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David L. Debertin, Angelos Pagoulatos, and
Garnett L. Bradford

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Biographical Sketch of Senior Author

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Computer Graphics: An Educational Tool for Understanding Agricultural Production Functions*

David L. Debertin, Angelos Pagoulatos, and Garnett L. Bradford

Undergraduate and graduate students often have difficulty in developing an understanding of the nature of surfaces of multi-input agricultural production functions. Wooden or plastic models sometimes used to illustrate production surfaces are difficult to build or expensive to purchase. These models usually are supposed to represent a two-input production function called the "textbook" case. However, the mathematical specification of the production function underlying the plastic or wooden model is often ambiguous. But the most serious disadvantage of these models is that even if a mathematical specification of the production function underlying the model is known, it is impossible with a single model to change the parameters of the underlying function and observe the resultant impacts on the surface of the function.

In this article we propose the use of computer graphics as a tool for teaching students about the nature of agricultural production function surfaces. A plotter linked to a computer is used to generate three dimensional illustrations of two-input agricultural production functions. Production economics problems often require the maximization or minimization of a function. Computer graphics is also used as a tool for developing in students an understanding of the mathematical conditions necessary and sufficient for a maximum or minimum. Computer graphics is useful as a tool for developing in students an understanding of production economics because:

(1) With computer graphics it is possible to generate three dimensional illustrations of surfaces for a host of production functions, not just the usual textbook case. Parameters of the production function can be varied and the resultant impact on the surface be observed. Students can compare surfaces among production functions widely used in agricultural economics research such as the Cobb-Douglas, Transcendental, Spillman, CES, polynomial forms and others. Shapes of agricultural production function surfaces can be discerned at least as easily from a computer generated illustration as from a wooden or plastic model, and yet copies of the illustrations can be carried by students in a notebook!

(2) Computer graphics, when used as a tool to supplement instruction dealing with optimization problems, provides vivid illustrations of the meaning of mathematical conditions necessary and sufficient for a maximum or minimum. Students can easily grasp the importance of sufficient conditions when a mathematical presentation is supplemented with computer-generated illustrations.

(3) Computer graphics is quite inexpensive and simple to use. The only equipment required is a computer linked to a CALCOMP (or other) plotter, which is available at nearly all university computing centers. The computer program developed at Kentucky for using the CALCOMP plotter is simple enough to operate so that beginning graduate students may run their own computer graphics programs.[1] The cost of each illustration is approximately $1.50 on the University of Kentucky IBM 370-165 system.

Alternative Agricultural Production Functions

The Cobb-Douglas. The most widely used function in production economics literature is the Cobb-Douglas (1928). A general Cobb-Douglas is represented by [2]

$$(1) \quad y = Ax_1^{\alpha}x_2^{\beta}$$

The surface generated by (1) is illustrated in Figure 1 when $\alpha = .4$, $\beta = .6$, $A = 1$. Diminishing marginal returns to x_1 and x_2 and constant returns to scale are evident.

The Transcendental. Halter *et al.* popularized the "transcendental" production function. Appropriate selection of parameters yields a function which will represent all three stages of production.

The general form of the transcendental is

$$(2) \quad y = Ax_1^{\alpha_1} x_2^{\alpha_2} e^{(\gamma_1 x_1 + \gamma_2 x_2)}$$

Figure 2 illustrates the surface of the function when $\alpha_1 = \alpha_2 = 4$, $\gamma_1 = \gamma_2 = -2$, $A = 1$. No output (y) is produced if either x_1 or x_2 assume a value of zero, and the isoquants generated by the function are asymptotic to the axes. Ridge lines enclosing Stage II for both inputs are parallel to the axes[3] and intersect at the maximum of the function. An alternate view of equation 2 which more clearly reveals the three stages of production is obtained by rotating the figure in the x_1, x_2 plane (Figure 3).

Polynomials. A number of polynomial forms have been used in agricultural economics research. A general polynomial form is

$$(3) \quad y = \alpha_1 x_1 + \alpha_2 x_1^2 + \alpha_3 x_1^3 + \beta_1 x_2 + \beta_2 x_2^2 + \beta_3 x_2^2 + \gamma_1 x_1 x_2$$

Figure 4 illustrates equation (3) when

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1,$$
$$\alpha_3 = \beta_3 = -.05,$$
$$\gamma_1 = .40.$$

The resultant illustration is a close approximation of what has been referred to as the textbook case. In contrast with the transcendental (Figure 2) isoquants intersect the axes, and ridge lines are no longer parallel to the axes.

The Spillman. One of the earliest mathematical forms used to represent production relationships was due to Spillman (1923, 1924). A general form of the Spillman is

$$(4) \quad y = A(1-R_1^{x_1})(1-R_2^{x_2})$$

where

$$0 \leq R_i \leq 1 = \text{a constant}$$

The Spillman when $A = 1$, $R_1 = .4$, $R_2 = .6$ is illustrated in Figure 5. Although the function only exhibits diminishing marginal returns, the shape of the surface is somewhat different from that of the Cobb-Douglas (Figure 1).

The CES. Regardless of the values assumed for α and β, all Cobb-Douglas type production functions have elasticities of substitution equal to one. The CES (Arrow et al.) allows for constant elasticities of substitution other than one. The general form of the CES is

$$(5) \quad y = A[\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}]^{-\frac{1}{\rho}}$$

where the elasticity of substitution (σ) is related to the parameter ρ by the identity

$$(6) \quad \sigma = \frac{1}{1+\rho}$$

Henderson and Quandt outline five cases with alternative assumptions about the value of σ. Two of these cases are illustrated in Figures 6 and 7. As $\sigma \to 0$, $\rho \to \infty$ the isoquants become right angles and the production surface approaches a pyramid in shape. This is shown in Figure 6 by setting $\rho = 10$. If $\sigma > 1$ and $-1 < \rho < 0$, the isoquants intersect both axes. The production surface shown in Figure 7 was generated with $\rho = -.5$. Compare this figure with Figure 1.

Sufficient Conditions

Computer graphics can also be used to supplement the presentation of second order, or sufficient conditions.[4] For example, the function

(7) $y = -x_1^2 - x_2^2$

reaches a maximum at $x_1 = x_2 = 0$ since

(8) $|-2| < 0$

(9) $\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} > 0$

The above determinental conditions are sufficient for a maximum since principal minors of the relevant hessian alternate in sign starting with a minus. Figure 8 illustrates equation 7.

A *saddle point* is generated by the equation

(10) $y = x_1^2 - x_2^2$

The function reaches a minimum at $x_1 = 0$ along the x_1 axis and a maximum at $x_2 = 0$ along the x_2 axis since

(11) $| 2 | > 0$

(12) $\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} < 0$

The above determinental conditions violate sufficient conditions for a maximum since the first principal minor is positive. The saddle point is illustrated in Figure 9 Finally, computer graphics is used to illustrate differences between local and global optima. The function

$$(13) \quad y = 40x_1 - 12x_1^2 + 1.2x_1^3 - .035x_1^4$$
$$+40x_2 - 12x_2^2 + 1.2x_1^3 - .035x_2^4$$

generates nine extreme values (four maxima, one minimum and four saddle points) occurring over the plotted range (Figure 10). Relevant second conditions are summarized in Table 1.

Applications to Instruction

Computer graphics was first used in 1976 as an educational tool in an advanced production economics course. Before this, production function analysis had been taught using only two-dimensional graphics, algebra and calculus. A 4-week period was devoted to examining algebraic forms, including those illustrated in this article, as alternative models of specific production processes. Two-input functions were compared with respect to characteristics such as cross-sectional slope, homogeneity, isoquants, ridgelines, and elasticities of substitution. Prior to the use of computer graphics, students had been able to only partially grasp the implications of algebraic characteristics of the production functions. But with the three dimensional illustrations generated via computer graphics, it is now possible for students to understand the impacts of each of these production function characteristics on the clear to students from the mathematics of the transcendental why that function is an imperfect representation of the textbook case. Examination of Figure 3 by students reveals that ridgelines for the transcendental must always be parallel to the axes. By comparing Figure 7 with Figure 1, students have been able to quickly grasp an understanding of what happens to production surfaces when isoquants intersect rather than become asymptotic to the axes, and what happens to a production surface when the elasticity of substitution changes. The shape of the production surface that results when isoquants become right angles is obvious to students only when they examine Figure 6. The abstract rules with regard to signs on principal minors of hessians sufficient for maxima or minima acquire meaning only when used in conjunction with Figures 8-10.

Table 1. Second Order Conditions for Figure 10.

x_2			
16.2434	*local maximum* $y = 232.3$ $\lvert H_1 \rvert < 0$[1] $\lvert H_2 \rvert > 0$	*saddle point* $y = 209.5$ $\lvert H_1 \rvert > 0$ $\lvert H_2 \rvert < 0$	*global maximum* $y = 379.8$ $\lvert H_1 \rvert < 0$ $\lvert H_2 \rvert > 0$
6.9342	*saddle point* $y = 61.9$ $\lvert H_1 \rvert < 0$ $\lvert H_2 \rvert < 0$	*local minimum* $y = 39.1$ $\lvert H_1 \rvert > 0$ $\lvert H_2 \rvert > 0$	*saddle point* $y = 209.5$ $\lvert H_1 \rvert > 0$ $\lvert H_2 \rvert < 0$
2.5366	*local maximum* $y = 84.8$ $\lvert H_1 \rvert < 0$ $\lvert H_2 \rvert > 0$	*saddle point* $y = 61.9$ $\lvert H_1 \rvert > 0$ $\lvert H_2 \rvert < 0$	*local maximum* $y = 232.5$ $\lvert H_1 \rvert > 0$ $\lvert H_2 \rvert > 0$
	2.5366	6.9342	16.2434
		x_1	

[1]

$$|H_1| = \left| \frac{\partial^2 y}{\partial x_1^2} \right|$$

$$|H_2| = \begin{vmatrix} \dfrac{\partial^2 y}{\partial x_1^2} & \dfrac{\partial^2 y}{\partial x_2 \partial x_1} \\[2ex] \dfrac{\partial^2 y}{\partial x_1 \partial x_2} & \dfrac{\partial^2 y}{\partial x_2^2} \end{vmatrix}$$

References

Arrow, K., H.B. Chenery, B. Menhas and R.M. Solow. "Capital Labor Substitution and Economic Efficiency." Review of Economics and Statistics. 43 (1929):228-232.

Chaing, Alpha C. Fundamental Methods of Mathematical Economics. New York: McGraw-Hill, 1967.

Cobb, Charles W. and Paul H. Douglas. "A Theory of Production." American Economic Review. 18 (Supplement, 1928):139-156.

Debertin, David L., Angelos Pagoulatos and Garnett L. Bradford. Computer Graphics: A Technique for the Analysis of Agricultural Production Functions, Part I: The Surfaces of Agricultural Production Functions, University of Kentucky Research Report. Department of Agr. Econ. forthcoming, 1976a.

Debertin, David L., Angelos Pagoulatos and Garnett L. Bradford. Computer Graphics: A Technique for the Analysis of Agricultural Production Functions, Part II: An Analysis of Sufficient Conditions, University of Kentucky Research Report. Department of Agr. Econ. forthcoming, 1976b.

Halter, A.N., H.O. Carter and J.G. Hocking. "A Note on the Transcendental Production Function." J. Farm Econ. 39 (1957):966-974.

Henderson, James M. and Richard E. Quandt. Microeconomic Theory: A Mathematical Approach. 2nd ed., New York: McGraw-Hill, 1971, pp. 87-88.

Intriligator, Michael D. Mathematical Optimization and Economic Theory. Englewood Cliffs, N.J.: Prentice Hall, 1971.

Spillman, W.J. "Application of the Law of Diminishing Returns to Some Fertilizer and Feed Data." J. Farm Econ. 5 (1923):36-52.

Spillman, W.J. "Law of the Diminishing Increment in the Fattening of Steers and Hogs." J. Farm Econ. 6 (1924):166-178.

Footnotes

*Debertin is Associate Professor of Agricultural Economics; Pagoulatos is Assistant Professor of Agricultural Economics; Bradford is Professor of Agricultural Economics, all at the University of Kentucky. This paper is an overview of work reported in two forthcoming University of Kentucky Research Reports by the authors (1976a and 1976b). Herman C. Collins wrote the PL-1 plotter program.

[1]A copy of the PL-1 plotter program is available in the forthcoming research reports. Program users select the function to be plotted, function parameters, and values over which the function is to be plotted.

[2]Throughout this article, y is used to designate an output, x_1, x_2 are two inputs or factors of production, e refers to the base of the natural logarithm, the letter A and the Greek letters α, β, γ, and ρ designate production function parameters.

[3]The function

$$y = Ax_1^{\alpha_1} x_2^{\alpha_2} e^{(\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1 x_2)}$$

will generate ridge lines which are not parallel to the axes.

[4]See the discussion on sufficient conditions in Chaing and Intrilligator.

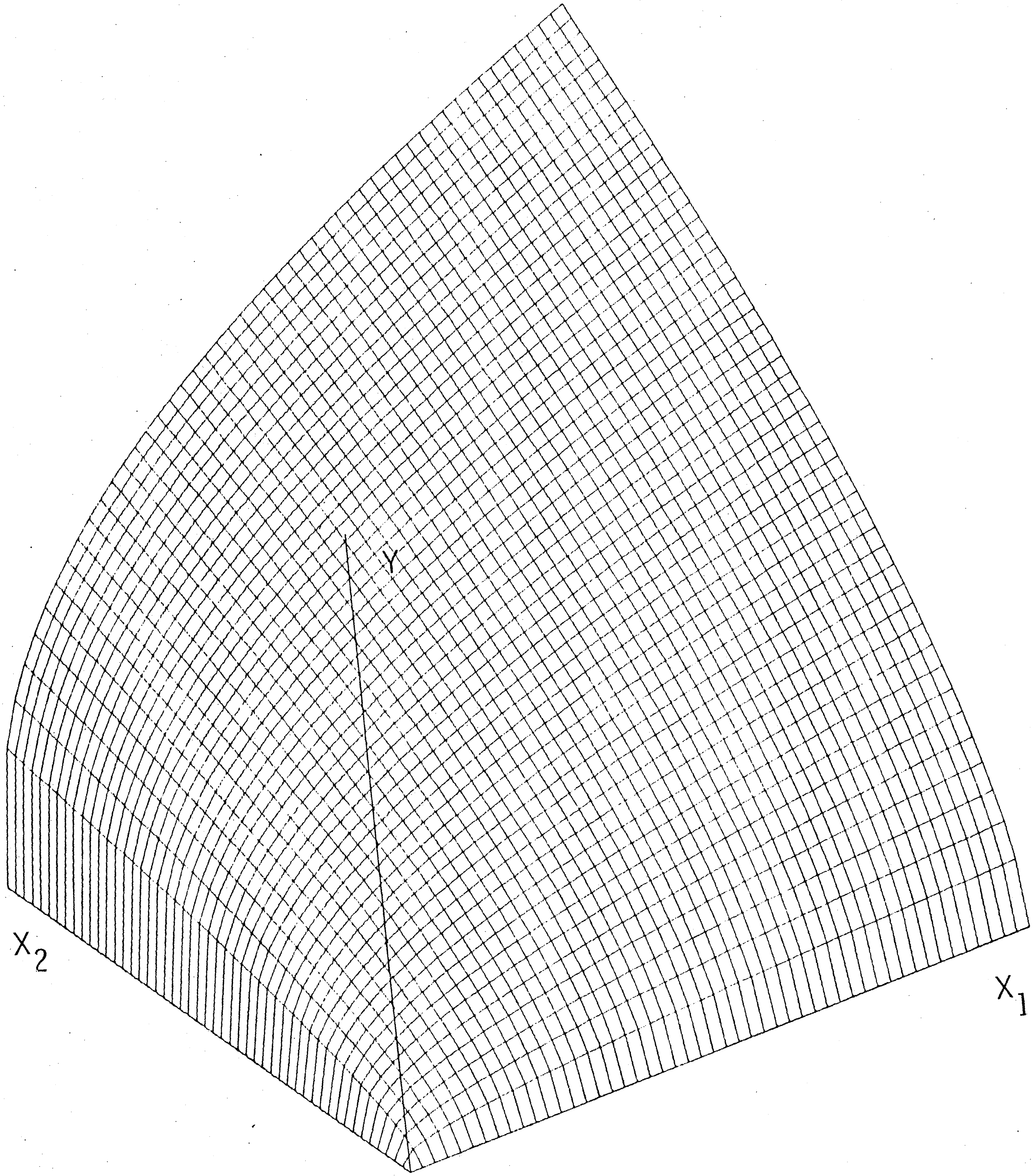

Figure 1. $Y = X_1^{.4} X_2^{.6}$

"The Cobb-Douglas"

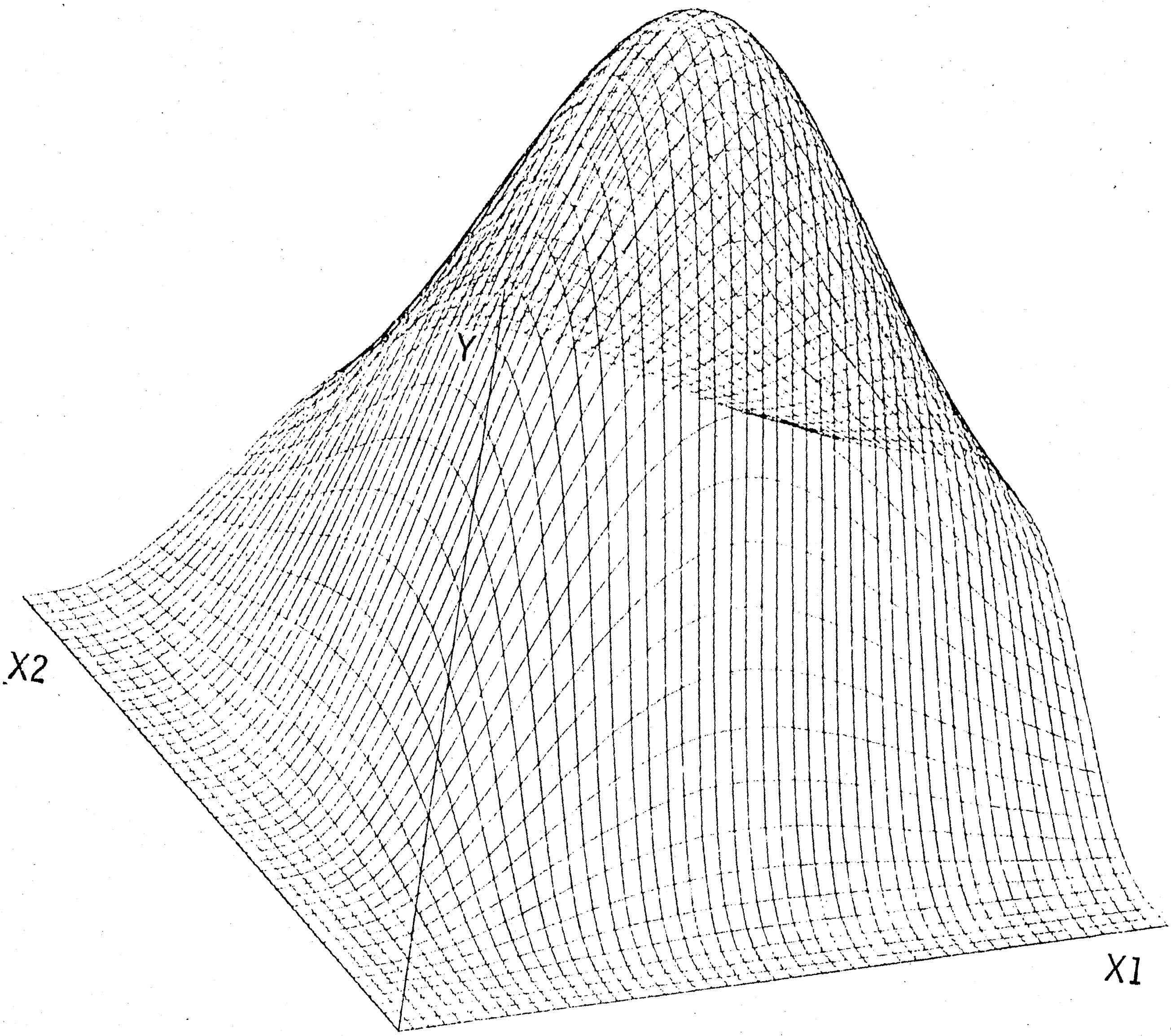

Figure 2. $Y = X_1^4 X_2^4 e^{(-2X_1 - 2X_2)}$

"The Transcendental"

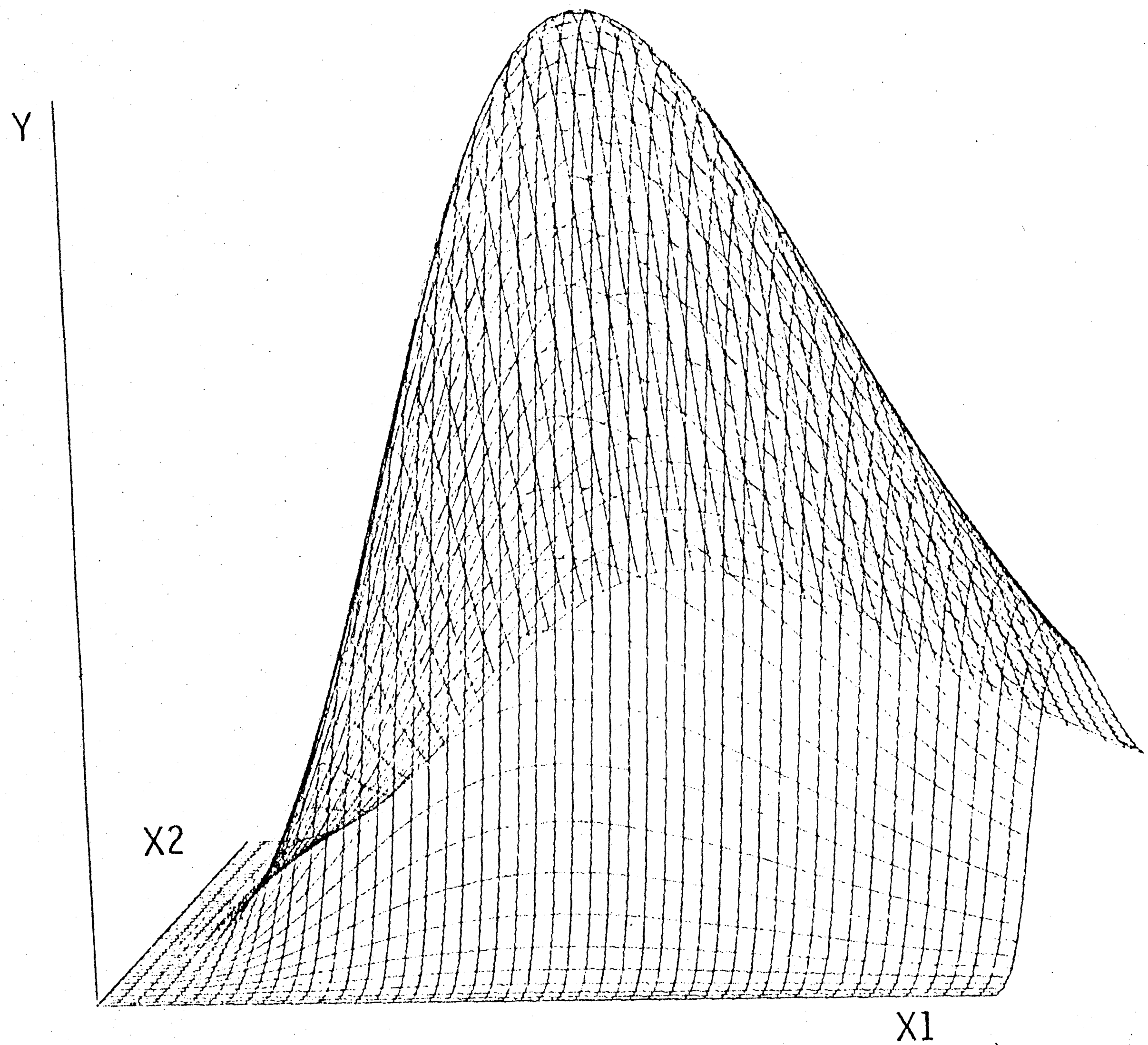

Figure 3. $Y = X_1^4 X_2^4 e^{(-2X_1 - 2X_2)}$

Alternate View of the Transcendental

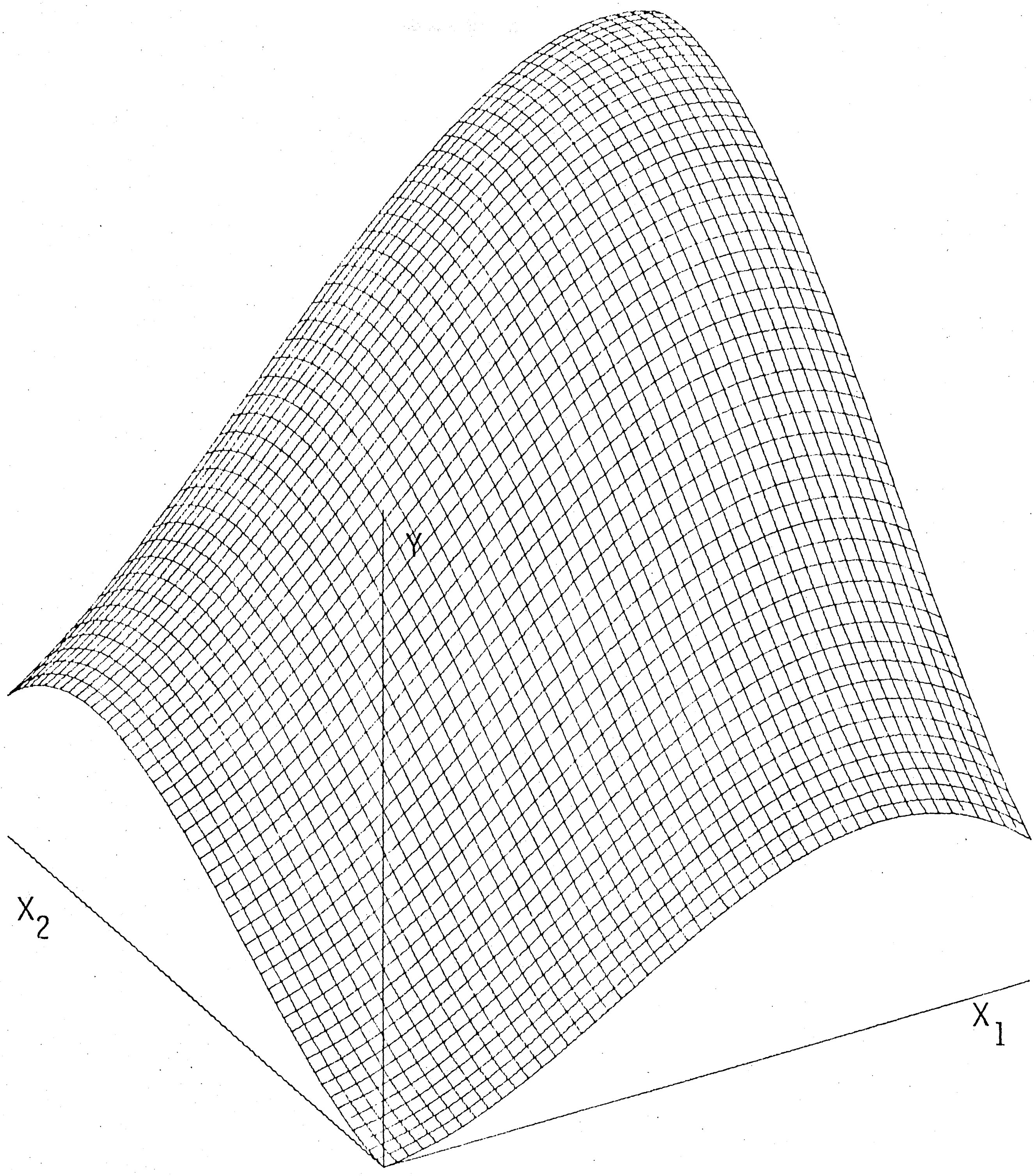

Figure 4. $Y = X_1 + X_1^2 - .05\,X_1^3$

$$+ X_2 + X_2^2 - .05\,X_2^3 + .40\,X_1 X_2$$

"A Polynomial"

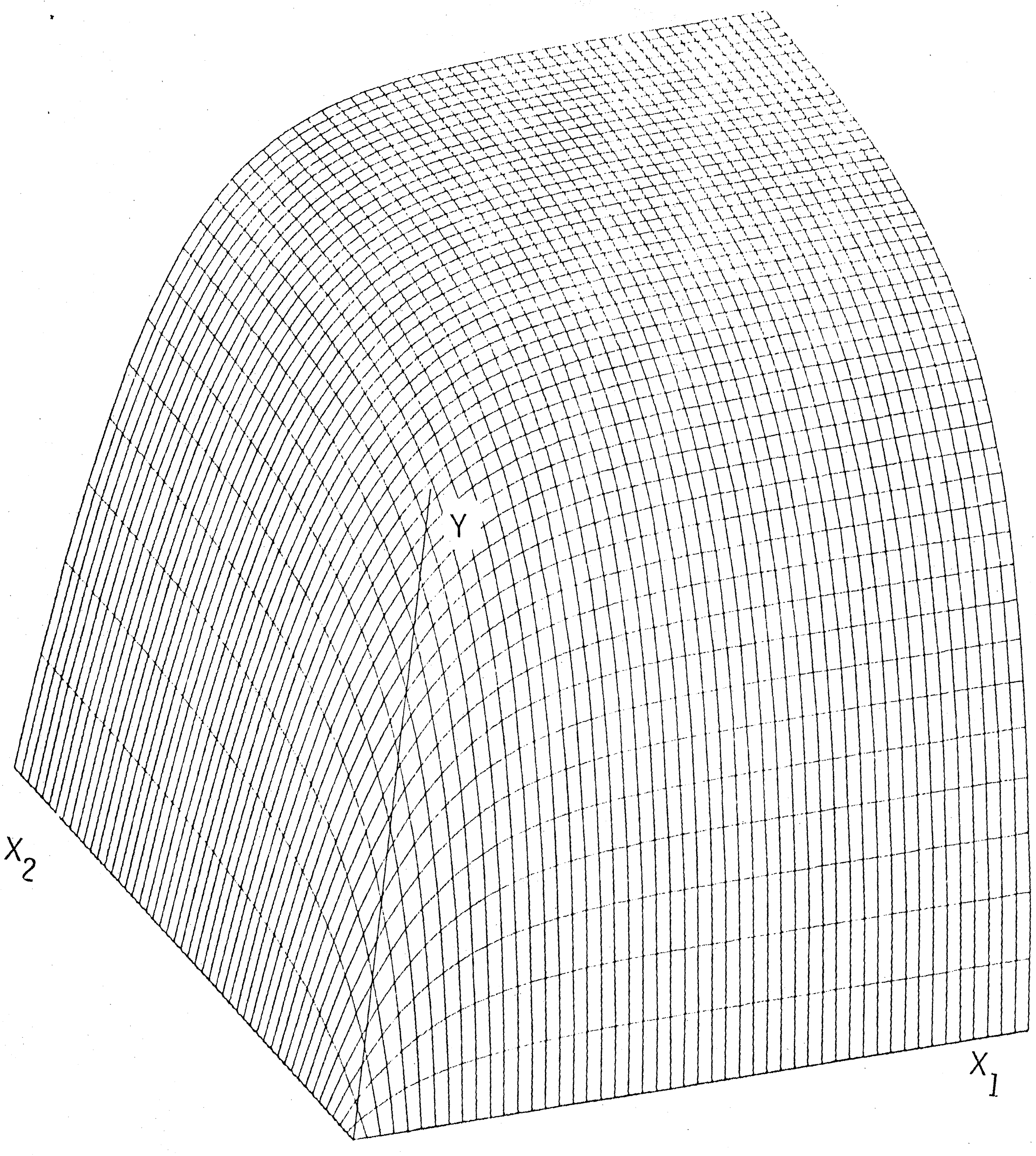

Figure 5. $Y = (1 - .4^{X_1}) \cdot (1 - .6^{X_2})$
"The Spillman"

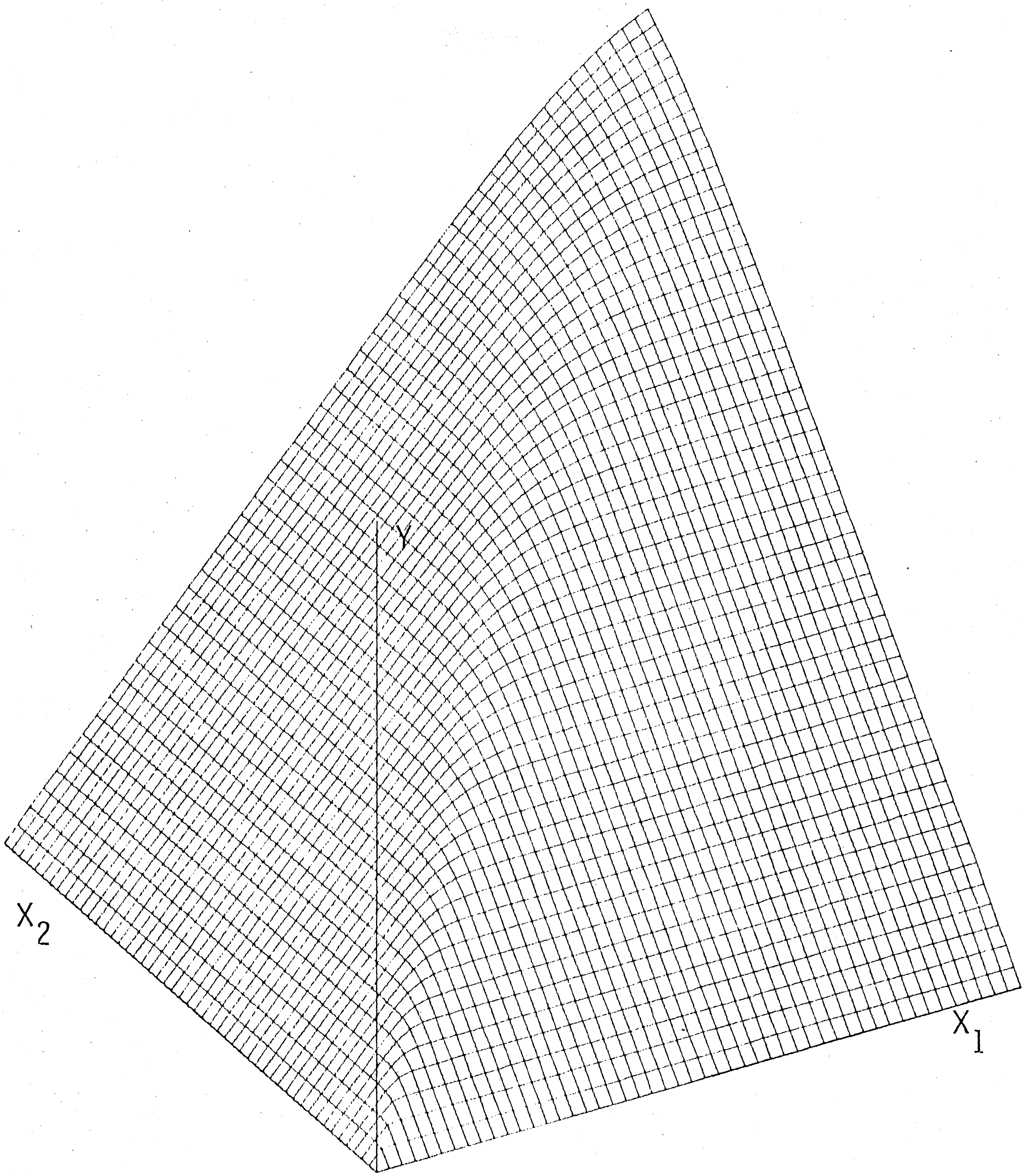

Figure 6. $Y = (.4X_1^{-10} + .6X_2^{-10})^{-\frac{1}{10}}$

"The C E S" (Case 1)

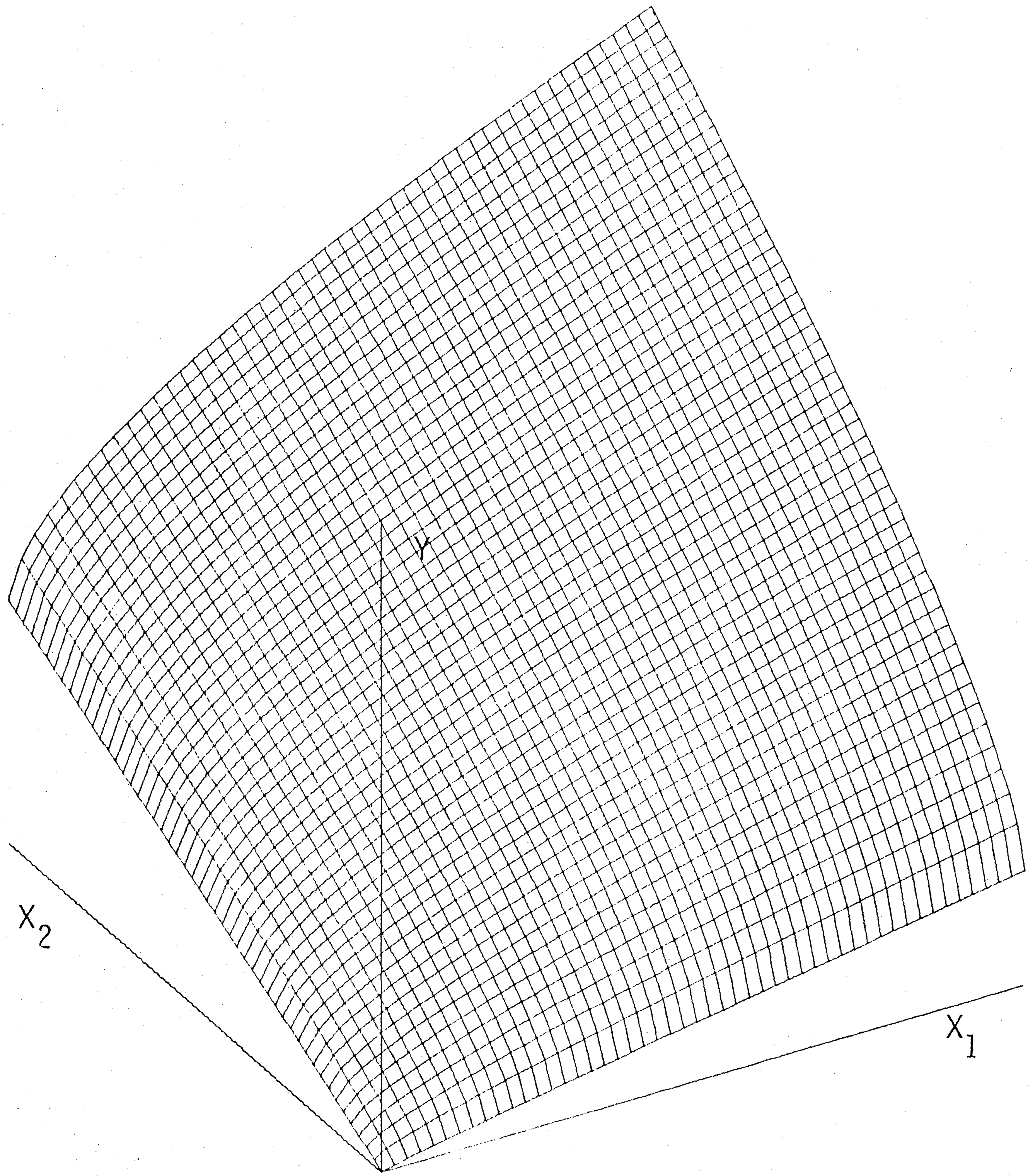

Figure 7. $Y = (.4X_1^{.5} + .6X_2^{.5})^{\frac{1}{.5}}$

"The C E S" (Case 2)

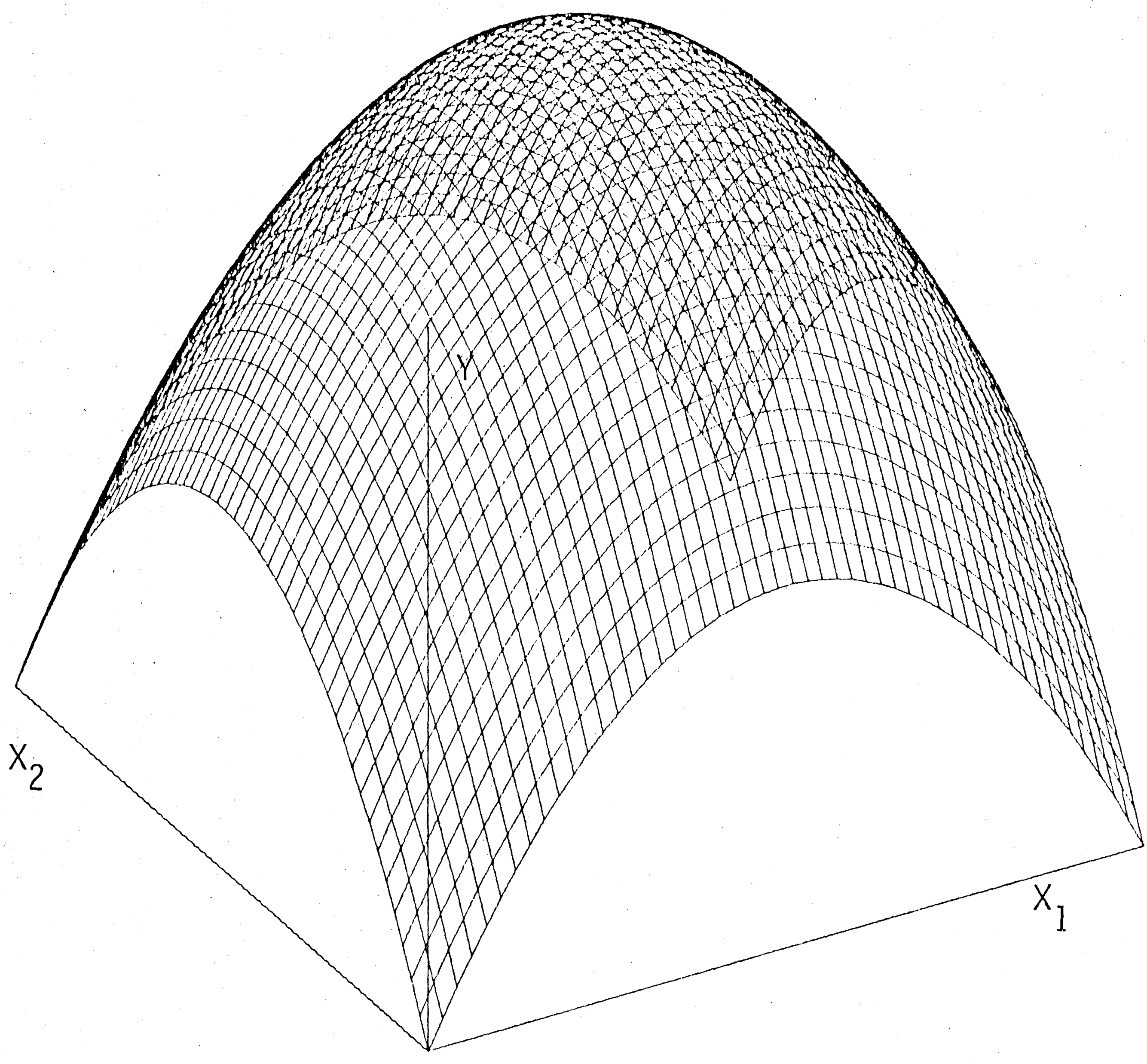

Figure 8. $Y = -X_1^2 - X_2^2$

"A Maximum"

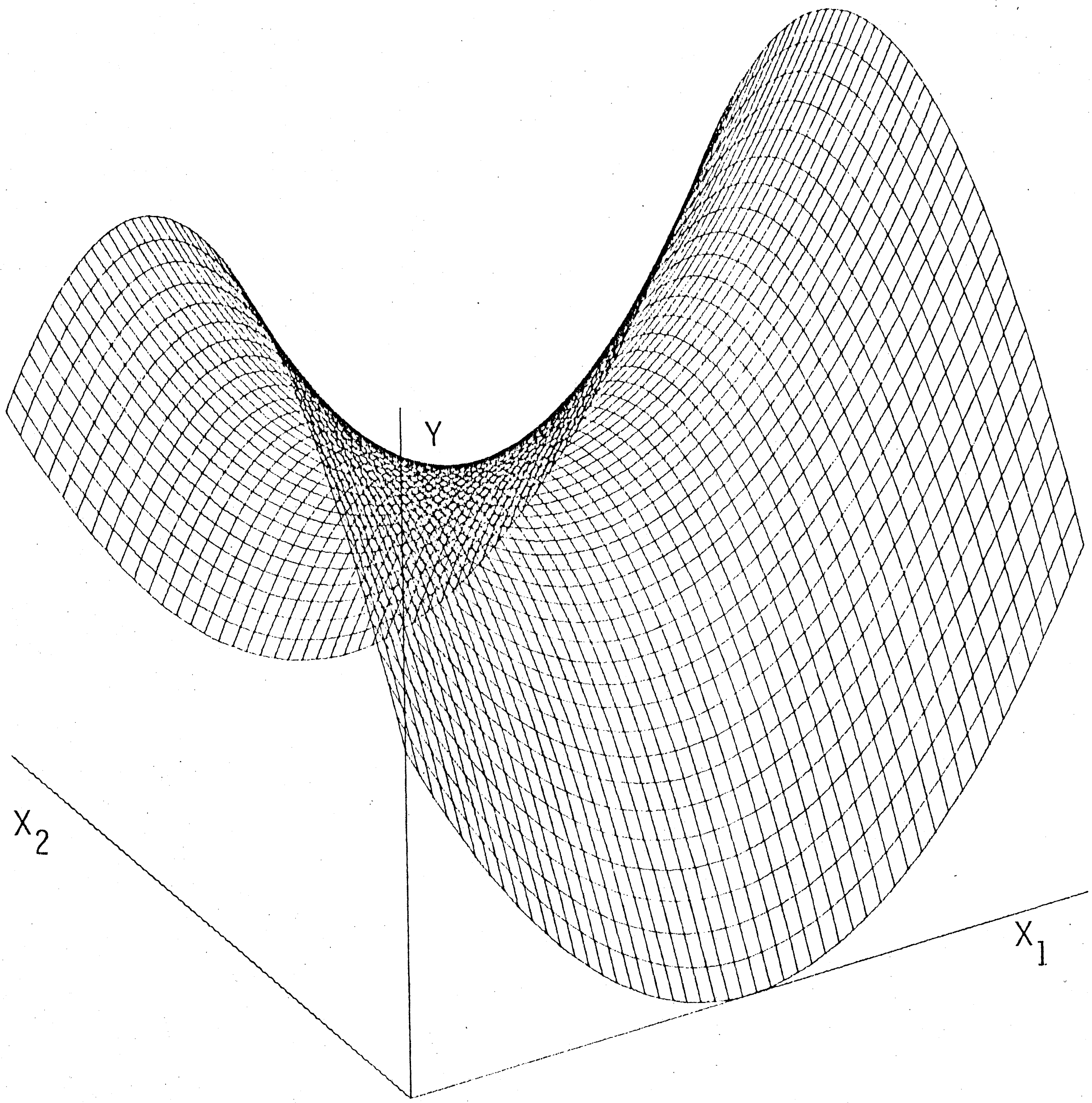

Figure 9. $Y = X_1^2 - X_2^2$

"A Saddle Point"

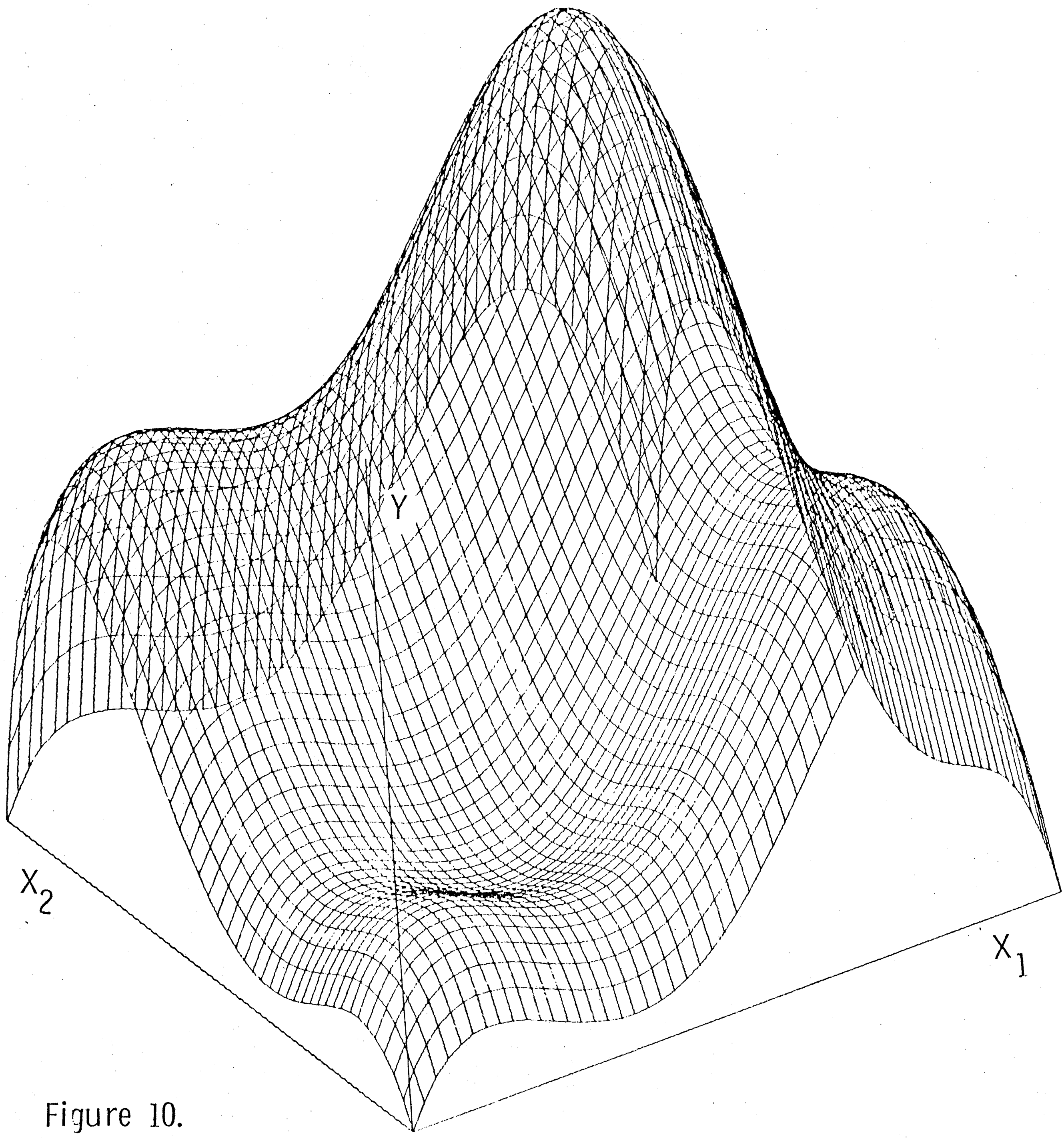

Figure 10.

$$Y = 40X_1 - 12X_1^2 + 1.2X_1^3 - .035X_1^4$$
$$+ 40X_2 - 12X_2^2 + 1.2X_2^3 - .035X_2^4$$

"Local and Global Optima"