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AGGREGATE EFFICIENCY, MARKET DEMAND, AND THE SUSTAINABILITY OF COLLUSION

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ABSTRACT

The effects of firms' aggregate productive efficiency and the level of demand on collusion are studied by considering very general collections of Cournot markets that differ in these variables. The results are very general. Firms are not required to be identical within nor across markets and the number of firms and level of demand are allowed to vary without restriction across markets. It is found that collusion can be ruled out in markets where aggregate efficiency (i.e. the ability of firms to profitably produce at prices exceeding the Cournot price) is large relative to the level of demand.

1. Introduction

In a well known passage Adam Smith [8] echoed a sentiment that was undoubtedly felt even in that dim past when market activity began. "People of the same trade seldom meet together, even for merriment and diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices." Of course each participant has a short run incentive to cheat on the conspiracy by expanding output or reducing price. However economists have understood at least since Friedman [3] that sellers may maintain collusion by means of credible threats if interest rates are sufficiently low. Friedman pointed out that simply the threat of refusing to cooperate after a defection may be enough to sustain collusion. More recently, Abreu [1] has shown that even more severe threats may be credible.

Given these possibilities it is important to understand which factors facilitate collusion. It has been widely believed, for example, that collusion is impossible when there are many firms. However an example by Green [5] demonstrated that if firms' actions are observable collusion may be sustainable in arbitrarily large markets. (Green mentioned medical doctors and other professionals as producers who are often believed to behave noncompetitively despite their large numbers.) Hence a large number of firms is not sufficient to eliminate the possibility of collusion. To understand which factors affect collusion one must turn elsewhere. Green turned elsewhere by assuming that firms' outputs are not directly observable and adding a stochastic element to demand. Under such assumptions firms observing a drop in price cannot know whether it resulted from an increase in output by one of the firms or simply from a low level of demand. This uncertainty reduces the feasiblity of collusion, implying that the availability of information is an important determinant of the firms' ability to collude.

This paper explores two other factors: the aggregate efficiency of the firms and the role of demand. To loosely describe what is to be made precise below, high aggregate efficiency tends to reduce firms' ability to collude while high levels of demand either have no effect or tend to increase the firms' ability to collude. Aggregate efficiency refers to the capacity of firms to profitably produce given prices exceeding the noncollusive price.

Section 2 describes the modeling techniques to be used. To study the effects of aggregate efficiency and market demand, collections of markets differing in these variables are considered. The main theorem is stated and discussed in section 3; its proof is relegated to the appendix. The result is very general. Specifically, firms need not be identical within nor across the markets to be compared. Indeed, they need not be related in any way. Furthermore, neither the level of demand nor the number of firms need vary in any systematic way across markets. Section 4 contains concluding remarks.

2. Definitions and Assumptions

Consider an infinite collection of markets indexed by $m = 1,2,\ldots$. In each market the firms exist for an infinite number of periods and make output decisions simultaneously at the beginning of each period, t. (The t will usually be suppressed for notational simplicity.) No legally binding agreements are permitted. Consider an arbitrary market m. In market m there are N_m firms indexed by $i = 1,\ldots,N_m$. Firm i in market m has cost function $C_{im}(\cdot)$. In each period q_{im} is the i^{th} firm's output, $Q_m \equiv \Sigma_i q_{im}$, $Q_{im} \equiv \Sigma_{j\neq i} q_{jm}$ and q_m is the vector $(q_{1m}, \ldots, q_{N_m m})$. Firm i's profit is $\pi_{im}(q_{im},Q_{im}) = q_{im}P_m(Q_m) - C_{im}(q_{im})$ where $P_m(\cdot)$ is the inverse demand function. It is assumed that $P_m(\cdot)$ is of the form $P_m(Q_m) = f(Q_m/S_m)$ for some function f where S_m is a positive number. S_m may vary in any way across markets. (Thus if $S_A > S_B$ then the level of demand is greater in market A than in market B.) Similarly the number of firms may vary in any way across markets. S_m and N_m need not be correlated with m nor need they be correlated with each other. The only restrictions are that N_m be an integer strictly greater than one and that S_m be positive for all m.

Cournot [2] suggested an outcome that is often asociated with an absence of collusion. A Cournot output vector for market m is denoted q_m^c and satisfies $\pi_{im}(q_{im}^c, Q_{im}^c) = \max_{q_{im}} \pi_{im}(q_{im}, Q_{im}^c)$ for all i; that is each firm maximizes current profit given the output choices by the other firms. The Cournot equilibrium arises when each firm, i, adopts the strategy of choosing q_{im}^c in each period. The Cournot equilibrium, however, is not Pareto efficient from the firms' point of view. Small reductions in output by all firms would increase the profits of all active firms. (See e.g. Friedman [4, page 25].) Hence there is an incentive for firms to collude.

A vector q_m^* is a collusive output vector if $\pi_{im}(q_{im}^*, Q_{im}^*) \geq \pi_{im}(q_{im}^c, Q_{im}^c)$ for all i with strict inequality for some i. The corresponding price, $f(Q_m^*/S_m)$, is a collusive price. All firms are at least as well off when q_m^* is chosen as when q_m^c is chosen, but since q_m^* is not a Cournot output vector some firms have an incentive to change output and increase current profit. Since binding agreements to produce q_m^* are prohibited, q_m^* can only be achieved if each firm realizes that the current gain from deviating will be outweighed by future losses when the other firms retaliate. It facilitates the discussion to think of collusion as arising from a nonbinding agreement (although the results clearly do not depend on this device).

Each firm agrees (either explicitly or tacitly) to produce according to q_m^* as long as everyone else does. Firms also agree on what measures to take if some firm cheats on the agreement. In the absence of legally binding agreements the threatened measures must be self-enforcing (i.e. subgame perfect in the sense of Selten [7]). That is, in the wake of a defection each firm must find it in its own interest to go through with the threat given that the other firms do so. There are many varieties of self-enforcing threats. (See Abreu [1] for a fuller, more formal, discussion.) For illustrative purposes attention is here restricted to grim trigger strategy equilibria because of their familiarity and tractability. As is made clear in section 4, however, the results are easily extended to include all subgame perfect threats. Grim trigger strategy equilibria will now be defined. For each firm i define

(2.1) $\pi^c_{im} \equiv \pi_{im}(q^c_{im}, Q^c_{im})$ as Cournot profit,

$\pi^*_{im} \equiv \pi_{im}(q^*_{im}, Q^*_{im})$ as collusive profit, and

$\pi^d_{im} \equiv \max_{q_{im}} \pi_{im}(q_{im}, Q^*_{im})$ as (optimal) deviation profit.

A trigger strategy equilibrium is a list of strategies of the form

$$q_{im}(1) = q^*_{im}$$

(2.2) $$q_{im}(t) = q^*_{im} \text{ if } q_{jm}(T) = q^*_{jm},\ j \neq i,\ T=1, \ldots, t-1;\ t=2,3, \ldots$$

$$q_{im}(t) = q^*_{im} \text{ otherwise}$$

such that $\pi^*_{im} \geq \pi^c_{im}$ for all i, $\pi^*_{im} > \pi^c_{im}$ for some i, and

(2.3) $$(\alpha/1-\alpha)[\pi^*_{im} - \pi^c_{im}] \geq \pi^d_{im} - \pi^*_{im} \text{ for all i}$$

where α is the discount factor used by the firms. (α is independent of m.)

The trigger strategy described in (2.2) says that firm i will abide by an agreement to produce according to q^*_m for as long as all other firms do so, but if any firm deviates from its agreed upon output firm i will "retaliate" by producing q^c_{im} in all later periods. Condition (2.3) says that the current gain to firm i should it deviate from the agreement is outweighed by the discounted sum of the foregone gains from future collusion. Hence a trigger strategy equilibrium will exhibit production of q^*_m in all periods. It is easy to see that the threat to produce at Cournot levels is self-enforcing. Given that all other firms produce at Cournot levels in all periods, the definition of q^c_m guarantees that an individual firm can do no better by doing otherwise.

A collusive price $f(Q^*_m/S_m)$ is said to be <u>sustainable (by α)</u> if there exists an output vector q^*_m satisfying (2.3) and such that $\Sigma_i q^*_{im} = Q^*_m$. It is said that <u>K-collusion is sustainable in all markets</u> if for some $\alpha < 1$ there is, in each market, a sustainable collusive price at least K larger than the Cournot price, where $K > 0$. Note that if K-collusion is <u>not</u> sustainable in all markets for <u>any</u> $K > 0$ then, for any $\alpha < 1$, in all but a finite number of markets only collusive outcomes arbitrarily close to Cournot outcomes are sustainable. These "trivial" collusive outcomes that are sustainable simply because they are not significantly different from Cournot outcomes are ignored.

The following assumptions are common in the Cournot literature. They are employed here because they simplify the proofs and the exposition. They also have the virtue of guaranteeing that a Cournot equilibrium exists in each market. No attempt is made to use the weakest possible assumptions.

A1. There exists $Z > 0$ such that if $(Q_m/S_m) \geq Z$, $f(Q_m/S_m) = 0$.

A2. $f(\cdot)$ is differentiable on $[0,Z)$ and $f'(\cdot) < 0$.

A3. For all i and m, $C_{im}(\cdot)$ is twice differentiable, $C_{im}(0) = 0$, $C'_{im}(\cdot) \geq 0$, and $C''_{im}(\cdot) \geq 0$.

A4. $\pi_{im}(\cdot,\cdot)$ is quasiconcave in q_{im} for all i and all m.

This paper reports a condition which is both necessary and sufficient for K-collusion to be sustainable in all markets. Only A1-A4 are used to prove that this condition is necessary. To prove that it is also sufficient an additional technical assumption is required. Let $B(q_m^c)$ be the set of collusive output vectors and define the set $\Omega(\epsilon)$ as follows:

$$\Omega(\epsilon) \equiv \{\{q_m\} \mid \text{(a) } q_m \epsilon B(q_m^c) \text{ for all } m;$$
$$\text{(b) } \epsilon > f(Q_m/S_m) - f(Q_m^c/S_m) > \delta^1 \text{ for all } m \text{ and some } \delta^1 \epsilon (0,\epsilon);$$
$$\text{(c) } [\Sigma_i \pi_{im}(q_{im},Q_{im}) - \Sigma_i \pi_{im}(q_{im}^c,Q_{im}^c)]/S_m > \delta^2 \text{ for all } m \text{ and some } \delta^2 > 0\}$$

where $\{q_m\}$ denotes a collection of output vectors corresponding to the collection of markets. Then the additional assumption is:

A5. $\Omega(\epsilon)$ is nonempty for all $\epsilon > 0$.

A5 rules out pathological cases where total profits cannot be significantly increased above Cournot levels in all markets by significant reductions in output. For an example of this possibility see Lambson [6, section 3]. That paper contains a special case of the model employed here: $N_m = m$, $S_m = m^s$ where $s \geq 0$, and all firms are identical within and across markets.

3. The Sustainability of Collusion

For firm i in market m let $\bar{q}_{im}(\epsilon)$ be the maximum value of q_{im} that satisfies

$$(3.1) \qquad f(Q_m^c/S_m) + \epsilon \geq C'_{im}(q_{im})$$

If (3.1) is not satisfied for any $q_{im} \geq 0$ then let $\bar{q}_{im}(\epsilon) = 0$. (Note that $\bar{q}_{im}(\epsilon)$ may be infinite.) Then $\bar{q}_{im}(\epsilon)$ would maximize firm i's profit if firm i were a pricetaker facing $f(Q_m^C/S_m) + \epsilon$. Define $e_{im}(\epsilon) \equiv \min[\bar{q}_{im}(\epsilon), S_m Z]$. $S_m Z$ is the upper bound on demand given by A1; a firm's ability to produce above that level is always irrelevant. The sum $\Sigma_i e_{im}(\epsilon)$ will be used to measure the "aggregate efficiency" of the firms, i.e. their aggregate ability to produce at prices exceeding the Cournot price. (The measure depends, of course, on ϵ. The sum $\Sigma_i e_{im}(\epsilon)$ will be called the aggregate ϵ-efficiency in market m.) The theorem can now be stated.

Theorem: K-collusion is sustainable in all markets for some $K > 0$ if and only if for some $\epsilon > 0$ $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded independently of m.

Proof: See appendix.

The theorem is proved by showing that if $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is unbounded K-collusion is not sustainable in all markets if $K > \epsilon$ while if $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded then K-collusion is sustainable in all markets if $K < \epsilon$. Less formally, given the discount factor $\alpha < 1$, a collusive price cannot be sustained if the firms' aggregate ability to produce at prices below the collusive price is large relative to demand. Hence high aggregate efficiency tends to reduce the range of sustainable prices while high demand either tends to increase it or has no effect. (High demand has no effect if $\bar{q}_{im}(\epsilon) = S_m Z$ for all i and ϵ, in which case $\Sigma_i e_{im}(\epsilon)$ varies with S_m. This occurs, for example, when all firms have identical constant marginal cost, as will be discussed below.) The intuition for these results is as follows: If a firm is to be enticed to abide by a collusive agreement, its gains from collusion must not be too small relative to its gains from breaking the agreement. High levels of aggregate efficiency imply that the gains from cheating are large, making it impossible to allocate to every firm sufficient collusive profit to outweigh the incentives to leave the cartel. By contrast, high levels of demand may help by increasing the gains to collusion. It is now clear that the existence of a large number of firms is not inconsistent with the sustainability of K-collusion. Indeed, even when S_m is constant there can be K-collusion by an unbounded number of firms. (An example is provided below.) Note, however, that if N_m (the number of firms in market m) is bounded independently of m then $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded no matter how the S_m are chosen. This indicates that having a small number of firms is sufficient (but not necessary) for K-collusion to be sustainable in all markets.

Some insights suggested by the theorem will be illustrated with examples. The theorem makes clear the importance of the interplay between aggregate efficiency and demand levels for the sustainability of collusion. Consider the special case where $N_m = m$ in all markets and $S_m = m^s$ where s is a real number. Then s varies less than proportionally to, proportionally to, and more than proportionally to changes in the number of firms as $s < 1$, $s = 1$, and $s > 1$, respectively. For $s = 0$ demand is independent of m and for $s < 0$ demand is lower in markets with more firms. Consider two types of firms (type A and type B) and let A_m and B_m denote their sets in market m. Let $\#A_m$ and $\#B_m$ be the number of type A and type B firms, respectively, in market m and define their cost functions by

$$C_{im}(q_{im}) = 0 \qquad i \in A_m$$
$$C_{im}(q_{im}) = (1/2)(q_{im})^2 \qquad i \in B_m$$

Let inverse demand be linear:

$$f(Q_m/S_m) = 1 - (Q_m/m^s)$$

Consider any $\epsilon > 0$ such that $f(Q_m^c/S_m) + \epsilon$ is a collusive price in all markets. (Clearly $\epsilon < 1$.) Then

$$(3.2) \qquad e_{im}(\epsilon) = m^s \qquad i \in A_m$$

For $i \in B_m$ the marginal cost of producing ϵ is ϵ, so $\bar{q}_{im}(\epsilon) \geq \epsilon$ since $f(Q_m^c/S_m) + \epsilon > \epsilon$. Hence the definition of $e_{im}(\epsilon)$ with $S_m Z = m^s$ implies

$$(3.3) \qquad e_{im}(\epsilon) \geq \min[\epsilon, m^s] \quad i \in B_m$$

Of course

$$(3.4) \qquad e_{im}(\epsilon) < 1 \qquad i \in B_m$$

Consider $s < 1$. If $\min[\epsilon, m^s] = m^s$ then (3.2) and (3.3) imply

$$(3.5) \qquad \Sigma_i e_{im}(\epsilon)/m^s \geq m(m^s)/m^s = m$$

while if $\min[\epsilon, m^s] = \epsilon$ they imply

$$\begin{aligned}(3.6) \qquad \Sigma_i e_{im}(\epsilon)/m^s &\geq [(\#A_m)m^s + (\#B_m)\epsilon]/m^s \\ &= [(\#A_m)m^s + (m - (\#A))\epsilon]/m^s \\ &= [(\#A_m)(m^s - \epsilon) + m\epsilon]/m^s \\ &\geq m\epsilon/m^s\end{aligned}$$

With $s < 1$ both (3.5) and (3.6) imply that $\{\Sigma_i e_{im}(\epsilon)/m^s\}$ is unbounded so K-collusion is not sustainable in all markets. Specifically, for any $K > 0$ and any $\alpha < 1$ exists a number of firms $m^*(K,\alpha)$ such that if $m > m^*(K,\alpha)$ then prices K higher than Cournot prices are not sustainable by the discount factor α. If $s \geq 1$, however, different results emerge. Note that (3.2) and (3.4) imply

$$(3.7) \qquad \Sigma_i e_{im}(\epsilon)/m^s \leq (\#A_m) + [(\#B_m)/m^s]$$

Since $s \geq 1$, $\#B_m \leq m^s$ so if $\{\#A_m\}$ is bounded K-collusion is sustainable in all markets. Of course $\Sigma_i e_{im}(\epsilon)/m^s \geq (\#A_m)$ so if $\{\#A_m\}$ is unbounded K-collusion is not possible in all markets, but as long as the number of type A firms does not grow too large demand can keep up with aggregate efficiency if it grows at least as fast as the number of firms.

This example suggests that having a large number of identical firms producing at constant marginal cost is damaging to collusion. Indeed, this is a general result. Returning to the general framework, if all firms produce at the same constant marginal cost then since Cournot price will exceed marginal cost $e_{im}(\epsilon) = S_m Z$ for all $\epsilon > 0$ in all markets. Hence $\Sigma_i e_{im}(\epsilon)/S_m = N_m Z$. Clearly this is bounded if and only if the number of firms, N_m, is bounded independently of m. Note that the demand parameter drops out -- demand is irrelevant. When firms are identical and produce at constant marginal cost demand can never keep up with aggregate efficiency if the number of firms grows large.

The first example (where with unbounded N_m K-collusion cannot be sustained in all markets when $s < 1$ but can sometimes be sustained in all markets when $s \geq 1$) should not be construed to imply that with unbounded N_m K-collusion can only be sustained in all markets if demand varies with the number of firms. Consider a case where $S_m = 1$ for all m, i.e. the level of demand is constant across markets. Let $N_m = m$ in each market and assume that the i^{th} firms have the same cost functions in all markets $m \geq i$. If all firms are identical it is known from Lambson [6] that K-collusion is not sustainable in all markets. However richer results are possible when firms need not be the same. Not only can K-collusion be sustainable in all markets when the number of firms is unbounded, but such collusion can exhibit restraint by all firms. Hence it need not be the case that collusion by an unbounded number of firms take the form of a small number of firms restricting output while a "competitive fringe" maximizes current profits. Specifically, let cost functions be

$$C_{im}(q_{im}) = 0 \qquad i = 1,2$$
$$= [18 - (1/2)^{i-2}]q_{im} \qquad i \geq 3$$

and let inverse demand be

$$f(Q_m) = 36 - Q_m \qquad 0 \leq Q_m \leq 36$$
$$= 0 \qquad Q_m > 36$$

The usual calculations verify that in Cournot equilibrium in each market firms 1 and 2 both produce 12 while the other firms produce nothing. The resulting price is 12 so firms 1 and 2 earn 144. Clearly $\{e_{im}(\epsilon)\}$ is bounded for all $\epsilon < 6$, hence the condition of the theorem is satisfied and K-collusion is sustainable in all markets. It is trivial, for example, that any collusive price less than 17½ is sustainable because marginal cost is at least 17½ for all $i \geq 3$ so those firms can be ignored. What is interesting is that a price of 18 is also sustainable even though marginal cost is less than 18 for all firms. To construct such an arrangement let $q^*_{im} = (1/2)^{i-2}$ for $i \geq 3$ where $Q^*_m = 18$. Then $\pi^*_{im} = (1/2)^{2i-3}$ and, of course, $\pi^c_{im} = 0$. Facing $Q^*_{im} = 18 - (1/2)^{i-2}$ the i^{th} firm maximizes current profit by producing $q_{im} = 3(1/2)^i$ and achieves $\pi^d_{im} = (9/8)(1/2)^{2i-3}$. Substituting these values into (2.3) verifies that (2.3) holds if $\alpha \geq 1/9$. Now $\Sigma^{\infty}_{i=3}q^*_{iN} = 1/2$, so to complete the example it must be verified that firms 1 and 2 can cut output back to 8.75 (so $Q^*_{im} = 9.25$, $i = 1,2$) and still satisfy (2.3). Straightforward calculations verify that this can be done if $\alpha > .62$. (For finite m, $\Sigma^m_{i=3}q^*_{im} < 1/2$ so q^*_{im} can exceed 8.75 for firms 1 and 2, making it easier for them to satisfy (2.3).) Note that $\pi^*_{im} < \pi^d_{im}$ for all i in all markets, implying that all firms are restricting output.

4. Concluding Remarks

This paper has investigated the effects of two factors on the sustainability of collusion, aggregate efficiency and market demand. High levels of aggregate efficiency (for which a large number of firms is necessary but not sufficient) tend to weaken the sustainability of collusion. This effect can sometimes be offset by higher levels of demand, but not always. Specifically, when for some $\epsilon > 0$, $\min[\bar{q}_{im}(\epsilon), ZS_m]$ can be made to equal $\bar{q}_{im}(\epsilon)$ for all but a bounded number of firms by choosing a large enough S_m in each market then high enough levels of demand in each market can guarantee that $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded independently of m. (An example appears in section 3.) As was seen, such choices of S_m are impossible if all firms have identical, constant marginal cost. However they will always be possible if, for example, marginal cost can take on values over ϵ greater than Cournot price in all markets for all but a finite number of firms.

It is easy to see that the results reported above still hold when subgame perfect punishments other than grim trigger strategy punishments are considered. The definition of "sustainable (by α)" need only be changed to include sustainability using other punishments. The statements of the theorem need not be changed. That the proof of sufficiency in the appendix is still valid is trivial. As for necessity, since there are no fixed costs the most severe subgame perfect punishment cannot impose negative discounted profits on a firm after deviation. So setting $\pi^c_{iN} = 0$ in the proof of necessity and verifying that the proof still goes through establishes the theorem.

Finally, the assumptions A1-A4 allow shorter proofs and cleaner exposition, but the analysis can be extended in a straightforward manner to allow differentiable, negatively sloped demand curves that need not cut the axes but which guarantee that total revenue is bounded in each market. (Of course the bound need not be independent of m.) Other assumptions can also be weakened. No new economic insights seem to be gained.

Appendix

Theorem: K-collusion is sustainable in all markets for some $K > 0$ if and only if for some $\epsilon > 0$ $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded independently of m.

Proof of Sufficiency: Choose any $\epsilon > 0$ such that $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded and choose any collection of output vectors $\{q_m^o\}$ in $\Omega(\epsilon)$. It will be shown that $Q_m^* \equiv \Sigma_i q_m^o$ can be allocated in each market to construct a collection of collusive output vectors $\{q_m^*\}$ such that for some $\alpha < 1$ (2.3) is satisfied for all m. Then K-collusion is sustainable in all markets for $K < \delta^1$. (δ^1 is from the definition of $\Omega(\epsilon)$.)

Define $d_{im}(x)$ as the maximal element of $\arg_z \max \pi_{im}(z,x)$. Clearly $d_{im}(Q_m^*) \leq e_{im}(\epsilon)$ so $\{\Sigma_i d_{im}(Q_m^*)/S_m\}$ is bounded. Define q_{im}^a to be the minimum the i^{th} firm in market m must produce to achieve its Cournot profit if it receives $f(Q_m^*/S_m)$ for its output, i.e. $\pi_{im}(q_{im}^a, Q_m^* - q_{im}^a) = \pi_{im}^c$. Define $E_m \equiv Q_m^* - Q_m^a$ and $\Pi_m(q_m) \equiv \Sigma_i \pi_{im}(q_{im}, Q_{im})$. From the definitions, in each market

$$\text{(A.1)} \qquad \Pi_m(q_m^*) - \Pi_m(q_m^c) = [(Q_m^* - Q_m^a) f(Q_m^*/S_m)] - \Sigma_i [C_{im}(q_{im}^*) - C_{im}(q_{im}^a)]$$

Since $q_{im}^* \geq q_{im}^a$ (A.1) implies (dividing by S_m and substituting using the definition of E_m)

$$\text{(A.2)} \qquad [\Pi_m(q_m^*) - \Pi_m(q_m^c)]/S_m \leq f(Q_m^*/S_m) E_m/S_m$$

Because $\{q_m^o\} \epsilon \Omega(\epsilon)$, $[\Pi_m(q_m^*) - \Pi_m(q_m^c)]/S_m > \delta^2$ so (A.2) and the boundedness of $f(\cdot)$ imply that $\{E_m/S_m\}$ is bounded strictly away from zero. Let $\bar{q}_{im}$ be defined by

(A.3) $\bar{q}_{im} = d_{im}(Q_m^* - \bar{q}_{im})$

Construct $\{q_m^*\}$ by setting

(A.4) $q_{im}^* = \min\{\bar{q}_{im}, \max[(1+k_m)q_{im}^a, k_m d_{im}(Q_m^*)]\}$

where k_m is chosen so that $\Sigma_i q_{im}^* = Q_m^*$. Now k_m is bounded strictly away from zero. To see this assume otherwise. By (A.4)

$$\begin{aligned}(A.5)\quad \Sigma_i q_{im}^*/S_m &\leq \Sigma_i \max[(1+k_m)q_{im}^a, k_m d_{im}(Q_m^*)]/S_m \\ &\leq \Sigma_i(1+k_m)q_{im}^a/S_m + \Sigma_i k_m d_{im}(Q_m^*)/S_m \\ &= \Sigma_i q_{im}^*/S_m - E_m/S_m + k_m[\Sigma_i q_{im}^a/S_m + \Sigma_i d_{im}(Q_m^*)/S_m]\end{aligned}$$

The last equality follows because $E_m = \Sigma_i q_{im}^* - \Sigma_i q_{im}^a$. So (A.5) implies

(A.6) $E_m/S_m \leq k_m[\Sigma_i q_{im}^a/S_m + \Sigma_i d_{im}(Q_m^*)/S_m]$

$\{E_m/S_m\}$ is bounded away from zero and the bracketed term on the right hand side of (A.6) is bounded above. So (A.6) cannot hold for all m if k_m approaches zero. Hence k_m must be bounded strictly away from zero.

It remains to be verfied that $\{q_m^*\}$ satisfies (2.3) for all m given sufficiently large $\alpha < 1$. It will first be shown that for firms satisfying $d_{im}(Q_m^*) > 0$

(A.7) $\sup_i \sup_m \pi_{im}^d/\pi_{im}^* < \infty$ and

(A.8) $\sup_i \sup_m \pi_{im}^c/\pi_{im}^* < 1$

(If $d_{im}(Q_m^*) = 0$ firm i cannot profitably produce at collusive prices and can thus be ignored.) First consider (A.7). If $q_{im}^* = \bar{q}_{im}$ then $(\pi_{im}^d/\pi_{im}^*) = 1$.

Otherwise,

$$\text{(A.9)} \qquad \pi^d_{im} \leq [q^*_{im} + d_{im}(Q^*_m)][f(Q^*_m/S_m) - A_{im}(q^*_{im})]$$

(where $A_{im}(x) = C_{im}(x)/x$ is average cost) because optimal net deviation cannot exceed $d_{im}(Q^*_m)$ and post-deviation per unit profit cannot exceed collusive per unit profit. (The former is true because marginal revenue is lower for all levels of net deviation if $q^*_{im} > 0$ than if $q^*_{im} = 0$ and $d_{im}(Q^*_m)$ is optimal net deviation if $q^*_{im} = 0$. The latter is obvious.) Furthermore, (A.4) ensures that

$$\text{(A.10)} \qquad \pi^*_{im} \geq k_m d_{im}(Q^*_m)[f(Q^*_m/S_m) - A_{im}(q^*_{im})]$$

Now (A.9) and (A.10) imply that $(\pi^d_{im}/\pi^*_{im}) \leq 1 + (1/k_m)$ for all i and m. Hence (A.7) is satisfied because k_m is bounded strictly away from zero.

Now consider (A.8). If $\pi^c_{im} = 0$ but $d_{im}(Q^*_m) > 0$ then (A.4) ensures that $\pi^*_{im} > 0$. Hence $(\pi^c_{im}/\pi^*_{im}) = 0$. For firms with $\pi^c_{im} > 0$ in market m define T(m), U(m) and V(m) as the sets such that $q^*_{im} \neq \bar{q}_{im}$ and $q^*_{im} < q^c_{im}$, $q^*_{im} \neq \bar{q}_{im}$ and $q^*_{im} \geq q^c_{im}$, and $q^*_{im} = \bar{q}_{im}$, respectively.

First consider $i \epsilon T(m)$ and note that, due to nondecreasing marginal cost,

$$\text{(A.11)} \qquad \pi^*_{im} - \pi^c_{im} \geq (q^*_{im} - q^a_{im})[f(Q^*_m/S_m) - C'_{im}(q^*_{im})]$$

which can be manipulated to yield

$$\text{(A.12)} \qquad \pi^c_{im}/\pi^*_{im} \leq 1 - \{(q^*_{im} - q^a_{im})[f(Q^*_m/S_m) - C'_{im}(q^*_{im})]/\pi^*_{im}\}$$

Since $q^*_{im} < q^c_{im}$ for $i \epsilon T(m)$, $C'_{im}(q^*_{im}) \leq C'_{im}(q^c_{im}) < f(Q^c_m/S_m)$ so, by the definition of $\Omega(\epsilon)$, $[f(Q^*_m/S_m) - C'_{im}(q^*_{im})] > \delta^1$. Clearly $\pi^*_{im} < q^*_{im}f(0)$ while (A.4) specifies that $(q^*_{im} - q^a_{im})/q^*_{im} > k_m/(1+k_m)$. These relationships and (A.12) imply

(A.13) $\pi^c_{im}/\pi^*_{im} \leq 1 - \delta^1 k_m/f(0)(1+k_m)$

Since k_m is bounded away from zero, $\sup_{i\epsilon T(m)}\sup_m(\pi^c_{im}/\pi^*_{im}) < 1$.

For $i\epsilon U(m)$ remember that

(A.14) $\pi^c_{im} = q^c_{im}[f(Q^c_m/S_m) - A_{im}(q^c_{im})]$

and, since $q^*_{im} \geq q^c_{im}$ (and π^*_{im} increases in q_{im} up to $\bar{q}_{im}$ given Q^*_m),

(A.15) $\pi^*_{im} \geq q^c_{im}[f(Q^*_m/S_m) - A_{im}(q^c_{im})]$

Then (A.14) and (A.15) imply $\sup_{i\epsilon U(m)}\sup_m(\pi^c_{im}/\pi^*_{im}) < 1$ because $f(Q^*_m/S_m)-f(Q^c_m/S_m) > \delta^1$.

Finally, for $i\epsilon V(m)$, $\pi_{im}(\bar{q}_{im},Q^*_m - \bar{q}_{im})$ cannot be less than $\pi_{im}(q^c_{im},Q^*_m - q^c_{im})$ by the definition of $\bar{q}_{im}$. So (A.14) and (A.15) imply $\sup_{i\epsilon V(m)}\sup_m(\pi^c_{im}/\pi^*_{im}) < 1$ as before. Hence (A.8) holds.

To see that (A.7) and (A.8) imply that (2.3) is satisfied for sufficiently large $\alpha < 1$, rewrite (2.3) as follows:

(A.16) $\alpha/(1-\alpha) \geq [(\pi^d_{im}/\pi^*_{im}) - 1]/[1 - (\pi^c_{im}/\pi^*_{im})]$

<u>Proof of Necessity</u>: Summing (2.3) over all firms and manipulating yields

(A.17) $(\alpha/1-\alpha) \geq [(\Sigma_i\pi^d_{im}/\Sigma_i\pi^*_{im})-1]/[1-(\Sigma_i\pi^c_{im}/\Sigma_i\pi^*_{im})]$

The denominator on the right hand side is bounded between zero and one so if (A.17) holds for all m and some $\alpha < 1$ the numerator must be bounded. A1 and A2 imply that $\{\Sigma_i\pi^*_{im}/S_m\}$ is bounded so, if (A.17) holds for all m, $\{\Sigma_i\pi^d_{im}/S_m\}$ must be bounded. It will be shown that unless $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is bounded for some $\epsilon > 0$, $\{\Sigma_i\pi^d_{im}/S_m\}$ is unbounded for any $\alpha < 1$ and any $\{q^*_m\}$ satisfying

$f(Q_m^*/S_m) - f(Q_m^c/S_m) > K$ for $K > 0$.

Assume $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ is unbounded for all $\epsilon > 0$. Choose $\epsilon < K$ and let q_{im}^e satisfy

$$\text{(A.18)} \quad f[(Q_{im}^* + q_{im}^e)/S_m] = f(Q_m^c/S_m) + [(K+\epsilon)/2]$$

i.e. q_{im}^e is the production that will drive price from $f(Q_{im}^*/S_m)$ to $f(Q_m^c/S_m) + [(K+\epsilon)/2]$. Since $f(Q_{im}^*/S_m) \geq f(Q_m^*/S_m)$ for all i, if $f(Q_m^*/S_m) - f(Q_m^c/S_m) > K$ the boundedness of $f'(\cdot)$ implies that

$$\text{(A.19)} \quad q_{im}^e/S_m > \delta$$

for all i and some $\delta > 0$. Define $b_{im} \equiv \min[e_{im}(\epsilon), q_{im}^e]$. Then

$$\text{(A.20)} \quad \pi_{im}(b_{im}, Q_{im}^*) \geq b_{im}(K-\epsilon)/2$$

(To see this consider two cases: (1) For $e_{im}(\epsilon) = 0$, $b_{im} = 0$ and the result is obvious. (2) For $e_{im}(\epsilon) > 0$ note that a firm producing b_{im} cannot drive price below $[f(Q_{im}^*+q_{im}^e)/S_m]$ and cannot have average cost exceeding $C_{im}'[e_{im}(\epsilon)]$. Hence

$$\text{(A.21)} \quad \pi_{im}(b_{im}, Q_{im}^*) \geq b_{im}\{[f(Q_{im}^* + q_{im}^e)/S_m - C_{im}[e_{im}(\epsilon)]\}$$

Substituting (A.18) into (A.21) and noting that $e_{im}(\epsilon) > 0$ implies $f(Q_m^c/S_m) + \epsilon \geq C_{im}'[e_{im}(\epsilon)]$ yields the result.) By definition $\pi_{im}^d \geq \pi_{im}(b_{im}, Q_{im}^*)$ so (A.20) implies

$$\text{(A.22)} \quad \Sigma_i \pi_{im}^d/S_m \geq [(K-\epsilon)/2]\Sigma_i b_{im}/S_m$$

Let A_m be the subset of firms with $e_{im}(\epsilon) > q_{im}^e$ and let B_m be its complement. Let $\#A_m$ and $\#B_m$ be the cardinality of A_m and B_m, respectively. Then,

$$\text{(A.23)} \quad \Sigma_i b_{im}/S_m = \Sigma_A q^e_{im}/S_m + \Sigma_B e_{im}(\epsilon)/S_m$$

$$= \Sigma_A q^e_{im}/S_m + \Sigma_i e_{im}(\epsilon)/S_m - \Sigma_A e_{im}(\epsilon)/S_m$$

$$> (\#A_m)\delta + \Sigma_i e_{im}(\epsilon)/S_m - \Sigma_A e_{im}(\epsilon)/S_m \qquad \text{(by (A.19))}$$

Now (A.23) implies that $\{\Sigma_i b_{im}/S_m\}$ is unbounded. (To see this note that if $\{\#A_m\}$ is unbounded so is $\{\Sigma_i b_{im}/S_m\}$. If $\{\#A_m\}$ is bounded so is $\{\Sigma_A e_{im}(\epsilon)/S_m\}$ so the assumed unboundedness of $\{\Sigma_i e_{im}(\epsilon)/S_m\}$ yields the result.) Then (A.22) implies that $\{\Sigma\pi^d_{im}/S_m\}$ is unbounded and (A.17) implies that (2.3) cannot be satisfied. A contradiction has been reached. <u>Q.E.D.</u>

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