381	1.15670	0.57727	1.23334	0.23140	1.31971
	5	0.72805	1.20782	0.51813	1.31433	0.19525	1.43699
n=20	2	0.91100	1.05066	0.80554	1.07358	0.46932	1.09858
	3	0.86641	1.07958	0.72117	1.11616	0.35298	1.15630
	4	0.82374	1.11108	0.64945	1.16322	0.28310	1.22091
	5	0.78395	1.14533	0.58998	1.21534	0.23810	1.29366
n=25	2	0.92875	1.03982	0.84126	1.05772	0.53035	1.07720
	3	0.89243	1.06201	0.76842	1.09017	0.40877	1.12093
	4	0.85682	1.08581	0.70337	1.12532	0.33101	1.16868
	5	0.82262	1.11131	0.64672	1.16341	1.27896	1.22099
n=30	2	0.94071	1.03279	0.86630	1.04748	0.58058	1.06344
	3	0.91025	1.05075	0.80289	1.07366	0.45789	1.09861
	4	0.87999	1.06983	0.74438	1.10163	0.37502	1.13640
	5	0.85040	1.09004	0.69165	1.13154	0.31752	1.17710
n=50	2	0.96472	1.01919	0.91881	1.02775	0.71205	1.03703
	3	0.94651	1.02935	0.87833	1.04247	0.60209	1.05671
	4	0.92811	1.03989	0.83879	1.05778	0.51557	1.07722
	5	0.90965	1.05083	0.80071	1.07372	0.44821	1.09863
n=70	2	0.97495	1.01356	0.94198	1.01960	0.78375	1.02614
	3	0.96207	1.02063	0.91271	1.02982	0.69130	1.03979
	4	0.94902	1.02789	0.88363	1.04034	0.61210	1.05385
	5	0.93586	1.03534	0.85497	1.05116	0.54556	1.06833

Table 2

Bounds for $\frac{1}{\sigma^2} E\frac{e'e}{n-k}$ for various n, k and ρ

(Established by using Anderson's approximation method)

n	k	$\rho = 0.3$ lower bound	$\rho = 0.3$ upper bound	$\rho = 0.5$ lower bound	$\rho = 0.5$ upper bound	$\rho = 0.8$ lower bound	$\rho = 0.8$ upper bound
n=10	2	0.81537	1.09791	0.63711	1.14067	0.26819	1.49764
	3	0.72690	1.17156	0.50970	1.25105	0.18546	1.69385
	4	0.67849	1.25109	0.45939	1.38228	0.16273	1.94923
	5	0.62172	1.35688	0.40265	1.56092	0.13790	2.30469
n=15	2	0.86821	1.05709	0.72235	1.07402	0.34389	1.17761
	3	0.80610	1.09759	0.61511	1.13361	0.24611	1.26569
	4	0.75830	1.14443	0.55186	1.20320	0.20956	1.36947
	5	0.70845	1.19480	0.48986	1.28186	0.17630	1.49213
n=20	2	0.89736	1.03782	0.77563	1.04498	0.40569	1.06545
	3	0.85047	1.06648	0.68691	1.08629	0.30154	1.12139
	4	0.80987	1.09744	0.62228	1.13174	0.25367	1.18393
	5	0.76737	1.13178	0.56184	1.18265	0.21409	1.25458
n=25	2	0.91543	1.02704	0.81130	1.02888	0.45669	1.01443
	3	0.87805	1.04878	0.73682	1.06013	0.35139	1.05536
	4	0.84278	1.07228	0.67546	1.09411	0.29438	1.10009
	5	0.80775	1.09734	0.61908	1.13087	0.25019	1.14906
n=30	2	0.92761	1.02002	0.83651	1.01852	0.49935	0.98659
	3	0.89664	1.03763	0.77278	1.04372	0.39583	1.01895
	4	0.86637	1.05633	0.71615	1.07065	0.33171	1.05372
	5	0.83648	1.07623	0.66443	1.09950	0.28429	1.09119
n=50	2	0.95201	1.00658	0.88858	0.99882	0.61571	0.94208
	3	0.93382	1.01649	0.84995	1.01293	0.52813	0.95974
	4	0.91520	1.02679	0.81031	1.02764	0.45154	0.97817
	5	0.89692	1.03746	0.77388	1.04292	0.39994	0.99738
n=70	2	0.96239	1.00104	0.91300	0.99075	0.68252	0.92605
	3	0.94956	1.00792	0.88449	1.00054	0.61003	0.93820
	4	0.93634	1.01501	0.85512	1.01063	0.53549	0.95073
	5	0.92334	1.02227	0.82779	1.02010	0.48607	0.96362

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