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ON THE DISPERSION MATRIX OF A MATRIX QUADRATIC FORM CONNECTED WITH THE NONCENTRAL WISHART DISTRIBUTION

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Abstract

Recently Magnus and Neudecker [3] derived the dispersion matrix of vec X'X, when X' is a pxn random matrix (n>p) and vec X' has the distribution $N_{np}(\text{vec } M', I_n \otimes V)$.
This note is concerned with the matrix quadratic form X'AX, where X' is as defined above and A is a nonrandom (not necessarily symmetric) matrix. The dispersion matrix of vec X'AX will then be derived by applying results of Magnus and Neudecker [3] and Neudecker and Wansbeek [4].
It will be shown that an earlier partial and special result of Giguère and Styan [2] which assumes a symmetric A agrees with our result.

1. INTRODUCTION

Let x_i for $i=1,\dots,n$ be px1 random vectors that are jointly independent with (normal) distribution $\mathcal{N}_p(\mu_i,V)$.
If we define $X' := (x_1,\dots,x_n)$ and $M' := (\mu_1,\dots,\mu_n)$, then $S_I := X'X$ will have a (noncentral Wishart) distribution $W_p(n,V,M'M)$, provided $n>p$.
Magnus and Neudecker [3] derived the dispersion matrix of vec S_I, viz.

$$\mathcal{D}(\text{vec } S_I) = (I_{p^2} + K_{pp})[n(V\otimes V) + V\otimes M'M + M'M\otimes V] \qquad (1.1)$$

where K_{pp} is a $p^2 \times p^2$ commutation matrix.
Although V was taken to be positive definite in their derivation, it is easy to prove that the result generally holds for nonnegative definite V.
In this paper we shall consider the matrix quadratic form $S_A := X'AX$, where X'X is distributed as stated above and A is a nonrandom (not necessarily symmetric matrix) and derive the dispersion matrix of vec S_A.
After establishing this result in an earlier version [5] of this paper, we became aware of a paper [2] by Giguère and Styan, who presented [see their (2.2.12)] an expression for the cross-covariance matrix T_{ij}, say, between the i-th and j-th columns of S_A $(i,j=1,\dots,p)$ for symmetric A.
One can relatively easily derive the complete dispersion matrix $T=\{T_{ij}\}$ if one uses the commutation matrix and properties of Kronecker multiplication.
Whereas the procedure followed by Giguère and Styan is unclear and their result is a partial and special one, we shall give a full derivation of the complete dispersion matrix.
We shall apply some earlier results concerning Kronecker multiplication and the commutation matrix, viz.

(1) $\text{vec } ABC = (C' \otimes A)\text{vec } B$, for compatible matrices A, B and C (1.1)

(2) $K_{mn} \text{ vec } A = \text{vec } A'$, where A is an mxn matrix (1.2)

(3) $K_{pm}(A \otimes B)K_{nq} = B \otimes A$, where A and B are mxn and pxq matrices (1.3)

(4) $(\text{vec } A)' \text{ vec } B = \text{tr } A'B$. (1.4)

These results are collected in Magnus and Neudecker [3].

(5) $\text{vec}(A \otimes B) = (I_n \otimes K_{qm} \otimes I_p)(\text{vec } A \otimes \text{vec } B)$, where A and B are arbitrary mxn and pxq matrices. (1.5)

This is Theorem 3.1(i) of Neudecker and Wansbeek [4].

(6) $$\mathcal{D}\big(\mathrm{vec}(X \otimes X)\big) = (I_{m^2n^2} + K_{nn} \otimes K_{mm})(I_n \otimes K_{nm} \otimes I_m)\times$$
$$[V \otimes V + V \otimes \mathrm{vec}\, M(\mathrm{vec}\, M)' + \mathrm{vec}\, M(\mathrm{vec}\, M)'\otimes V](I_n \otimes K_{mn} \otimes I_m)\,, \qquad (1.6)$$
when X is an $m\times n$ matrix and $\mathrm{vec}\, X$ has distribution $\mathcal{N}_{mn}(\mathrm{vec}\, M, V)$.

This is application 3 of Neudecker and Wansbeek [4].

The result will be reached in stages. First an intermediate result will be derived.

2. AN INTERMEDIATE RESULT

LEMMA.

Let the $p\times 1$ random vectors x_i be independently distributed each as $\mathcal{N}_p(0,V)$ for $i=1,\dots,n$. Let

$$X := (x_1,\dots,x_n)' \quad \text{and} \quad M := (\mu_1\dots,\mu_n)'.$$

Consider the quadratic form $S_A := X'AX$, where the $n\times n$ matrix A is a random (not necessarily symmetric) matrix. Then the dispersion matrix

$$\mathcal{D}(\mathrm{vec}\, S_A) = (\mathrm{tr}\, A'A)(I_{p^2} + K_{pp})(V \otimes V). \qquad (2.1)$$

Proof. We write

$$\mathrm{vec}\, S_A = \mathrm{vec}\, X'AX = (X' \otimes X')\mathrm{vec}\, A \qquad (2.2)$$

$$= \mathrm{vec}[I_{p^2}(X' \otimes X')\mathrm{vec}\, A] = (\mathrm{vec}\, A \otimes I_{p^2})'\, \mathrm{vec}(X' \otimes X')\,, \qquad (2.3)$$

by means of (1.1).

Using (1.6), (1.2), (1.3) and (1.4), we get

$$\mathcal{D}(\mathrm{vec}\, S_A) = (\mathrm{vec}\, A \otimes I_{p^2})'(I_{n^2p^2} + K_{nn} \otimes K_{pp})(I_n \otimes K_{np} \otimes I_p)\ \times$$
$$(I_n \otimes V \otimes I_n \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\mathrm{vec}\, A \otimes I_{p^2}) \qquad (2.4)$$
$$= [\mathrm{vec}\, A \otimes (I_{p^2} + K_{pp})]'(I_{n^2} \otimes V \otimes V)(\mathrm{vec}\, A \otimes I_{p^2}) \qquad (2.5)$$
$$= (\mathrm{vec}\, A)'\, \mathrm{vec}\, A\ .\ (I_{p^2} + K_{pp})(V \otimes V) \qquad (2.6)$$
$$= (\mathrm{tr}\, A'A)(I_{p^2}+K_{pp})(V \otimes V). \quad \blacksquare \qquad (2.7)$$

3. THE MAIN RESULT

THEOREM

Let the px1 random vectors x_i be independently distributed each as $N_p(\mu_i, V)$ for i=1,...,n. Let

$$X := (x_1, \ldots, x_n)' \quad \text{and} \quad M := (\mu_1, \ldots, \mu_n)' .$$

Consider the matrix quadratic form $S_A := X'AX$, where the nxn matrix A is a random (not necessarily symmetric) matrix. Then the dispersion matrix

$$\mathcal{D}(\text{vec } S_A) = (I_{p^2} + K_{pp})[(\text{tr } A'A)(V \otimes V) + M'A'AM \otimes V + V \otimes M'A'AM]. \tag{3.1}$$

Proof. We write $Y := X - M$. Hence

$$S_A = (Y + M)'A(Y + M) \tag{3.2}$$

$$= Y'AY + Y'AM + M'AY + M'AM . \tag{3.3}$$

As vec Y' has distribution $N_{np}(0, I_n \otimes V)$, any third moment about the mean is zero. See Anderson [1,p.39] . Therefore

$$\mathcal{D}(\text{vec } S_A) = \mathcal{D}(\text{vec } Y'AY) + \mathcal{D}\{(I_{p^2} + K_{pp})(M'A' \otimes I_p) \text{ vec } Y'\} \tag{3.4}$$

$$= (\text{tr } A'A)(I_{p^2} + K_{pp})(V \otimes V)$$

$$+ (I_{p^2} + K_{pp})(M'A' \otimes I_p)(I \otimes V)(AM \otimes I_p)(I_{p^2} + K_{pp}) \tag{3.5}$$

$$= (I_{p^2} + K_{pp})[\tfrac{1}{2}(\text{tr } A'A)(V \otimes V) + M'A'AM \otimes V)](I_{p^2} + K_{pp}) \tag{3.6}$$

$$= (I_{p^2} + K_{pp})[(\text{tr } A'A)(V \otimes V) + M'A'AM \otimes V + V \otimes M'A'AM], \tag{3.7}$$

by virtue of the Lemma, (1.2) and (1.3).

4. THE RESULT OF GIGUÈRE AND STYAN

Giguère and Styan presented the cross-covariance matrix T_{ij} between the i-th and j-th columns of S_A for <u>symmetric</u> A, viz.

$$T_{ij} = (\text{tr } A^2)(v_j v_i' + v_{ij} V) + h_j v_i' + v_j h_i' + v_{ij} H + h_{ij} V , \tag{4.1}$$

where

$$H = \{h_{ij}\} = (h_1, \ldots, h_p) := M'A^2M$$

and

$$V = \{v_{ij}\} = (v_1, \ldots, v_p) .$$

Clearly the (i,j)th submatrix of $V \otimes H$ is $v_{ij}H$.
Further

$$K_{pp}(V \otimes H) = K_{pp}(v_1 \otimes H, \ldots, v_p \otimes H) = [K_{pp}(v_1 \otimes H) \ldots K_{pp}(v_p \otimes H)]$$

$$= (H \otimes v_1 \ldots H \otimes v_p) .$$

Its (i,j)th submatrix is $h_i' \otimes v_j = v_j \otimes h_i' = v_j h_i'$, etcetera.
From this follows

$$T = \{T_{ij}\} = (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + H \otimes V + V \otimes H]. \qquad (4.2)$$

5. COMMENT I

When A is symmetric idempotent of sufficient rank, r say, then S_A will have a (noncentral) Wishart distribution.
In this case we get

$$\mathcal{D}(\text{vec } S_A) = (I_{p^2} + K_{pp})[r(V \otimes V) + M'AM \otimes V + V \otimes M'AM] . \qquad (5.1)$$

The result of Magnus and Neudecker [3] is a special case of (5.1) for $A = I_n$. Another special case arises for $A = N$, $N := I_n - \frac{1}{n} s_n s_n'$, where $s_n' := (1,\ldots,1)$ is the n-dimensional summation vector. We then find

$$\mathcal{D}(\text{vec } S_N) = (I_{p^2} + K_{pp})[(n-1)(V \otimes V) + M'NM \otimes V + V \otimes M'NM]. \qquad (5.2)$$

6. COMMENT II

The dispersion matrix for the symmetric case can, of course, be found very easily.
Let $R'AR = \Lambda$ be the diagonal representation of A. Let further $Y' := X'R$, and $Y' = (y_1,\ldots,y_n)$. Then $\mathcal{E}(Y') = M'R$ and $\mathcal{D}(\text{vec } Y') = I_n \otimes V$. Hence the vectors y_i are jointly independent with (normal) distribution $\mathcal{N}_p(M'R_{\cdot i}, V)$, where $R_{\cdot i}$ is the i-th column of R. Then $X'AX = X'R\Lambda R'X = Y'\Lambda Y = \sum_i \lambda_i y_i y_i'$.

Further $\mathcal{D}(\text{vec } X'AX) = \mathcal{D}(\sum_i \lambda_i y_i \otimes y_i)$

$$= \sum_i \lambda_i^2 \mathcal{D}(y_i \otimes y_i) = \sum_i \lambda_i^2 (I_{p^2} + K_{pp})(V \otimes V + V \otimes M'R_{\cdot i}(R_{\cdot i})'M$$

$$+ M'R_{\cdot i}(R_{\cdot i})'M \otimes V) = (I_{p^2} + K_{pp})[(\text{tr } A^2)(V \otimes V) + V \otimes M'A^2M$$

$$+ M'A^2M \otimes V],$$ by virtue of Theorem 4.3(iv) of Magnus and Neudecker [3].

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