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A CONSISTENCY CONDITION FOR EXPECTED UTILITY AND MEAN VARIANCE ANALYSIS

by

Jack Meyer
Department of Economics
Michigan State University
East Lansing, Michigan 48824

Lindon J. Robison
Department of Agricultural Economics
Michigan State University
East Lansing, Michigan 48824

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Introduction

Expected utility (EU) and mean variance models (MV) are the two main approaches used to represent an agent's preferences over risky alternatives. The EU model, the better theoretically grounded of the two, has made available a rich set of risk aversion definitions which allow one to categorize agents by preference type. Agricultural economists have used the absolute risk aversion function obtained from the EU model to measure risk attitudes in many different settings (e.g., King and Robison, Binswanger, Love and Robison, and Wilson and Eidman).

Practical risk problems, on the other hand, have often been solved using the MV model. Work by Lin, Dean, and Moore, Scott and Baker, Boussard, Brink, and McCarl, and Robison and Barry are examples from the agricultural economics literature. The advantages of the MV model include its simplicity which allows for two-dimensional representation of risky alternatives and the computational ease which allows portfolio models to be solved using quadratic programming. More recently, its usefulness has been extended to analytic work because of its ability to describe relationships with more than one random variable (Chavas and Pope) and its convenient representation of income and substitution effects under risk (Cass and Stiglitz, Royama and Hamada, and Robison and Barry).

Because each approach has useful and unique features, much effort has been made to identify the conditions under which the two approaches are consistent; or at least the conditions under which the two approaches can be viewed as approximations of each other. If satisfactory consistency conditions could be found, researchers would have the advantage of being able to combine the best features of each approach in conducting decision theory research.

Until recently, no generally acceptable sufficient condition had been found. Restrictions on the probability density functions describing risky alternatives, such as normality, were rarely satisfied empirically. On the other hand, restricting the agent's preference function to the quadratic utility form implies perverse risk attitudes. Thus, most researchers modeling decisions with the MV model have abandoned the requirement of consistency with EU because to do so limits the flexibility of the MV model.

The paper has two purposes. The first is to report a new and more general sufficient condition for MV and EU consistency recently identified by Meyer. The second purpose is to use the new condition to perform comparative static analysis in MV models. In doing so, a number of apparently unrelated results from the literature are shown to arise from a common principle.

The remainder of this paper is organized as follows. First, the literature is reviewed with special attention given to the recently identified consistency condition called the ''location parameter consistency'' condition. The next section discusses how this location parameter consistency condition permits comparative static analysis using the MV model in a more general setting. Finally, the last section illustrates comparative static results for two well-studied economic models.

Literature Review

Available space limits our review of the known conditions under which EU and MV models are consistent. An up-to-date review is given in Meyer. We simply summarize by saying that without restricting either the class of admissible probability density functions or limiting the decision maker's preferences to an uninteresting set, so far no sufficiently flexible consistency condition has been found.

The location parameter consistency condition (LPC), recently identified by Meyer, is sufficient for MV and EU consistency. Most significant about the LPC is that the condition has both empirical and theoretical support; moreover, it does not restrict the preferences of the agent nor the specific form of the distribution describing the random alternatives. What instead it requires is that the risky alternatives satisfy a certain relationship relative to one another. A definition from Feller is useful in summarizing the nature of the relationship between risky alternatives required by the LPC.

Definition: Two cumulative distribution functions $G_1(.)$ and $G_2(.)$ are said to differ only by location parameters α and β if $G_1(x) = G_2(\alpha + \beta x)$ with $\beta > 0$.

Some two-parameter families of distribution functions, such as the normal or uniform families, are made up of members which differ from one another only by location parameters. In addition, an infinite number of unnamed two-parameter families are included among those families whose members differ from one another only by location parameters. Others, such as the lognormal family, are not. However, even in the case of lognormal distributions, transformations of variables may sometimes permit the distributions to satisfy the LPC. Using the above definition, we state the LPC condition which has the generality described above.

Proposition: If the set of random alternatives are each described by cumulative distribution functions which differ from one another only by location parameters, then any expected utility ranking of those alternatives can be represented by a MV ranking instead.

Notice that the LPC proposition imposes no restriction on agents' preferences nor are members of the choice set restricted to a particular probability distribution type. Instead, the distributions in the choice set are required to be related to one another in the manner described.

To prove the LPC proposition, one recognizes that random variables are described by distribution functions which differ from one another only by location parameters if and only if they are equal in distribution to some positive linear transformation of the same variable. Normalizing this random variable so that its mean is zero and variance is one, this means that each random alternative Y_i is equal in distribution to $\mu_i+\sigma_i x$ where μ_i and σ_i are the mean and standard deviation of Y_i and x is some arbitrary random variable. Of course, this means that an expected utility ranking of these random alternatives is given by:

$$(1) \quad \int_a^b U(\mu_i+\sigma_i x)dF(x)$$

where F(x) is the CDF describing random variable x and [a,b] contains its support. Further details and a formal proof of this proposition are available elsewhere (Meyer). Important to notice, however, is that the expected utility model is now a function of the mean and standard deviation (square root of the variance).

With the mean and variance embedded in the expected utility model we are now prepared to do comparative static analysis on those models which meet the LPC requirements. Many simple economic models do, in fact, meet the LPC requirement including Sandmo's model of the competitive firm and Tobin's pure liquidity model. These two economic models, for example, satisfy LPC no matter how the random variable is distributed and for any utility function because all risky alternatives available to the decision maker are positive linear transformations of a random variable outside the agent's control. In the Sandmo model, random profit alternatives are each written as px-C(x)-B where p is an exogeneously given random output price, x is output level selected by the firm, and C(x) and B are the variable and fixed costs of production. The firm, in choosing output level x, is choosing the parameters in the positive linear transformation of random output price p which determine its profits. All alternative profit distributions in the Sandmo model satisfy the condition of being related to one another by location parameters. Thus, to analyze the behavior of this firm in a comparative static sense, one can use either MV or EU techniques and must get the same results. Sandmo and Hawawini do indeed carry out the analysis under the two different approaches and do prove the same theorems, but before this explanation, no reason for this was available. A very similar point can be made for the pure liquidity preference model.

In addition to this theoretical support for the LPC, preliminary indications are that empirical support exists in the only data set examined thus far. To test for this sufficient consistency condition, one must test the hypothesis that except for location parameters all cumulative distribution functions describing the random alternatives are the same. Notice that this is a far easier condition to meet than tests for normality or uniformity, for instance, which require all CDFs to be the same except for location parameters <u>and</u> to be of a specified functional form. It is the flexibility to choose <u>any</u> CDF in formulating the test for the LPC that makes that condition more likely to be satisfied in a given data set. Preliminary work using a variation on the n-sample Kolmogorov-Smirnov test for the mutual fund data used by Levy and Markowitz indicates that one cannot reject the location parameter consistency condition as a hypothesis.

The final point to be discussed before going on to comparative statics is that the ranking function:

$$(2) \quad \int_a^b U(\mu_i+\sigma_i x)dF(x) = \emptyset(\sigma_i^2,\mu_i)$$

which was presented earlier defines preferences in MV space as indicated. The MV preference function, $\emptyset(\sigma^2,\mu)$, is not overly restricted. That is, by choosing various utility functions and distribution functions, one can obtain a wide variety of functions $\emptyset(\sigma^2,\mu)$. Important to the comparative static results agricultural economists desire, one can incorporate decreasing, constant, or increasing absolute risk aversion and other such properties into a MV model by deriving those properties of $\emptyset(\sigma^2,\mu)$ which arise from the imposition these conditions on the utility function in the expression defining $\emptyset$. A number of these results are available elsewhere (Meyer).

In summary, the LPC is one with theoretical and empirical support, yields flexible MV decision models, and hence allows one to combine the best features of the EU and MV approaches in conducting analysis of various models where this condition is met. Such analysis is carried out in a general model in the next section.

Comparative Static Analysis

In this section, we deal with a MV decision model and the comparative static results that one can derive within it due to the LPC assumption discussed in the previous section. The impact of imposing this condition lies in the particular MV preference functions which can arise and how their properties correspond to properties of preferences in the expected utility model. The linkage between MV and EU models was given earlier in equation (2).

Using this expression, one can relate such concepts as constant or decreasing absolute risk aversion from the expected utility framework, to various properties of the MV preference function $\emptyset$. We will use certain of those properties of $\emptyset$ to discuss the intuitively appealing ''income'' and ''substitution'' effects often developed in MV comparative static analysis.

Assume an agent with preference function $\emptyset(\sigma^2,\mu)$ chooses σ^2 and μ subject to the generally specified restriction $\mu = g(\sigma^2,\alpha)+I$ where α and I are exogenous parameters. To solve a constrained maximization of this form, it is easiest to replace μ in the objective function, thereby eliminating the constraint. Thus, the maximization becomes:

$$\max_{\sigma^2} \emptyset[\sigma^2, g(\sigma^2,\alpha)+I]$$

The first-order condition for this problem is:

$$\emptyset_1 + \emptyset_2 \, g_1 = 0$$

where the arguments of these functions have been suppressed for notational simplicity. The sufficient second-order condition is:

$$\emptyset_{11} + 2\,\emptyset_{12}\, g_1 + \emptyset_{22}\, g_1^2 + \emptyset_2\, g_{11} < 0$$

Let D denote the left side of this inequality. One can show that assuming u(.) is increasing and concave (i.e., displays risk aversion) is enough to imply that D 0 holds for the Sandmo firm model and the Tobin liquidity preference model to be discussed soon.

Assuming the condition $D < 0$ is satisfied, one can then use the first-order condition to derive the effect of changes in parameters in the model on the optimal choice of σ^2. Doing so for parameter I yields:

$$\frac{d\sigma^2}{dI} = \frac{-(\emptyset_{12} + \emptyset_{22}\, g_1)}{D}$$

Now changes in parameter I represent vertically parallel shifts in the opportunity set in (σ^2, μ) space and can be thought of as representing ''income'' changes. That is, changes that make the agent better or worse off, but do not affect the rate at which σ^2 and μ can be exchanged for one another. In the model of the competitive firm, such a parameter is the fixed cost parameter and in the portfolio model it is the initial wealth parameter.

Of course, one can calculate $d\sigma^2/dI$, as was done above, without ever imposing the LPC. The value of imposing the LPC lies in its ability to determine the sign of $d\sigma^2/dI$. Specifically, one can show that the sign of $d\sigma^2/dI$ depends only on whether the agent in question displays increasing, constant, or decreasing absolute risk aversion (Meyer). This result is not a big surprise, it has been assumed by others, illustrated by example, and proven in certain specific economic models, but it has never been formally demonstrated as being true under the general conditions used here.

The main point we wish to make concerning the above income effect, is that it allows one to quite easily divide the effect of any change in the opportunity set into two components, the income effect term and the substitution effect term. As shall be illustrated, these terms play roles similar to those of the corresponding terms in basic two-good consumer theory. To see this, calculate $d\sigma^2/d\alpha$ where α is some parameter which shifts the opportunity set in an unspecified fashion.

$$\frac{d\sigma^2}{d\alpha} = \frac{-(\emptyset_{12}\, g_2 + \emptyset_{22}\, g_1\, g_2 + \emptyset_2\, g_{12})}{D}$$

$$= g_2 \left(\frac{d\sigma^2}{dI}\right) + \frac{(-\emptyset_2\, g_{12})}{D}$$

Simple calculations show that $-g_2$ is how much I, the ''income'' parameter, must change in order to offset in utility terms a one unit change in parameter α. Assume for discussion purposes that $g_2 > 0$, thus increases in α increase the agent's opportunities. The term $(g_2 \frac{d\sigma^2}{dI})$ can be called the income effect resulting from the change in α. It is composed of both a term which indicates an ''equivalent'' income change, and a term which indicates how this income change would affect the decision made by the agent. This is completely analogous to the standard consumer theory discussion.

The remaining term, $(\frac{-\emptyset_2\, g_{12}}{D})$, represents the effect of changing α if the parameter I had been adjusted to keep utility constant. It can be termed a substitution effect. Since one can easily show that $\emptyset_2$ is positive, this term is signable if the constraint defining the opportunity set is such that g_{12} is of one sign. For the examples we discuss, this is the case. Obviously, much more can be done within this framework; space does not permit us to carry this out at this time. Next, two examples of models satisfying the location parameter consistency condition are briefly analyzed using the general comparative static results just obtained.

The Sandmo Model

In the Sandmo model, output x is selected so as to maximize expected utility from profits which were earlier defined as $\pi = px-C(x)-B$ where $C(x)$ and B are variable and fixed costs, respectively. As mentioned earlier, all alternatives to the firm are positive linear transformations of the given random variable p; hence, the LPC proposition applies. The mean and variance for each profit alternative in the Sandmo firm can be written as:

$$\mu = {}_p x - C(x) - B$$

$$\sigma^2 = \sigma_p^2 \, x^2$$

where μ_p and σ_p^2 are the mean and variance of the random variable p.

Using the second equation to eliminate x from the first equation allows us to write:

$$\mu = \mu_p \left(\frac{\sigma^2}{\sigma_p^2}\right)^{1/2} - C\left[\left(\frac{\sigma^2}{\sigma_p^2}\right)^{1/2}\right] - B$$

which is in the form $\mu = g(\sigma^2,\alpha) + I$.

One can now determine the effect of changing μ_p or σ_p^2 or B in this model using the general methodology presented in the previous section. As mentioned earlier, shifts in B are ''income'' changes and hence signable on the basis of decreasing, constant, or increasing absolute risk aversion.

Space does permit us to show this. Sandmo, Hawawani, and Robison and Barry have found similar results concerning a change in B.

To consider the effect of an increase in σ_p^2, we write the result as:

$$\frac{d\sigma^2}{d\sigma_p^2} = -g_2 \frac{d\sigma^2}{dB} - \frac{\emptyset_2 g_{12}}{D}$$

The Tobin Model

Tobin developed the pure theory of liquidity preference in the context of a portfolio model involving a single risky and riskless asset. In that model, terminal wealth, W, is expressed as $W = \alpha W_0 \rho + (1-\alpha)W_0 r$ where α is the proportion of initial wealth W_0 invested in the riskless asset, and ρ and r are the returns to the riskless and risky assets. All alternatives to the firm are positive linear transformations of the random variable r; hence the LPC proposition applies. The mean and variance for each terminal wealth alternative in the Tobin model can be written as:

$$\mu = \alpha W_0 \rho + (1-\alpha)\ W_0 \mu r \qquad \text{and}$$

$$\sigma^2 = [(1-\alpha)W_0]^2\ \sigma_r^2$$

where μ_r and σ_r^2 are the mean and variance of the random variable r.

Using the second equation to eliminate α from the first equation allows us to write:

$$\mu = W_0 \rho + (\mu_r - \rho)\ \left(\frac{\sigma^2}{\sigma_r^2}\right)^{1/2}$$

which like the Sandmo model is in the form $\mu = g(\sigma^2, \alpha) + I$.

One can now determine the effect of changing μ_r, σ_r^2, ρ, or W_0 in this model using the general methodology presented in the previous section. Changes in W_0 are income changes and hence are signable on the basis of increasing, constant, or decreasing absolute risk aversion. Cass and Stiglitz and Robison and Barry (1986) have found similar results.

Summary

This paper has introduced and illustrated a new condition which allows for consistency between EU and MV models under much more general conditions than heretofore known. The condition, the LPC, requires only that distributions be related to one another in a particular way, namely that the distributions differ only by location parameters.

This condition has important implications for both theoretical and empirical risk work. On the empirical side, it suggests the need to develop tests to determine statistically when distributions differ by location parameters. This, however, will be a much less restrictive test than the test to determine whether all distributions are members of the same specified two-parameter family of distribution functions.

On the theoretical side, much remains to be done also. We have shown that the LPC applies to two specific examples, namely the Sandmo and Tobin models. It is believed the LPC applies to a large number of theoretical models and can be used to deduce original risk models in setting not yet examined. Moreover, preliminary results suggest that income and substitution effects, with correspondence to consumer theory models, may be more easily defined in MV models than in the EU model. Thus, it may be that the MV model has some advantage over the EU model because of its mathematical tractibility.

Finally, some effort is needed to work backwards, when the Ø function is known to find the corresponding U(.) function. For example, Robison and Barry and Chavas and Pope have used particular MV models to duplicate EU models. The framework developed in this model might allow for a richer interpretation of their models.

References

Binswanger, H.P. ''Attitudes Toward Risk: Experimental Measures in Rural India.'' Amer. J. Agr. Econ. 62(1980):395-407.

Boussard, J.M. ''The Introduction of Risk into a Programming Model: Different Criteria and Actual Behavior of Farmers,'' European Economic Review 1(1969):92-121.

Brink, L. and B. McCarl. ''The Trade-off Between Expected Returns and Risk Among Corn Belt Farmers,'' Amer. J. Agr. Econ. 60(1978):259-263.

Cass, D. and J.E. Stiglitz. ''Risk Aversion and Wealth Effects on Portfolios with Many Assets,'' Review of Economic Studies 39(1972):331-354.

Chavas, Jean-Paul and Rulon Pope. ''Hedging and Production Decisions Under a Linear Mean-Variance Preference Function,'' Western Journal of Agricultural Economics, July 1982, pp. 99-110.

Feller, W. An Introduction to Probability Theory and Its Application: Vol. II, John Wiley and Sons, 19___.

Hawawini, G.A. ''A Mean-Standard Deviation Exposition of the Theory of the Firm Under Uncertainty: A Pedagogical Note,'' American Economic Review 68(1978):194-202.

King, R.P. and L.J. Robison. ''An Interval Approach to the Measurement of Decision Maker Preferences,'' Amer. J. Agr. Econ. 63(1981):510-520.

Levy, H. and H. Markowitz. ''Approximating Expected Utility by a Function of Mean and Variance,'' American Economic Review, 69(1979):308-317.

Lin, W., G. Dean, and C. Moore. ''An Empirical Test of Utility Versus Profit Maximization in Agricultural Production,'' Amer. J. Agr. Econ. 56(1974):497-508.

Love, R.O. and L.J. Robison. ''An Empirical Analysis of the Intertemporal Stability of Risk Preference,'' Southern Journal of Agricultural Economics, July 1984, pp. 159-165.

Meyer, Jack. ''Two Moment Decision Models and Expected Utility Maximization.'' Mimeo, Department of Economics, 1985.

Robison, L.J. and P.J. Barry. ''Portfolio Adjustments: An Application to Rural Banking,'' Amer. J. Agr. Econ. 59(1977):311-320.

___________. The Competitive Firm's Response to Risk. Macmillan, New York, 1986.

Royama, S. and K. Hamada. ''Substitution and Complementarity in the Choice of Risky Assets,'' in D.D. Hester and J. Tobin, eds. Risk Aversion and Portfolio Choices. Monogram 19, New Haven, 1966, pp. 27-40.

Sandmo, A. ''On the Theory of the Competitive Firm Under Price Uncertainty,'' American Economic Review 61(1971):65-73.

Scott, J.T. and C.B. Baker. ''A Practical Way to Select an Optimum Farm Plan Under Risk,'' Amer. J. Agr. Econ. 54(1972):657-660.

Tobin, J. ''Liquidity Preference as Behavior Towards Risk,'' Review of Economic Studies 25(1958):65-86.

Wilson, Paul and Vernon Eidman. ''An Empirical Test of the Interval Approach for Estimating Risk Preferences,'' West. J. of Agr. Econ. 8(1983):170-182.