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The Optimal Mechanism for Selling to Budget-Constrained Consumers

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The Optimal Mechanism for Selling to Budget-Constrained Consumers

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Abstract. This paper finds an optimal mechanism for selling an indivisible good to consumers who may be budget-constrained. Unlike the standard case, where buyers are not budget-constrained, a single posted price is not optimal. An optimal mechanism generally consists of a continuum of lotteries indexed by the probability of consumption and the entry fee.

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Key words: budget constraints, lotteries, mechanism design.

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1. Introduction

Sellers often face the problem of screening consumers who differ in their willingness to pay for a good and in their ability to pay. In the existing literature, it is typically assumed that consumers have resources exceeding their individual valuations of the good. In this context, the simplest form of screening — posting a single price — is optimal for a seller with constant marginal cost, if consumers are risk neutral and desire at most one unit of the good (Harris and Raviv [1981]; Riley and Zeckhauser [1983]).[1] In practice, however, consumers may be budget constrained for a number of reasons. Imperfect capital markets can limit a consumer's ability to borrow against future income. Or the consumer could be a bureaucratic agent who only internalizes the benefit side of the acquisition, so his superiors must put budgetary constraints on him. The practical importance of budget constraints is clear given casual observation of sales practices. For instance, sellers of consumer durables frequently offer low-interest financing.[2] Yet, virtually no attention has been paid to consumers' limited purchasing power in this context.[3]

One possible explanation for the lack of attention is that recognizing a budget constraint might simply involve relabeling each consumer's reservation price for a unit as the minimum of his valuation and budget, for example. Budget constraints would then simply scale down the demand curve facing the seller, so the existing analyses that focus on valuations would still apply. We show, however, that revenue and welfare analysis must distinguish willingness to pay and ability to pay. In particular, in many settings where a single posted price would be optimal for a seller in the absence of budget constraints, a strategy involving a menu of lotteries is optimal if some consumers face budget constraints.

An example illustrates the basic point of this paper. Suppose that a seller has an unlimited

[1] If consumers have concave utility, then the optimal selling mechanism may involve a non-linear pricing scheme. See Mussa and Rosen [1978]; Maskin and Riley [1984].

[2] In government auctions, anecdotal evidence suggests that the limited financial resources of small firms are a major concern. For example, the U.S. government has limited the length and size of mineral leases. In timber rights auctions, "setaside sales" have been made available to small firms (Bergsten et al. [1987]). Also, the fact that joint ventures are allowed for small firms in OCS auctions can be explained from this perspective (Hendricks and Porter [1992]; McDonald [1979]).

[3] One exception is Sen (1995) who considers seller-provided financing as a means of price discrimination when otherwise identical consumers face income fluctuations.

supply of an indivisible good. There is a unit mass of consumers, each of whom desires one unit of the good. The typical consumer realizes a gross surplus of v from the purchase of one unit, and has a budget of w, which is the most he can spend. The (v, w) pairs are consumers' private information, and are drawn from $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$ with positive density. Let $\underline{w} < \overline{v}$, so some consumers are unable to pay their valuations.

A single price $p > \underline{w}$ extracts revenue from consumers with $v, w \geq p$ (the dark-shaded square in Figure 1). The seller has a better strategy, however. In addition to offering the good at p, she can offer a lottery $\langle t, x \rangle$, which charges t for the right to receive the good with probability $x \in (0, 1)$, where $\underline{w} < t < p < \frac{t}{x} < \overline{v}$. The lottery implicitly charges a price per unit probability equal to $t/x > p$, so those consumers with $(v, w) \geq (p, p)$ prefer to pay p and receive the good with certainty. Consumers with low budgets (i.e., $t < w < p$) cannot pay p, but will purchase the lottery if $v > t/x$. Thus, the monopolist can extract additional revenue (depicted as the light-shaded area in Figure 1). This example suggests that private information about a consumer's budget is qualitatively different from private information about his reservation value. Consumers with different budgets are easier to screen (for example, through a menu of lotteries) than consumers with different valuations.

With the suboptimality of standard monopoly pricing as our point of departure, section 2 identifies an optimal selling mechanism. Section 3 discusses implications of our findings.

2. The Optimal Selling Mechanism

We consider the setting introduced in the previous section: a single seller has an unlimited supply of an indivisible good, which is available at zero cost, and there is a unit mass of risk-neutral consumers with inelastic demand for one unit of the good. (The model can alternatively be interpreted as one where the seller faces a single buyer of unknown type.) Each consumer is indexed by his gross surplus from the consumption of one unit, v, and his budget, w. The (v, w) pairs are drawn independently and identically from $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$. The marginal distribution of w and the conditional distribution of v given w are $G(w)$ and $F(v|w)$, respectively, with corresponding densities $g(w)$ and $f(v|w)$. (Note that v and w need not be independent.) The densities are positive and continuously differentiable for all $(v,w) \in (\underline{v},\overline{v}) \times (\underline{w},\overline{w})$. To make the presence of budget constraints meaningful, we assume that $\underline{w} < \overline{v}$ throughout the paper.

By the Revelation Principle, we can restrict attention to direct revelation mechanisms in which each consumer has an incentive to report his private information truthfully. Thus, without loss of generality, the seller specifies the probability, x, that a consumer receives the good and the expected transfer, t, that he must pay, as functions of his reported valuation and budget. Since the seller and buyers are risk neutral, the probability distribution of the transfer has no payoff relevance, once its expected value is fixed, so we assume henceforth that transfers are deterministic.[4] Let $\langle t(\tilde{v},\tilde{w}), x(\tilde{v},\tilde{w})\rangle$ be the contract given to a consumer who has reported $(\tilde{v},\tilde{w})$. Formally, a "mechanism" $\langle t,x\rangle$ is a mapping from $[\underline{v},\overline{v}] \times [\underline{w},\overline{w}]$ into $(-\infty,\infty) \times [0,1]$.

A feasible mechanism satisfies the following constraints. First, each type (v,w) must have an incentive to report his type truthfully from among the types that he can feasibly report given his budget:

$$(IC) \qquad vx(v,w) - t(v,w) \geq vx(\tilde{v},\tilde{w}) - t(\tilde{v},\tilde{w}),\ \forall (v,w),(\tilde{v},\tilde{w}) \text{ s.t. } \tilde{w} \leq w.$$

The restriction on the set of reports entails no loss of generality since the seller can costlessly prevent buyers from exaggerating their budgets. This is achieved by requiring buyers to post a cash bond equal to their reported budgets, and then refunding the difference between the bond and the required transfer. Alternatively, the seller may require a random transfer, with the maximum realized transfer equal to the reported budget.[5] Bonding requirements and random transfers are not common sales practices. Therefore, we are also interested in mechanisms that do not involve bonding or random transfers. For these mechanisms, the incentive constraint is stronger, and it is given by

$$(IC') \qquad vx(v,w) - t(v,w) \geq vx(\tilde{v},\tilde{w}) - t(\tilde{v},\tilde{w}),\ \forall (v,w),(\tilde{v},\tilde{w}) \text{ s.t. } t(\tilde{v},\tilde{w}) \leq w.$$

[4] Random payments can be used to reveal low-budget buyers, as will be discussed subsequently.

[5] The random transfer method may not be as effective as the bonding requirement, since the former requires a sufficiently severe penalty for lying, which is limited because of fixed budgets. Clearly, bonding can achieve whatever a random transfer can. For this reason, we assume that the transfer is deterministic.

A fundamental question that we address subsequently is when (IC′) entails no loss for the seller; i.e., when the bonding requirement is superfluous.

Next, each consumer must earn at least his reservation payoff, which we take to be zero:

$$(IR) \qquad vx(v,w) - t(v,w) \geq 0 \ \forall (v,w).$$

Furthermore, the mechanism must satisfy each consumer's budget constraint:

$$(BC) \qquad t(v,w) \leq w \ \forall (v,w).$$

Probability constraints must also be satisfied:

$$(PR) \qquad x(v,w) \in [0,1] \ \forall (v,w).$$

Finally, the mechanism must maximize the seller's expected revenue. That is, the seller faces the following program:

$$[S] \qquad \max_{x(\cdot,\cdot),t(\cdot,\cdot)} \int_{\underline{w}}^{\overline{w}} \int_{\underline{v}}^{\overline{v}} t(v,w) f(v|w)\, dv\, g(w)\, dw$$

s.t. (IC), (IR), (BC), and (PR).

Optimal mechanisms with two-dimensional uncertainty have been characterized by several authors (see McAfee and McMillan [1988]; Laffont, Maskin and Rochet [1987]; and Rochet [1987]). Our problem differs because of (BC) and the possible use of bonding. Without these two features, the optimal mechanism forms a pricing *function* since, by incentive compatibility, no two consumers will pay different (expected) amounts for the same probability of obtaining the good. So, one can simply look for the pricing function that the optimal mechanism induces.[6] In our problem, however, bonding may cause a lower-budget consumer to pay more for a given probability of obtaining the good than would a higher-budget consumer. Nevertheless, if the optimal mechanism induces the higher-budget consumers to pay a larger expected transfer than lower-budget consumers, bonding plays no role, and we can focus on the pricing function, as will be shown subsequently.

[6] We thank a referee for suggesting this approach.

Our methodology is described formally as follows. Consider a mapping $T : [0,1] \rightarrow \Re$, that is increasing and convex, with $T(0) = 0$. This mapping is interpreted as a non-linear pricing scheme that is offered to all consumers. For all $v \in [\underline{v}, \overline{v}]$, let $x(v) \equiv \max\{x | x \in \text{argmax}_{x' \in [0,1]} vx' - T(x')\}$. Clearly, $x(\cdot)$ is non-decreasing. Note that if $T(x(v)) \leq w$, then the consumer with (v, w) would be willing and able to purchase $x(v)$ and pay $T(x(v))$. If $T(x(v)) > w$, the consumer would want to spend w and get $x = T^{-1}(w)$, since $vx - T(x)$ is concave in x.

Now, consider the mechanism defined as follows:

$$\langle \overline{t}, \overline{x} \rangle (v, w) = \begin{cases} \langle T(x(v)), x(v) \rangle & \text{if } T(x(v)) \leq w; \\ \langle w, T^{-1}(w) \rangle & \text{if } T(x(v)) > w. \end{cases} \tag{1}$$

This mechanism assigns each consumer to his preferred, *feasible* contract from the underlying pricing curve $T(\cdot)$, as depicted in Figure 2. In particular, a consumer with valuation v is assigned to a point on the curve with subgradient equal to v, if he has a sufficient budget. Otherwise, the consumer is assigned to the contract that exhausts his budget (see Figure 2).

Definition: A simple mechanism (with an associated mapping $T(\cdot)$) is a mechanism $\langle \overline{t}, \overline{x} \rangle$ of the form (1), that has non-random transfers and does not require bonding. A simple mechanism is **optimal** if it satisfies (IC′), (IR), (BC) and (PR) and yields the same expected revenue as the solution to [S].

If there exists a simple mechanism that is optimal, then we can focus on the non-linear pricing problem:

$$[S1] \qquad \max_{T(\cdot)} \iint \min\{T(x(v)), w\} f(v|w)\, dv g(w)\, dw$$

$$\text{s.t.} \qquad T : [0,1] \rightarrow \Re \text{ is strictly increasing and convex, and } T(0) = 0.$$

The following lemma formally shows the conditions under which the original, two-dimensional problem can be reduced to the one-dimensional problem represented by [S1].

Lemma 1. Suppose that a mechanism $\langle t, x \rangle$ solves [S], with $t(v, \cdot)$ non-decreasing for all v. Then, a simple mechanism $\langle \overline{t}, \overline{x} \rangle$ whose associated mapping $T(\cdot)$ solves [S1] is optimal.

Proof. See Appendix A.

The rough intuition is that, when the high-budget consumers pay more than low-budget consumers (in expectation), the seller can implement the optimal mechanism without bonding. Instead, she can design a (non-linear) pricing function based on the contracts offered to consumers with the highest budget. (This function is increasing and convex, by incentive compatibility for these consumers.) As shown in Figure 2, a consumer with valuation v then pays the same transfer as a consumer with the same valuation but the highest budget, or else he pays his budget. Clearly, such a transfer is the maximum the consumer can pay in the optimal mechanism.

The next step is to explore when the hypothesis holds. To this end, we make the following two assumptions.

Assumption 1 (Regularity). $H(v|w) \equiv vf(v|w) - (1 - F(v|w))$ is strictly increasing in v for all w.

Assumption 2 (Monotone Likelihood Ratio Property). For any $v_1 \geq v_2$ and $w_1 \geq w_2$, $f(v_1|w_1)/f(v_1|w_2) \geq f(v_2|w_1)/f(v_2|w_2)$.

Assumption 1 is a standard regularity condition, that is satisfied by many well-known distributions such as the uniform distribution. Assumption 2 implies that a higher w makes a higher v (weakly) more likely. Let $m(w) \equiv \text{argmax } p[1 - F(p|w)]$ be the "monopoly price" against buyers with w, ignoring the effect of their budget constraint. Assumption 2 implies that $m(\cdot)$ is non-decreasing, which includes the case where v and w are independent. We now establish that expected payments are non-decreasing in w.

Lemma 2. Given Assumptions 1 and 2, a simple mechanism $\langle t, x \rangle$ whose associated $T(\cdot)$ solves [S1] is optimal.

Proof. See Appendix B.

Here, we offer some intuition for the impact of the latter assumption. Assumption 2 implies that the marginal revenue curve is higher for buyers with higher budgets, so the seller would like to charge them higher prices. In this case, bonding is unnecessary, since it helps only when the seller wants to extract more surplus from buyers with lower budgets.[7]

Given Assumptions 1 and 2, we solve the program [S1]. The following proposition characterizes the optimal mechanism based on the solution to [S1]. To aid in the characterization, we define

$$\hat{v}(t) \equiv \text{argmax}_p \int_{\max\{t,\underline{w}\}}^{\overline{w}} p[1 - F(p|w)]g(w)dw.$$

Roughly speaking, $\hat{v}(t)$ is the monopoly price against buyers whose budgets exceed t, again ignoring the effect of the budget constraint. If no buyers are constrained, the optimal mechanism involves setting a single price (Harris and Raviv [1981]; Riley and Zeckhauser [1983]). In particular, if $\underline{w} \geq \hat{v}(\underline{w})$, the seller sets $p = \hat{v}(\underline{w})$. Otherwise, the seller offers a continuum of lotteries.[8]

Proposition 1. Assume $\overline{w} \geq \hat{v}(\underline{w})$. Given Assumptions 1 and 2, a solution to [S] exists and is characterized by the following simple mechanism:

i) If $\underline{w} \geq \hat{v}(\underline{w})$,

$$\langle t, x\rangle(v,w) = \begin{cases} \langle \hat{v}(\underline{w}), 1\rangle & \text{if } (v,w) \in [\hat{v}(\underline{w}), \overline{v}] \times [\underline{w}, \overline{w}]; \\ \langle 0, 0\rangle & \text{if } (v,w) \in [\underline{v}, \hat{v}(\underline{w})] \times [\underline{w}, \overline{w}]. \end{cases}$$

ii) If $\underline{w} < \hat{v}(\underline{w})$, then there exists a strictly increasing and convex function, $T(\cdot)$, and constants $v^* \in [\max\{\underline{w}, \underline{v}\}, T(1)]$ and $x^* \in [\underline{w}/v^*, 1]$, such that

$$\langle t, x\rangle(v,w) = \begin{cases} \langle T(1), 1\rangle & \text{if } (v,w) \in [\hat{v}(T(1)), \overline{v}] \times [T(1), \overline{w}]; \\ \langle T(x(v)), x(v)\rangle & \text{if } (v,w) \in [v^*, \hat{v}(T(1))] \times [T(x(v)), \overline{w}]; \\ \langle w, T^{-1}(w)\rangle & \text{if } (v,w) \in [v^*, \overline{v}] \times [\underline{w}, T(x(v))]; \\ \langle 0, 0\rangle & \text{if } (v,w) \in [\underline{v}, v^*) \times [\underline{w}, \overline{w}], \end{cases}$$

where $T(\cdot)$ solves the following second-order differential equation:

$$\frac{T'(x)^2}{T''(x)} f(T'(x)|T(x)) - \frac{\int_{T(x)}^{\overline{w}} \{2f(T'(x)|w) + T'(x)f'(T'(x)|w)\}\, g(w)\, dw}{g(T(x))} = 0$$

if $x \in [x^*, 1]$, and $T(x) = v^*x$ if $x \in [0, x^*]$.

[7] When Assumption 2 is violated, bonding can strictly increase the seller's revenue, in which case the simple mechanism in (1) is suboptimal. Suppose $\overline{v} < \underline{w}$, and $m(\cdot)$ is strictly decreasing. Then, the seller can charge the monopoly price $m(w)$ to buyers with w, by revealing w through bonding. The buyers pay different amounts for the same probability (=1) of obtaining the good in the optimal mechanism. So, the nonlinear pricing approach, which assigns a single payment to a given probability of obtaining the good, is suboptimal in this case.

[8] The seller cannot charge $m(w)$ to buyers with w, if $m(\cdot)$ is strictly increasing, since a buyer with a higher budget would then mimic a buyer with a lower budget.

If $\hat{v}(\underline{w}) > \underline{v}$, then $v^* < \hat{v}(T(1))$ and $x^* < 1$.

Proof. See Appendix C.

In case i), the budget constraints do not play any role. Budget constraints bind in case ii). In the latter case, consumers with sufficiently high valuations and budgets choose non-random sales contracts, while consumers with lower valuations or lower budgets receive lottery contracts or are excluded.

The features of the optimal mechanism are illustrated by the following example.

Example: Suppose that (v, w) follows a uniform distribution on $[0,1]^2$. Then, $\hat{v}(t) = 1/2$ for all t and case ii) is relevant. The differential equation simplifies to $2T''(x)[1 - T(x)] - T'(x)^2 = 0$. Solving the equation gives

$$T(x) = 1 - \left[\frac{2(7 - 4T(1)) - 6x}{1 - T(1)}\right]^{2/3} \left(\frac{1 - T(1)}{4}\right),$$

where $T(1) = .4279$ and $v^* = .3782$. Thus, in the optimal mechanism,

$$x(v, w) = \begin{cases} 1 & \text{if } v \geq 1/2, w \geq .4279; \\ 1.7628 - .0953/v^3 & \text{if } .3782 \leq v < 1/2, w \geq 1 - .143/v^2; \\ 1.7628 - 1.7616(1 - w)^{3/2} & \text{if } v \geq .3782, w < 1 - .143/v^2; \\ 0 & \text{otherwise.} \end{cases}$$

Remark: If $\overline{w} < \hat{v}$, it is possible that $t(v, \overline{w}) = \overline{w}$, in which case an optimal mechanism is more difficult to characterize. One case, however, is clear-cut. Suppose that all consumers are budget constrained (i.e., $\overline{w} < \underline{v}$). In this case, the mechanism: $\langle t, x\rangle(v, w) = \langle w, w/\underline{v}\rangle$ for all (v, w) is optimal. It achieves the upper bound on revenue since all consumers pay their budgets. Furthermore, it satisfies all the constraints of [S] since $v\,(w/\underline{v}) - w = w(v/\underline{v} - 1)$ is non-negative and non-decreasing in w for all v. Thus, no consumer wants to understate his budget.

3. Conclusion

This paper has shown that simple posted-price mechanisms are not optimal when some consumers are budget constrained and goods are indivisible. The optimal selling mechanism in this environment involves a continuum of lotteries. This result is robust to the availability of credit, as long as consumers' access to the credit market is imperfect. Our model corresponds to the case where the difference between borrowing and lending interest rates is infinite, but lotteries remain optimal for a range of rates, depending on the underlying distribution.

In reality, legal and strategic problems limit the practicality of lotteries. In most U.S. states, the use of lotteries by private firms is prohibited. An obvious strategic problem is that the seller could award the object to a confederate. Nevertheless, simple, disguised forms of lotteries have surfaced during cyclical downturns when liquidity constraints are most acute.[9]

The restrictions on the use of lotteries raise both normative and positive questions. Allowing lotteries presents an interesting welfare trade-off. On the one hand, lotteries make it profitable for sellers to serve (with positive probability) low-budget, high-valuation consumers who would otherwise be excluded. On the other hand, an improved ability to screen consumers through lotteries may cause the seller to exclude some low-valuation consumers who otherwise would be served. In the uniform distribution example, the net effect is positive.

Restrictions on the use of lotteries can elicit several responses from the seller. She may provide loans at a below-market interest rate. Indeed, unlimited zero-percent financing will eliminate buyers' budget constraints, in which case a single monopoly price is again optimal for the seller. While this observation may provide a rationale for seller-provided financing, this strategy has drawbacks. First, borrowers may default. Although the good can be used as collateral, buyers' moral hazard and the costs associated with confiscation makes this an imperfect solution. For example, zero-percent financing, which is prevalent in the sale of consumer durables, is often very short term, because of the possible depreciation of the collateral due to moral hazard. (Many zero-percent loans offered in the sale of consumer durables are actually deferred interest loans, so

[9] Essay contests — a recent fad in selling houses and businesses — provide one example. In the essay contests, a participant pays an entry fee, typically around $100, and writes an essay on why he would like to own the property. The property is awarded to the buyer who submitted the best essay. This scheme is not, strictly speaking, a lottery since selection of the winner is skill-based. But, given the subjectivity of judging essays, the scheme is just a disguised lottery. Lotteries have been explicitly used in the sale of some government-owned assets. The U.S. federal government has used lotteries to allocate radio spectrum and onshore oil and gas leases. However, the fees were nominal, and the use of lotteries does not appear to have been motivated by revenue considerations.

the effective interest rate can be substantial relative to other commercial loans.) Second, the seller may face a significant opportunity cost of providing financing. She may be financially constrained herself, or she may have better investment opportunities. In this case, a financial intermediary may offer loans. Financing by a third party is often less than perfect, however, because of adverse selection problems.[10]

The restriction on lotteries may also create incentives for reducing the size and value of the sales unit. The key insight of the paper is that consumers are able to pay relatively more for smaller units, in the presence of budget constraints. The use of a lottery allows the sale of small "probability" units. When lotteries are infeasible, the seller will have an incentive to divide the good, either physically or intertemporally, or to offer a variety of quality levels.[11] The optimal non-linear pricing (identified in Proposition 1) would then be applied to these units.[12] As noted above, such division of the good may enhance welfare as well. Schemes such as installment payments and royalty payments, which are often used in government auctions, have a similar effect. Likewise, rotating savings and credit assocations (ROSCAs) can enhance welfare by relaxing the budget constraints of the poor.

[10] For instance, "good" borrowers may pay a premium to compensate lenders for the possibility of their funding "bad" borrowers (Myers and Majluf [1984], Greenwald, Stiglitz, and Weiss [1984]).

[11] Deneckere and McAfee (1994) document many episodes where sellers incur costs to use non-linear pricing.

[12] In the extreme case, where the good is perfectly divisible, the seller can implement the optimal mechanism in Proposition 1 through a non-linear pricing scheme that exihibits quantity or quality premia. On the implementation of quantity premia, see Katz (1984).

Appendix A: Proof of Lemma 1

The proof has two separate steps.

Step 1: A simple mechanism $\langle \bar{t}, \bar{x} \rangle$ satisfies (IC′), (IR), (BC) and (PR) if the mapping $T : [0,1] \rightarrow \Re$ is strictly increasing and convex, and $T(0) = 0$.

Proof: By construction of $\langle \bar{t}, \bar{x} \rangle$ in (1), (BC) and (PR) hold trivially. We now show that (IC′) and (IR) hold. First, consider a type (v, w) such that $T(x(v)) \leq w$. By the definition of $x(v)$,

$$vx(v) - T(x(v)) \geq vx - T(x),$$

for all $x \in [0,1]$. In particular, since $T(0) = 0$, the LHS must be non-negative. So, both (IR) and (IC′) hold for such a type.

Second, consider a type (v, w) such that $T(x(v)) > w$. Since $vx - T(x)$ is concave in x and maximized at $x(v)$, it is non-decreasing in x for all $x < x(v)$. In particular, $vx - T(x)$ is non-decreasing for all x such that $T(x) \leq w$ since $w < T(x(v))$. (Recall that $T(\cdot)$ is strictly increasing.) This, along with $T(0) = 0$, implies that the type (v, w) picks $\langle w, T^{-1}(w) \rangle$ and earns non-negative expected surplus. □

Step 2: If a mechanism $\langle t, x \rangle$ solves [S], with $t(v, \cdot)$ non-decreasing for all v, then there exists a simple mechanism $\langle \bar{t}, \bar{x} \rangle$ that is optimal.

Proof: Suppose that $\langle t, x \rangle$ solves [S]. Let $\xi(\cdot) \equiv x(\cdot, \overline{w})$, and define a mapping $\tau :$ range$(\xi) \rightarrow \Re$ such that $\tau(\xi(\cdot)) \equiv t(\cdot, \overline{w})$. Now, we define a new mapping $T : [0,1] \rightarrow \Re$ to be the extension of $\tau(\cdot)$ to [0,1], using linear interpolations (i.e., linking the neighboring ξ's by a linear segment). Clearly, $T(x)$ is uniquely determined for $x \in [\xi(\underline{v}), \xi(\overline{v})]$. It is determined elsewhere as follows. If $\xi(\underline{v}) = 0$, then $T(0)$ is already defined and must equal zero since $\tau(\xi(\underline{v})) = 0$. ((IR) requires $\tau(0) \leq 0$, and revenue can only increase by raising $\tau(\xi)$ to zero.) If $\xi(\underline{v}) > 0$, then we define $T(0) = 0$. Likewise, if $\xi(\overline{v}) < 1$, then we define $T(1) = \tau(\xi(\overline{v})) + (1 - \xi(\overline{v}))\overline{v}$. We again use linear interpolations to complete the definition of $T(\cdot)$.

We now show that $T(\cdot)$ is convex and strictly increasing in (0,1). Let $\xi \equiv \xi(v)$ and $\xi' \equiv \xi(v')$ for arbitrary $v, v' \in [\underline{v}, \overline{v}]$. Then, by (IC), $\tau(\xi') \geq \tau(\xi) + v(\xi' - \xi)$. Fixing ξ, this inequality holds for all ξ'. Thus, there is a supporting hyperplane at ξ with slope equal to v. Therefore, $\tau(\cdot)$ is convex. It is clearly strictly increasing. The linear extensions preserve convexity and strict monotonicty in $(\xi(\underline{v}), \xi(\overline{v}))$. So does the extension in the remaining regions, since (IR) (for $\langle x, t\rangle$) means that the supporting hyperplane at $\xi(\underline{v})$ is (weakly) steeper than the ray through the origin, and since all supporting hyperplanes have slopes (weakly) less than $\overline{v}$. Thus, $T(\cdot)$ is convex and is strictly increasing.

By construction, $x(v) = \xi(v)$ for all v. Then, by Step 1, a simple mechanism $\langle \overline{t}, \overline{x}\rangle$ based on such a mapping satisfies (IC′), (IR), (BC) and (PR). Further, since $t(v, \cdot)$ is non-decreasing for all v, $\overline{t}(v, w) = \min\{T(x(v)), w\} = \min\{t(v, \overline{w}), w\} \geq t(v, w)$ for all (v, w), which implies the optimality of $\langle \overline{t}, \overline{x}\rangle$. □

Combining Steps 1 and 2, we conclude that a simple mechanism whose associated mapping $T(\cdot)$ solves [S1] is optimal. □

Appendix B: Proof of Lemma 2

For the proof, it is convenient to represent a mechanism as $\langle u, x\rangle$, where $u(v,w)$ denotes the expected utility of a type (v,w). Clearly, $t(v,w)$ is uniquely determined once $\langle u, x\rangle$ is determined. We now rewrite the program [S] as follows, using $\langle u, x\rangle$.

$$[AP] \qquad \max_{x(\cdot,\cdot),u(\cdot,\cdot)} \int_{\underline{w}}^{\overline{w}} \int_{\underline{v}}^{\overline{v}} [vx(v,w) - u(v,w)] f(v|w) dv g(w) dw$$

subject to

$$(ENV) \qquad u(v,w) = \int_{\underline{v}}^{v} x(s,w) ds + u(\underline{v}, w) \quad \forall v, w,$$

$$(IR) \qquad u(\underline{v}, \underline{w}) \geq 0$$

$$(XM) \qquad x(\cdot, w) \text{ is non-decreasing} \quad \forall w,$$

$$(UM) \qquad u(v, \cdot) \text{ is non-decreasing} \quad \forall v,$$

$$(BC) \qquad vx(v,w) - u(v,w) \leq w \quad \forall v, w,$$

$$(PR) \qquad 0 \leq x(\cdot,\cdot) \leq 1.$$

The incentive constraints for buyers with a given budget imply (ENV) and (XM). (UM) follows from incentive constraints for buyers with different budgets. Conversely, the above constraints are sufficient for a mechanism to satisfy (IC), (IR), (BC), and (PR) in [S]. It suffices to show that (IC) is satisfied everywhere. ((IR), (BC), and (PR) in [S] are satisfied trivially.) For all (v,w) and (v',w') with $w \geq w'$,

$$\begin{aligned} u(v,w) &\geq u(v,w') \\ &= \int_{\check{v}(w)^+}^{v} x(s,w') ds + u(\check{v}(w)^+, w') \\ &\geq (v - \check{v}(w)^+) x(\check{v}(w)^+, w') + u(\check{v}(w)^+, w') \\ &= vx(\check{v}(w)^+, w') - t(\check{v}(w)^+, w'), \end{aligned}$$

where the first inequality follows from (UM), and the second inequality follows from (XM).

Substituting in (ENV) makes the objective function of [AP]:

$$\int_{\underline{w}}^{\overline{w}} \left\{ \int_{\underline{v}}^{\overline{v}} [vf(v|w) - (1 - F(v|w))] x(v, w) dv - u(\underline{v}, w) \right\} g(w) dw. \tag{B1}$$

We first establish the following lemma.

Lemma B1. Given Assumption 1, there exists an optimal mechanism in which $x(v, \cdot)$ is non-decreasing.

Proof. We show that for any feasible mechanism $\langle u, x \rangle$ for which $x(v, \cdot)$ decreases somewhere, there exists an alternative feasible mechanism for which $x(v, \cdot)$ is non-decreasing, and revenue is (weakly) higher. Specifically, if there exist w_1 and w_2, $w_1 < w_2$, such that $x(v, w_1) > x(v, w_2)$ for some v, then there is a mechanism $\langle \overline{u}, \overline{x} \rangle$ with $\overline{x}(v, w_1) \leq \overline{x}(v, w_2)$ for all v, that yields (weakly) higher revenue than $\langle t, x \rangle$.

We construct the new mechanism in the following way. First, let $\overline{x}(\cdot, w) \equiv x(\cdot, w)$ and $\overline{u}(\underline{v}, w) \equiv u(\underline{v}, w_1)$, for all $w \notin (w_1, w_2]$. Next, we let $\overline{u}(\underline{v}, w_2) \equiv u(\underline{v}, w_1)$ and let

$$\overline{x}(v, w_2) \equiv \begin{cases} x(v, w_1) & \text{if } v < \tilde{v}; \\ \max\{x(v, w_1), x(v, w_2)\} & \text{if } v \geq \tilde{v}, \end{cases} \tag{B2}$$

such that

$$u(\overline{v}, w_2) = \int_{\underline{v}}^{\overline{v}} x(v, w_2) dv + u(\underline{v}, w_2) = \int_{\underline{v}}^{\overline{v}} \overline{x}(v, w_2) dv + u(\underline{v}, w_1) = \overline{u}(\overline{v}, w_2). \tag{B3}$$

If no such $\tilde{v}$ exists (i.e., $u(\overline{v}, w_2) > \overline{u}(\overline{v}, w_2)$ for all $\tilde{v} \in [\underline{v}, \overline{v}]$), then let $\tilde{v} \equiv \underline{v}$ and pick $\overline{u}(\underline{v}, w) \in [u(\underline{v}, w_1), u(\underline{v}, w_2)]$ so that $u(\overline{v}, w_2) = \overline{u}(\overline{v}, w_2)$.

For $w \in (w_1, w_2)$, we define

$$\overline{u}(\cdot, w) \equiv \text{conv}\{u(\cdot, w), \overline{u}(\cdot, w_2)\}.$$

That is, $\overline{u}(\cdot, w)$ is the highest convex function α such that $\alpha(v, w) \leq \min\{u(v, w), \overline{u}(v, w_2)\}$ for all v (see RockaFellar [1972], p. 37). Note that $\overline{u}(\overline{v}, w) = u(\overline{v}, w)$, by (UM). Since $\overline{u}(\cdot, w)$ is convex, it is almost everywhere differentiable. We define $\overline{x}(\cdot, w)$ to be an arbitrary subgradient of $\overline{u}(\cdot, w)$ for all w.

Now, we show that the new mechanism satisfies all the constraints.

(i) (XM): (XM) is satisfied for all $w \in (w_1, w_2)$ since $\overline{u}(\cdot, w)$ is convex. It is also satisfied for w_2 since both $x(\cdot, w_1)$ and $x(\cdot, w_2)$ are themselves non-decreasing.

(ii) (BC): We first show that, for all $w \in (w_1, w_2]$,

$$\overline{x}(\overline{v}, w) \leq \max\{x(\overline{v}, w_1), x(\overline{v}, w)\}.$$

This inequality holds for $w = w_2$, by definition. For a given $w \in (w_1, w_2)$, it can be shown as follows. If $\overline{u}(v, w_2) \geq u(v, w)$ for all v, then $\overline{u}(v, w) = u(v, w)$ for all v, so $\overline{x}(\overline{v}, w) = x(\overline{v}, w)$. Therefore, assume that $\overline{u}(v, w_2) < u(v, w)$ for some v and that $\overline{x}(\overline{v}, w) > x(\overline{v}, w)$. Let $V \equiv \sup\{v | \overline{u}(v, w_2) < u(v, w)\}$. Convexity of $\overline{u}(\cdot, w_2)$ implies that $\overline{x}(\overline{v}, w) \leq \overline{x}(V, w_2)$. Meanwhile, that $\overline{u}(v, w_2) < u(v, w) \leq u(v, w_2)$ for some $v > V - \epsilon$, $\epsilon > 0$, implies that $x(v', w_1) > x(v', w_2)$ for some $v' \geq V$. (This last step follows from (B2) and (B3).) Combining these two facts yields $\overline{x}(\overline{v}, w) \leq x(\overline{v}, w_1)$ in this case. We conclude that $\overline{x}(\overline{v}, w) \leq \max\{x(\overline{v}, w_1), x(\overline{v}, w)\}$.

Now consider the transfer under the new mechanism for all v and for $w \in (w_1, w_2]$:

$$\begin{aligned}
\overline{t}(v|w) \leq& \overline{t}(\overline{v}, w) \\
=& \overline{x}(\overline{v}, w)\overline{v} - \overline{u}(\overline{v}, w) \\
\leq& \max\{x(\overline{v}, w)\overline{v} - u(\overline{v}, w), x(\overline{v}, w_1)\overline{v} - u(\overline{v}, w)\} \\
\leq& \max\{x(\overline{v}, w)\overline{v} - u(\overline{v}, w), x(\overline{v}, w_1)\overline{v} - u(\overline{v}, w_1)\} \\
=& \max\{t(\overline{v}, w), t(\overline{v}, w_1)\} \\
\leq& w.
\end{aligned}$$

The first inequality follows since $\overline{x}(\cdot, w)$ is non-decreasing, the first equality follows from the definition, the second inequality follows from the above argument and the fact that $\overline{u}(\overline{v}, w) = u(\overline{v}, w)$, the third inequality follows from (UM) on the original mechanism, the second equality follows from definition, and the last inequality follows from the fact that $u(\overline{v}, w) \geq u(\overline{v}, w_1)$ and from (BC) for the original mechanism.

(iii) (PR): By construction, $\overline{x}(\underline{v}, w) \geq \min\{x(\underline{v}, w_1), x(\underline{v}, w_2), x(\underline{v}, w)\} \geq 0$. It also follows from an argument in (ii) that $\overline{x}(\overline{v}, w) \leq \max\{x(\overline{v}, w_1), x(\overline{v}, w)\} \leq 1$. Finally, (i) completes the proof.

(iv) (UM): Consider any w' and w'' such that $w_1 \leq w' \leq w'' \leq w_2$. Then, for all v, $\overline{u}(v, w') \leq \overline{u}(v, w'')$ since $\min\{u(v, w'), \overline{u}(v, w_2)\} \leq \min\{u(v, w''), \overline{u}(v, w_2)\}$ for all v, which shows that (UM) is satisfied over $[w_1, w_2]$. Next, observe that $\overline{u}(v, w) = u(v, w)$ for all $w \notin (w_1, w_2]$ and that $\overline{u}(v, w_2) \leq u(v, w_2)$ for all v. The latter fact follows since, for $v \leq \tilde{v}$, $\overline{u}(v, w_2) = u(v, w_1) \leq u(v, w_2)$, and for $v > \tilde{v}$,

$$\overline{u}(v, w_2) = \overline{u}(\overline{v}, w_2) - \int_v^{\overline{v}} \overline{x}(s, w_2) ds \leq u(\overline{v}, w_2) - \int_v^{\overline{v}} x(s, w_2) ds = u(v, w_2),$$

since $\overline{x}(v, w_2) \geq x(v, w_2)$ in that interval and $\overline{u}(\overline{v}, w_2) = u(\overline{v}, w_2)$. We conclude that (UM) is satisfied globally.

Finally, we show that the new mechanism yields (weakly) higher expected revenue than the original mechanism. Since the new mechanism assigns the same contracts to all $w \notin (w_1, w_2]$ as the old mechanism, we focus on $w \in (w_1, w_2]$. By construction, $\overline{u}(v, w) \leq u(v, w)$ for all v. Then, by (1) and integration by parts, the expected revenue extracted from consumers with budget w is

$$\begin{aligned}
&\int_{\underline{v}}^{\overline{v}} \overline{t}(v, w) f(v|w) dv \\
=& \int_{\underline{v}}^{\overline{v}} H(v|w) \overline{x}(v, w) dv - \overline{u}(\underline{v}, w) \\
=& H(\overline{v}|w) \overline{u}(\overline{v}, w) - \int_{\underline{v}}^{\overline{v}} \frac{\partial H(v|w)}{\partial v} \overline{u}(v, w) dv - [H(\underline{v}|w) + 1] \overline{u}(\underline{v}, w) \\
\geq& H(\overline{v}|w) u(\overline{v}, w) - \int_{\underline{v}}^{\overline{v}} \frac{\partial H(v|w)}{\partial v} u(v, w) dv - [H(\underline{v}|w) + 1] u(\underline{v}, w) \\
=& \int_{\underline{v}}^{\overline{v}} H(v|w) x(v, w) dv - u(\underline{v}, w) \\
=& \int_{\underline{v}}^{\overline{v}} t(v, w) f(v|w) dv,
\end{aligned}$$

since $\frac{\partial H(v|w)}{\partial v}$ is positive by Assumption 1, $\overline{u}(\overline{v}, w) = u(\overline{v}, w)$, and $\overline{u}(\cdot, w) \leq u(\cdot, w)$. □

By Lemma B1, we can replace (UM) by the following, stronger constraint:

$$x(v, \cdot) \text{ is non-decreasing } \forall v. \tag{XM$'$}$$

From now on, we work with program [AP], with (UM) replaced by (XM′). Given (XM′), we conclude that $u(\underline{v}, \cdot) = 0$ in the optimal mechanism. (Setting $u(\underline{v}, \cdot) = 0$ does not violate any constraints, while increasing the seller's expected revenue.)

Lemma B2. Given Assumptions 1 and 2, if a mechanism $\langle t, x\rangle$ solves [S], then $t(v, \cdot)$ is non-decreasing.

Proof. By Lemma B1, it suffices to show that the solution to [AP] has $t(v, \cdot)$ non-decreasing for all v. To this end, we fix an arbitrary (v, w^*), and show that $t(v, w) \geq t(v, w^*)$ for all $w > w^*$. Without loss of generality, we assume that the type-(v, w^*) buyer is not budget constrained. (We can focus on unconstrained of consumers since, for constrained consumers, expected payments are equal to their budgets and thus are increasing in budgets.)

Recall that Assumption 2 implies that $m(\cdot)$ (defined in the text) is non-decreasing. From (1) it follows that $x(v, w) = 1$ for all $(v, w) \geq (m(\overline{w}), w^*)$. (Setting the highest probability for these types maximizes revenue without violating any constraints.) Similarly, $x(v, w) = x(v, w^*)$ for all $v < m(w^*), w \geq w^*$. (Here, minimizing the probability maximizes the expected revenue, and the minimum probability is equal to $x(v, w^*)$, due to the (XM$'$) constraint.) Now consider $v \in (m(w^*), m(\overline{w}))$. We show that for each such v, $x(v, w)$ is constant for all $w \geq w^*$. Define $\hat{w}(v)$ such that $v = m(\hat{w}(v))$. Again, for all $w > \hat{w}(v)$, minimizing probabilities is optimal, so $x(v, w) = x(v, \hat{w}(v))$, while, for all $w \in (w^*, \hat{w}(v))$, maximizing probabilities is optimal, so $x(v, w) = x(v, \hat{w}(v))$. That is, in both cases, (XM') is binding. $x(\cdot, \cdot)$ thus constructed also satisfies (XM): for $v < m(w^*)$, $x(v, w) = x(v, w^*)$ is non-decreasing in v by (XM) in the original mechanism; for $v \in (m(w^*), m(\overline{w}))$, $x(v, w) = x(v, \hat{w}(v))$ is increasing in v since $\hat{w}(\cdot)$ is non-decreasing.

From (ENV), $x(v, w) = x(v, w^*)$ for all v and all $w \geq w^*$ implies that $t(v, w) = t(v, w^*)$ for all v and all $w \geq w^*$. Therefore, $t(v, \cdot)$ is non-decreasing. □

Appendix C: Proof of Proposition 1

We use a change of variables to simplify [S1]. Given a feasible function $T(\cdot)$, let $Q(x)$ be the probability that a randomly chosen consumer picks a contract on $T(\cdot)$ that entails a probability of receipt less than $x \in [0,1]$. This event occurs if $w < T(x)$, or if $w \geq T(x)$ and $x(v) < x$. Since $x(v) < x$ if and only if $v < T'(x)$, for almost every $x \in [0,1]$,

$$Q(x) = 1 - \text{Prob}\{v \geq T'(x) \text{ and } w \geq T(x)\} = 1 - D(T'(x), T(x)),$$

where $D(T'(x), T(x)) \equiv \int_{\max\{T(x),\underline{w}\}}^{\overline{w}} [1 - F(T'(x)|w)]g(w)dw$. ($T'(\cdot)$ is well-defined almost everywhere since $T(\cdot)$ is convex.)

Integrating by parts and using $T(0) = 0$, the seller's expected revenue can be expressed as

$$\int_0^1 T(x)dQ(x) + T(1)[1 - Q(1)] = \int_0^1 T'(x)D(T'(x), T(x))dx.$$

We can now rewrite [S1] as an optimal control problem:

$$\max_{u(\cdot)} \int_0^1 u(x)D(u(x), T(x))dx$$

$$\text{s.t.} \qquad u(\cdot) = T'(\cdot),$$

$$T(\cdot) \text{ is strictly increasing and convex, and } T(0) = 0.$$

In solving the problem, we confine attention to the case where $T(1) < \overline{w}$. We later consider a sufficient condition for this to hold. Furthermore, we ignore the constraint that $T(\cdot)$ be strictly increasing and convex, and instead consider a relaxed program with $T(0) = 0$ as the only constraint. We later prove that the constraints are satisfied by a solution to the relaxed problem, given Assumptions 1 and 2.

We first establish the existence of a solution to the relaxed program. Without loss of generality, the control variable can be restricted to a compact set $[0, \overline{v}]$. Furthermore, the integrand $uD(u,T)$ is concave in u. (It is linear for $u \in (0, \underline{v})$ and strictly concave for $u \in (\underline{v}, \overline{v})$, by Assumption 1.) Therefore, a solution to the relaxed program exists (see Fleming and Rishel [1975], pp. 68-70).

Lemmas 1 and 2 then imply that a simple mechanism based on the solution to the relaxed program solves [S], provided that the solution satisfies the covexity constraint.

We now characterize the necessary conditions for the solution. The Hamiltonian for the relaxed program can be written as

$$J \equiv uD(u,T) + \lambda u.$$

Its derivative with respect to u, $D + uD_1 + \lambda$, is strictly positive for $u < \underline{v}$ and is discontinuous at $u = \underline{v}$, where D_i denotes the partial derivative of D with respect to the i'th argument. ($D_1 < 0$ for $u > \underline{v}$ and $D_1 = 0$ for $u < \underline{v}$.) Therefore, the first-order necessary conditions are:

$$\frac{\partial J}{\partial u} = D + uD_1 + \lambda = 0 \text{ if } u \neq \underline{v} \qquad (C1)$$

$$\frac{\partial J}{\partial u} \leq 0 \text{ if } u = \underline{v},$$

$$\lambda' = -\frac{\partial J}{\partial T} = -uD_2. \qquad (C2)$$

The free-end condition, $\lambda(1) = 0$, and (C2) imply that $\lambda(x) = \int_x^1 u(x')D_2(u(x'), T(x'))\, dx'$. Substituting $\lambda(x)$ into (C1), differentiating with respect to x, and substituting $u(x) = T'(x)$ yields the differential equation:

$$\frac{d[D(T'(x), T(x)) + T'(x)D_1(T'(x), T(x))]}{dx} = T'(x)D_2(T'(x), T(x)), \qquad (C3)$$

with the initial condition $T(0) = 0$. The solution to [S1] satisfies (C3), unless $T'(x) = \underline{v}$. Likewise, (C1) and the free-end condition imply that

$$D(T'(1), T(1)) + T'(1)D_1(T'(1), T(1)) = -\int_{\max\{T(1),\underline{w}\}}^{\overline{w}} H(T'(1)|w)g(w)dw = 0,$$

or $T'(1) = \hat{v}(T(1))$.

Suppose that $T(1) \leq \underline{w}$. Then, the right-hand side of (C3) vanishes since $D_2 = 0$ for all $T(x) < \underline{w}$. Therefore, (C3) can be written as:

$$-T''(x)\int_{\underline{w}}^{\overline{w}} \left\{\frac{\partial H(T'(x)|w)}{\partial T'}\right\} g(w)dw = 0,$$

so $T(\cdot)$ is linear because the integral is strictly positive by Assumption 1. Since $T'(1) = \hat{v}(\underline{w})$ and $T(0) = 0$, $T(x) = \hat{v}(\underline{w})x \; \forall x$. $T(\cdot)$ is strictly increasing and convex. Clearly, this is the optimal mechanism if $\hat{v}(\underline{w}) \leq \underline{w}$ and it is implemented by the simple mechanism in (i).

Now consider case (ii). Again, (C3) and the initial condition $T(0) = 0$ characterize the optimal mechanism if $T'(\cdot) \neq \underline{v}$. As above, D_2 vanishes for $T(x) < \underline{w}$, so the solution to the differential equation is linear for that segment. For $T(x) \geq \underline{w}$ and $T'(x) > \underline{v}$, (C3) can be rewritten as

$$\frac{T'(x)^2}{T''(x)} f(T'(x)|T(x)) = \frac{\int_{T(x)}^{\overline{w}} \{2f(T'(x)|w) + T'(x)f'(T'(x)|w)\}\, g(w)\, dw}{g(T(x))}. \tag{C4}$$

In sum, there exists $v^* \in [\max\{\underline{w}, \underline{v}\}, T(1)]$ and $x^* \in [\underline{w}/v^*, 1]$ such that $T(x) = v^* x$ for $x < x^*$, and $T(x)$ obeys (C4) for $x \geq x^*$.

$T(\cdot)$ is strictly increasing, since (C1) is violated otherwise. To see that $T(\cdot)$ is convex, observe that the right-hand side of (C4) is strictly positive for $T(x) > \underline{w}$, which implies that $T''(x) > 0$.[13] It now suffices to show that the right-hand limit of $T'(x)$ at x^* is greater than or equal to v^*. This is clearly satisfied if $v^* = \underline{v}$, by (C1). Therefore, suppose $v^* > \underline{v}$. In this case, (C1) implies that $T'(\cdot)$ must be continuous everywhere, so we conclude that $T(\cdot)$ is convex. (Observe that D, D_1 and λ are all continuous, which implies that $u = T'$ is continuous.)

By the free-end condition ($T'(1) = \hat{v}(T(1))$), consumers with $v \geq \hat{v}(T(1))$ and $w \geq T(1)$ pick the contract $\langle T(1), 1\rangle$. All other consumers with $v \geq v^*$ pick contracts with $x < 1$, described by (ii). If $\hat{v}(\underline{w}) > \underline{v}$, then $v^* < \hat{v}(T(1))$. Suppose, to the contrary, that $v^* = \hat{v}(T(1))$. Then, by the above argument, $T(x) = \hat{v}(T(1))x$ for all $x \in [0, 1]$. If x is very close to 1, however, then $T(x) > \underline{w}$. It follows that the right-hand side of (C4) is strictly positive, which implies that $T''(x) > 0$, contradicting $v^* = \hat{v}(T(1))$.

In all cases, $\hat{v}(T(1)) \leq \hat{v}(\overline{w}) < \overline{w}$, so our restriction to the case where $T(1) < \overline{w}$ entails no loss of generality. □

[13] The integrand in the RHS can be rewritten as $\frac{\partial H(T'(x)|w)}{\partial T'}$, which is strictly positive because of Assumption 1.

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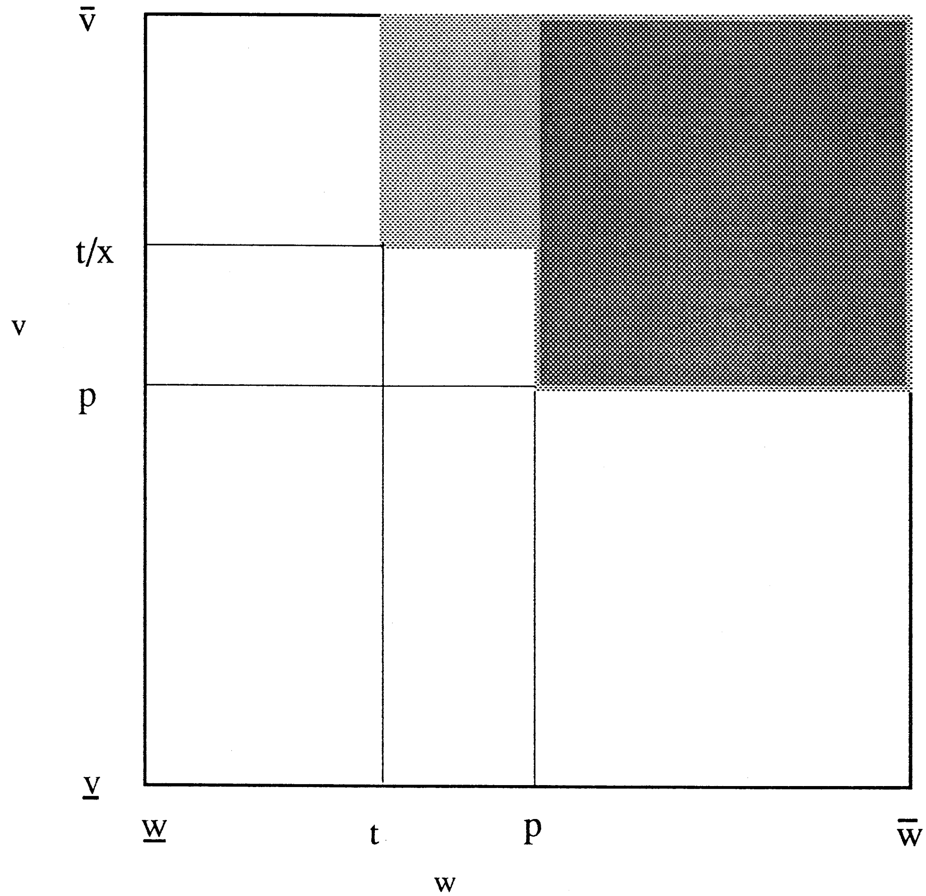

Figure 1

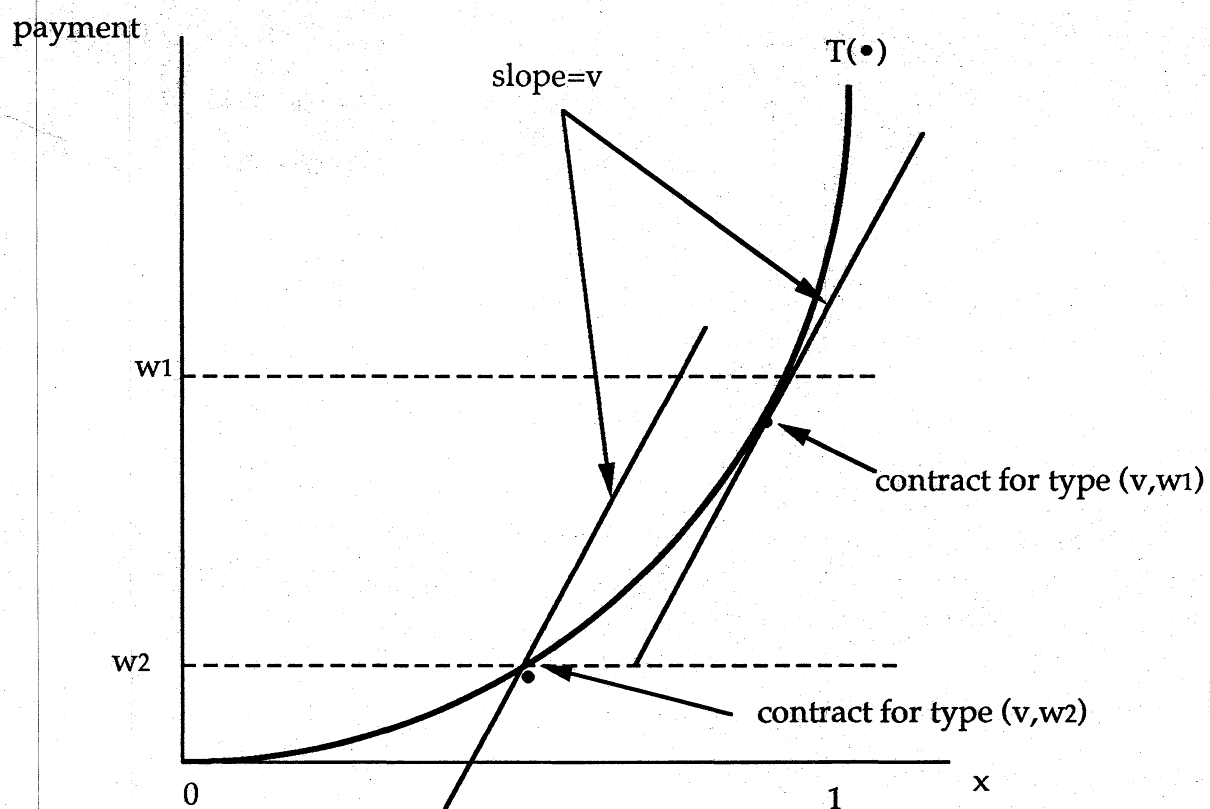

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Figure 2

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