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DYNAMIC BEHAVIOR IN LARGE
MARKETS FOR DIFFERENTIATED
PRODUCTS*

Val Eugene Lambson

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ABSTRACT

An important question is how well competitive models approximate models of large finite economies. This paper studies that question for models of differentiated products. Both static and dynamic Nash equilibria are considered. In the class of models analyzed it is shown that static Nash equilibria always converge to competition as the number of firms increases. Dynamic Nash equilibria need not so converge. Easily checked necessary and sufficient conditions for their convergence to competition are, however, established.

1. Introduction

Much effort has gone into investigating whether competitive models closely approximate large finite economies. Ruffin [9] analyzed Cournot equilibria in large markets with identical firms and found conditions for the Cournot equilibrium to converge to the competitive equilibrium as the number of firms increases. Roberts [8] considered the same question in a general equilibrium context. More recently Novshek [7] has analyzed more general sequences of partial equilibrium Cournot markets for convergence to competitive equilibrium. Similar efforts have been made by Allen and Hellwig [1] for Bertrand price games. In all of these studies only static Nash equilibrium concepts were considered. An example by Green [4], however, established that even in models where static Nash equilibria converge to competition, there may exist dynamic Nash equilibria which do not. He then considered how the addition of uncertainty and incomplete information to Cournot models might rule out such possibilities. Following a different line, Lambson [5] studied a class of full information Cournot models to derive conditions for convergence of dynamic Nash equilibria to competitive equilibria.

This paper considers the question of convergence to marginal cost pricing in a model of differentiated products. Both static Nash equilibria and general subgame perfect dynamic Nash equilibria are considered. In section 2 the framework for the analysis is constructed. It is shown in section 3 that in the class of models considered all sequences of static Nash equilibria converge to competition. Such is not the case for all sequences of dynamic Nash equilibria. Necessary and sufficient conditions for their convergence to competitive outcomes are derived in section 4. Section 5 contains some extensions and concluding remarks.

2. The Model

Consumers and firms are assumed to inhabit a circle of circumference Z for a countably infinite number of periods indexed by t. Consumers are uniformly distributed on this product space. In each period every consumer has the demand function $f[\min_i(P_i+\lambda(x_i))]$ where x_i is the distance (minimum arc length) of the consumer from the i^{th} firm, and λ is a positive constant. Each consumer buys only from the firm for which the delivered price $P_i+\lambda(x_i)$ is lowest. There exists a finite price, $k > 0$, such that $f(k) = 0$. On the interval $[0,k]$, $f(\cdot)$ is continuously differentiable, $\infty > f(0) > 0$, and $f'(\cdot) < 0$.

The N firms are located symmetrically on the circle and are assumed to simultaneously choose prices in each period. All firms have the same cost function, $C(D)$, where D is the level of output. The cost function is twice continuously differentiable on $[0,\infty)$, $C(0) = 0$, $C'(\cdot) \geqslant 0$, and $C''(\cdot) \geqslant 0$. Let P_{iN} be the price chosen by the i^{th} firm when there are N firms in the market. Let $P_N \equiv (P_{1N},\ldots,P_{NN})$ and define $\bar{P}_{iN}$ as P_N with the i^{th} element removed.

The demand curve faced by the i^{th} firm is

$$(2.1) \qquad D_{iN}(P_N) = \int_0^{\gamma^-} N^S f(P_{iN}+\lambda(\phi))d\phi + \int_0^{\gamma^+} N^S f(P_{iN}+\lambda(\phi))d\phi$$

where γ^- and γ^+ denote, respectively, the length of the interval serviced by firm i on each side of its location and are defined (for x = +, -) by

$$(2.2) \qquad \gamma^x = \min\left\{[k-P_{iN}]/\lambda,\ [P_{\hat{\ell}N}-P_{iN}+\lambda|\hat{\ell}-i|]/2\lambda\right\}$$

where $|\hat{\ell}-i|$ is the arc length between firms $\hat{\ell}$ and i measured on the appropriate side of firm i and $\hat{\ell}$ is the firm for which $[P_{\ell N}-P_{iN}+\lambda|\ell-i|]/2\lambda$ is minimized. (γ^x depends on both i and N, which are suppressed for notational convenience.) If

$\gamma^x = [k-P_{iN}]/\lambda$ firm i is a local monopolist on that side of its market. Otherwise firm i is in competition with firm $\hat{\ell}$. In the first case firm i's marginal customer is indifferent between buying and not buying. In the second case the marginal consumer prefers to buy but is indifferent between patronizing firms i and $\hat{\ell}$. If $\gamma^x \leq 0$ firm i is undercut and has zero sales. Firm $\hat{\ell}$ is then an undercutter.

The specification of demand employed allows demand to be replicated with the firms. When $s < 1$, $s = 1$, and $s > 1$ demand increases at a rate less than, equal to, and greater than, respectively, the rate of increase in the number of firms. When $s = 0$ demand is independent of the number of firms. Given P_N each firm is free to choose its output q_{iN}. It is usually the case in what follows that $q_{iN} = D_{iN}(P_N)$ is the most profitable output. When a firm undercuts another, however, servicing the discontinuously longer interval may drive marginal cost too high. Then $q_{iN} < D_{iN}(P_N)$ may be more profitable. Firm i's current period profit can thus be written

(2.3) $$\pi_{iN}(P_N) = \max_{q_{iN} \leq D_{iN}(P_N)} [P_{iN}q_{iN} - C(q_{iN})]$$

Firm i's strategy σ_{iN} is a sequence of functions $\{\sigma_{iN}(t)\}$ which specify firm i's price in each period as a function of the previous choices by all firms. So $\sigma_{iN}(1)\epsilon[0,k]$ and, for $t > 1$, $\sigma_{iN}(t):[0,k]^{N(t-1)} \rightarrow [0,k]$. Let σ_N denote the strategy profile $(\sigma_{1N},\ldots,\sigma_{NN})$. A sequence of price vectors $\{P_N(t)\}$ is called an outcome path. Let Ω denote the set of possible outcome paths, i.e. each element of Ω is an outcome path generated by some strategy profile as follows:

$$P_N(\sigma_N)(1) = [\sigma_{1N}(1),\ldots,\sigma_{NN}(1)]$$
$$P_N(\sigma_N)(t) = [\sigma_{1N}(t)(P_N(\sigma_N)(1),\ldots,P_N(\sigma_N)(t-1)),\ldots,$$
$$\sigma_{NN}(t)(P_N(\sigma_N)(1),\ldots,P_N(\sigma_N)(t-1))]$$

Let H_t be a history through time t, i.e. $H_t \equiv [P_N(1),\ldots,P_N(t)]$. Then $\sigma_N\big|_{H_t}$ will denote the strategy profile induced on the subgame following H_t, i.e. given two sequences H_t and H_u,

$$\sigma_N\Big|_{H_t}(u+1)(H_u) = \sigma_N(t+u+1)(H_t,H_u)$$

Define the payoff to firm i (when there are N firms) by

$$(2.4) \qquad V_{iN}(\sigma_N) \equiv \Sigma_{t=1}^{\infty}\beta^{t-1}\pi_{iN}[\sigma_N(t)(h_{t-1})]$$

where $\{h_t\}_{t=1}^{\infty}$ is the sequence of histories induced by σ_N. Then if Σ_{iN} is firm i's strategy set, σ_N is a Nash equilibrium iff $V_{iN}(\sigma_N) \geqslant V_{iN}(\sigma'_{iN},\bar{\sigma}_{iN})$ for all $\sigma'_{iN} \in \Sigma_{iN}$ where $\bar{\sigma}_{iN}$ denotes the vector σ_N with the i^{th} element removed. σ_N is a subgame perfect Nash equilibrium iff it is a Nash equilibrium and for all t and all histories H_t, $\sigma_N\big|_{H_t}$ is a Nash equilibrium. The question examined here is whether subgame perfect Nash equilibria are approximately competitive when there are many firms. To investigate this sequences of markets differing in the number of firms will be studied. Formally, a market is a sextuple $M_N = (N, C, f, Z, \beta, s)$. Sequences of markets $\{M_N\}_{N=2}^{\infty}$ differing only in N will be studied in what follows.

3. Non-Collusive Outcomes in Large Markets

A Nash equilibrium will be called non-collusive if the equilibrium strategies are simple, i.e. if $\sigma_{iN}(t)$ is independent of H_{t-1} for all i and t. Intuitively, this rules out the threats that make outcomes other than static Nash outcomes possible in each period, so a sequence of static Nash equilibria emerges as the non-collusive equilibrium. Formally, a static Nash price vector is a vector (P_N^r) such that for each i

(3.1) $\pi_{iN}(P^r_N) = \max_{P_{iN}} \pi_{iN}(P_{iN}, \bar{P}^r_{iN})$

The strategy profile (σ^r_N) where $\sigma^r_{iN}(t) = P^r_{iN}$ for all i is a non-collusive equilibrium. It follows immediately from the definitions that (σ^r_N) is a subgame perfect equilibrium. Clearly, (σ^r_N) will be the unique non-collusive equilibrium iff (P^r_N) is the unique static Nash price vector. If other static Nash price vectors exist other non-collusive equilibria, including nonstationary ones, will exist because any outcome exhibiting a static Nash price vector in each period is a non-collusive equilibrium. Neither uniqueness nor symmetry of (P^r_N) is assumed in what follows. (Of course, existence of a static Nash price vector is assumed. To see some sufficient, but not necesary, conditions for existence see Economides [3] and Novshek [6].) Issues of uniqueness and symmetry can be ignored because all sequences of static Nash price vectors are indistinguishable in the limit, as will be seen. Of interest in its own right is that in the limit price equals marginal cost for each firm, i.e. a version of the "classical limit theorem" holds in this model. The rest of this section is dedicated to the proof of that proposition.

Theorem 3.1: There exists $P^r_\infty \epsilon [0,k]$ such that for any sequence of static Nash price vectors $\{P^r_N\}$ and any $\epsilon > 0$ there exists N_ϵ such that if $N > N_\epsilon$

(3.2) $| P^r_\infty - P^r_{iN} | < \epsilon$ $\quad i = 1,\ldots,N$, and

(3.3) $| P^r_\infty - C'[D_{iN}(P^r_N)] | < 2\epsilon$ $\quad i = 1,\ldots,N$.

Proof: Note that $P^r_{iN} \geqslant C'(0)$ for all i and N because otherwise the lowest priced firm would make negative profits, contradicting the definition of P^r_N.

It also follows that no firm is undercut on either side, i.e. for all i, j, and N,

$$(3.4) \qquad P^r_{jN} \leqslant \min_{i \neq j} [P^r_{iN} + \lambda|i-j|]$$

This is because, since $P^r_{iN} \geq C'(0)$ for all i, positive profits can always be achieved by some P^r_{jN} satisfying (3.4) while profits are zero if firm j is undercut. Two cases will be treated separately: (1) Either $s \leqslant 1$, or else $s > 1$ and $C'(\infty) < k$. (2) $s > 1$ and $C'(\infty) \geqslant k$.

(1) It is shown in the appendix that all firms are in competition on both sides of their markets for all N greater than some $\bar{N}$. The theorem is proved by using this result along with the first order conditions that hold for competitors in equilibrium, i.e. for all i

$$(3.5) \qquad \{P^r_{iN} - C'[D_{iN}(P^r_N)]\} \{f[P^r_{iN} + \lambda(\gamma^-)](1/2) + f[P^r_{iN} + \lambda(\gamma^+)](1/2) - 2f(P^r_{iN})\}$$
$$+ \{F[P^r_{iN} + \lambda(\gamma^-)] + F[P^r_{iN} + \lambda(\gamma^+)] - 2F(P^r_{iN})\} = 0$$

where F is the antiderivative of f. Since undercutting cannot occur in equilibrium, γ^- and γ^+ must approach zero as N grows large. Hence the last bracketed term in (3.5) approaches zero but the second bracketed term is bounded strictly away from zero, implying that the first bracketed term must approach zero, i.e. price approaches marginal cost for each i. In particular, renumber firms so that firm j charges the highest price for each N. Then for any $\epsilon > 0$ there exists N^1_ϵ such that if $N > N^1_\epsilon$

$$(3.6) \qquad | P^r_{jN} - C'[D_{jN}(P^r_N)] | < \epsilon.$$

The monotonicity properties of $C'(\cdot)$ and $D_{iN}(\cdot)$ guarantee that for each N there is a unique vector P_N^{MC} (clearly symmetric) such that, for all i,

$$(3.7) \qquad C'[D_{iN}(P_N^{MC})] = P_{iN}^{MC}.$$

It will be shown that $\{P_{iN}^{MC}\}$ has a limit P_∞^r. First consider s < 1. Since total demand is bounded by $N^s Zf(0)$, average demand cannot exceed $N^s Zf(0)/N$ which approaches zero in N. Hence if s < 1

$$(3.8) \qquad \lim_{N\to\infty} P_{iN}^{MC} = C'(0).$$

Now consider $s \geq 1$. For $s \geq 1$, $P_{i,N+1}^{MC} \geq P_{iN}^{MC}$ because demand increases at least as quickly as the number of firms and the increased density of firms lowers transportation costs. Hence $\{P_{iN}^{MC}\}$ is monotonically nondecreasing on a compact set and must have a well defined limit. So for $\epsilon > 0$ there exists N_ϵ^2 such that if $N > N_\epsilon^2$

$$(3.9) \qquad |\, P_\infty^r - P_{iN}^{MC} \,| < \epsilon/2.$$

Assume, counterfactually, that $P_{jN}^r > P_{iN}^{MC} + \delta$ for some $\delta > 0$ and some sub-sequence. Then $C'[D_{jN}(P_N^r)]$ cannot exceed $C'[D_{jN}(P_N^{MC})]$ so

$$(3.10) \qquad P_{jN}^r - C'[D_{jN}(P_N^r)] > P_{iN}^{MC} + \delta - C'[D_{jN}(P_N^{MC})] = \delta.$$

This contradicts (3.6) for $\epsilon < \delta$ and $N > N_\epsilon^1$. So for any $\epsilon > 0$ there exists N_ϵ^3 such that if $N > N_\epsilon^3$ then

$$(3.11) \qquad |\, P_{jN}^r - P_{iN}^{MC} \,| < \epsilon/2.$$

Now $P_{iN}^r \geq P_{iN}^{MC}$ for all i and N, otherwise the lowest priced firm will face positive demand at price less than marginal cost. So since $P^r_{jN} \geq P^r_{iN}$, (3.9) and (3.11) establish (3.2) for $N_\epsilon > \max[N^2_\epsilon, N^3_\epsilon]$. Finally,

$$(3.12) \qquad P^r_{jN} \geq P^r_{iN} \geq C'[D_{iN}(P^r_N)] \geq C'[D_{jN}(P^r_N)]$$

so (3.2), (3.6), and (3.12) establish (3.3) for $N_\epsilon > \max[N^1_\epsilon, N^2_\epsilon, N^3_\epsilon]$.

(2) For this case $\{P^r_{iN}\}$ must approach k for all i, otherwise unbounded output would be sold by some firms; but $C'(\infty) \geqslant k$, implying unbounded losses.

If $\{D_{iN}(P^r_N)\}$ is unbounded for every subsequence then $C'(\infty) = k$ (for if $C'(\infty) > k$ unbounded losses would occur in the limit, violating the definition of P^r_N) and $C(D_{iN}(P^r_N))$ approaches k, proving the result.

If $\{D_{iN}(P^r_N)\}$ is bounded for some subsequence then this subsequence has a convergent subsequence with limit $\bar{D}$. It will be shown that $C(\bar{D}) = k$ for any such subsequence. Clearly $C'(\bar{D}) > k$ can be ruled out by the definition of P^r_N since for large N a unilateral increase in price would increase profit. To see that $C'(\bar{D}) < k$ can be ruled out, note that

$$(3.13) \qquad \lim_{N\to\infty} \pi_{iN}(P^r_N) = k\bar{D} - \int_0^{\bar{D}} C'(D)dD$$

Consider a corresponding subsequence $\{P^e_{iN}\}$ such that $P^e_{iN} > P^r_{iN}$ for all N in the subsequence, such that P^e_{iN} converges to k, and such that $D_{iN}(P^e_{iN}, P^r_{iN})$ converges to $\bar{D} + \delta$ where $C'(\bar{D} + \delta) < k$. Then

$$(3.14) \qquad \lim_{N\to\infty} \pi_{iN}(P^e_{iN}, \bar{P}^r_{iN}) = [k\bar{D} - \int_0^{\bar{D}} C'(D)dD] + [k\delta - \int_{\bar{D}}^{\bar{D}+\delta} C'(D)dD]$$

Since $C'(\bar{D} + \delta) < k$, the last term in (3.14) is positive so the comparison of of (3.13) and (3.14) contradicts the definition of $\{P^r_N\}$. QED

4. Self-Enforcing Collusion in Large Markets

The multiperiod structure of the model introduces a dynamic element ignored in the last section. Specifically, by colluding firms may achieve outcomes yielding higher profits than are attained in a static Nash equilibrium. Such collusion may be possible despite the noncooperative nature of the model since firms may adopt strategies which effectively punish any firm that deviates from a collusive outcome.

Let Ω_N be the set of subgame perfect Nash equilibria in M_N. For each i (given N) there exists a function $\hat{V}_{iN}:\Omega_N \to R$ which assigns a discounted payoff stream to firm i. ($\hat{V}_{iN}$ is V_{iN} with the domain restricted to Ω_N.) Let $\hat{\nu}_{iN}$ be the image of Ω_N in R. (A generic element of $\hat{\nu}_{iN}$ will be denoted by ν_{iN}.) Now $\hat{\nu}_{iN}$ can be interpreted as the set of values of possible punishments that can be inflicted on firm i, discounted to the first period of the punishment path. Let $\hat{\nu}_N \equiv (\hat{\nu}_{iN})^N$ and let ν_N denote a generic element of $\hat{\nu}_N$. The price vector P_N^C is a <u>sustainable collusive price vector</u> in M_N if, for all i, $P_{iN}^C > C'[D_{iN}(P_N^C)]$ and there exists $\nu_N \in \hat{\nu}_N$ such that

$$(4.1) \qquad \pi_{iN}^*(P_N^C) - \pi_{iN}(P_N^C) \leq (\beta/1-\beta)\pi_{iN}(P_N^C) - \beta\nu_{iN}$$

where $\pi_{iN}^*(P_N^C) \equiv \sup_{P_{iN}} \pi_{iN}(P_{iN},\bar{P}_{iN}^C)$. A <u>sequence of sustainable collusive price vectors</u> is a sequence of price vectors $\{P_N^C\}$ such that $P_{iN}^C > C'[D_{iN}(P_N^C)]$ for all i, and for which there is a sequence $\{\nu_N\}$ such that $\nu_N \in \hat{\nu}_N$ and (4.1) is satisfied for all N. A sequence of markets $\{M_N\}$ <u>converges to competition</u> if $\lim_{N\to\infty}[\max_i P_{iN}^C - P_\infty^r] = 0$ for all sequences of sustainable collusive price vectors. Note that this definition implies that $\lim_{N\to\infty} [P_{iN}^C - P_\infty^r] = 0$ for all i because otherwise the lowest priced firm would have marginal cost higher than its price (for large N) and the definitions of $\{P_N^C\}$ would be contradicted.

Theorem 4.1: If $0 \leq s < 1$ then $\{M_N\}$ converges to competition.

Proof: Let $\{P_N^C\}$ be any sequence of sustainable collusive price vectors. The sum of firms' outputs cannot exceed $N^s f(0)Z$. The highest priced firm, say j, will produce no more than average output, therefore

(4.2) $$D_{jN}(P_N^C) \leq N^s f(0)Z/N$$

Hence, since profit cannot exceed revenue and price will not exceed k,

(4.3) $$\pi_{jN}(P_N^C) < kD_{jN}(P_N^C) \leq kN^s f(0)Z/N$$

Renumber firms so j is the highest priced firm for all N. Then (4.3) implies

(4.4) $$\lim_{N\to\infty} \pi_{jN}(P_N^C) = 0$$

because $s < 1$. Since the left hand side of (4.1) is nonnegative, if (4.1) holds then

(4.5) $$(\beta/1-\beta)\pi_{jN}(P_N^C) \geq \beta\nu_{jN}$$

implying (since $\nu_{jN} \geq 0$ by subgame perfection)

(4.6) $$\lim_{N\to\infty} \nu_{jN} = 0$$

So for the highest priced firm every term in (4.1) but $\pi^*_{jN}(P_N^C)$ has been shown to approach zero. Assume, contrary to the theorem, that for some $K > 0$

(4.7) $$P_{jN}^C - P_\infty^r > K$$

for a subsequence. It will be shown that this implies that $\pi^*_{jN}(P_N^C)$ is bounded strictly away from zero for a subsequence. This will complete the proof since it will imply that (4.1) is not satisfied for large N in the subsequence, contradicting the hypothesis that $\{P_N^C\}$ is a sequence of sustainable collusive price vectors.

First note that $\pi^*_{jN}(P_N^C)$ is always positive since by matching the lowest priced firm's price firm j can achieve positive sales at a price exceeding marginal cost. Let $\Gamma_N(\delta)$ be the set of firms on both sides of j satisfying $|j-i| < \delta$. Then $P_{jN}^C \leq P_{iN}^C + \lambda(\delta)$ for all $i\epsilon\Gamma_N(\delta)$. (Otherwise the highest priced firm is undercut and $\pi_{jN}(P_N^C) = 0$. Since $\pi^*_{jN}(P_N^C) > 0$ this implies that (4.1) is violated.) Choose δ such that $\lambda(\delta) < K/2$. Then firm j can charge $P^*_{jN} = P^C_{jN} - 2\lambda(\delta)$ and undercut all firms in $\Gamma_N(\delta)$. By (4.7) and the choice P^*_{jN}, there exists a subsequence of outputs such that profits given the price subsequence $\{P^*_{jN}\}$ are bounded strictly away from zero. (The reasoning is similar to that found in the proof of the lemma in the appendix.) Hence $\{\pi^*_{jN}(P_N^C)\}$ is bounded strictly away from zero for the subsequence. Q.E.D.

<u>Theorem 4.2</u>: (a) When s = 1 $\{M_N\}$ converges to competition if $P^r_\infty = C'(\infty)$. (b) By contrast, if $P^r_\infty \neq C'(\infty)$ then there exists $\beta^* < 1$ such that if $\beta > \beta^*$ $\{M_N\}$ does not converge to competition.

<u>Proof of (a)</u>: Suppose $P^r_\infty = C'(\infty)$ but $\{M_N\}$ does not converge to competition. Then there exists K > 0 such that

(4.8) $$P^C_{jN} - P^r_\infty > K$$

for a subsequence of sustainable collusive price vectors $\{P_N^C\}$, where firms are renumbered so that j is the highest priced firm in M_N. In any period the sum of firms' outputs cannot exceed Nf(0)Z. Since the highest priced firm will not produce more than average output

(4.9) $D_{jN}(P_N^C) \leqslant f(0)Z$

Since price is less than k and costs are nonnegative

(4.10) $\pi_{jN}(P_N^C) < kf(0)Z$

Rewrite (4.1) as

(4.11) $\pi^*_{jN}(P_N^C) \leqslant [\pi_{jN}(P_N^C)/(1-\beta)] - \beta\nu_{jN}$

Since $\nu_{jN} \geq 0$, (4.10) implies the right hand side of (4.11) is bounded above. So if $\{\pi^*_{jN}(P_N^C)\}$ is unbounded in N (4.11) will imply that $\{P_N^C\}$ is not a sequence of <u>sustainable</u> collusive price vectors, establishing the contradiction that proves the theorem. Rewrite (4.8) as

(4.12) $P^C_{jN} > P^r_\infty + K$

Remember that, by previous reasoning,

(4.13) $P^C_{jN} \leqslant P^C_{iN} + \lambda(\delta)$

for all $i\epsilon\Gamma_N(\delta)$ so from (4.12), (4.13) and the hypothesis of the theorem

(4.14) $P^C_{iN} + \lambda(\delta) > P^r_\infty + K = C'(\infty) + K$

If firm j charges $\overset{*}{P}_{jN} = C'(\infty) + K - 2\lambda(\delta)$ it will undercut all firms in $\Gamma_N(\delta)$ (for the subsequence) and earn at least $K-2\lambda(\delta)$ per unit profit. So since no consumer within δ of firm j will buy less than $f(\overset{*}{P}_{jN}+\lambda(\delta))$,

(4.15) $\pi_{jN}(C'(\infty) + K - 2\lambda\delta,\ \bar{P}^C_{jN}) \geqslant [K-2\lambda(\delta)]N(2\delta)f(P^*_{jN}+\lambda(\delta))$

Choose δ so $K > 2\lambda\delta$ and $f(\overset{*}{P}_{jN}+ \lambda(\delta)) > 0$. Then the right hand side of (4.15) is unbounded in N, so $\pi^*_{jN}(P_N^C)$ is unbounded and the result established.

Proof of (b): If $P^r_\infty \neq C'(\infty)$ then $P^r_\infty < C'(\infty)$ by Theorem 3.1. Consider the punishment whereby all firms choose static Nash equilibrium prices in all periods subsequent to a deviation. (It is immediate that this is a subgame perfect punishment.) The theorem will be proved if it can be shown that there exists $\beta^* < 1$ and a sequence of collusive price vectors $\{P^c_N\}$ such that (4.1) is satisfied for all i and N, i.e.

$$(4.16) \qquad \pi^*_{iN}(P^c_N) - \pi_{iN}(P^c_N) \leqslant (\beta^*/1-\beta^*)[\pi_{iN}(P^c_N) - \pi_{iN}(P^r_N)],$$

and such that, for some $K > 0$, all i, and all $N > \bar{N}$,

$$(4.17) \qquad P^c_{iN} - P^r_{iN} > K.$$

(Firms are in competition for $N > \bar{N}$. See the appendix.) Choose $\epsilon > 0$ so

$$(4.18) \qquad \epsilon > P^c_{iN} - P^r_{iN} \text{ for all i and all } N > \bar{N}$$

implies that $\{\max_i P^c_{iN}\}$ is eventually bounded strictly above by $C'(\infty)$. Since firms are in competition, $\partial\pi_{iN}/\partial P_{iN} = 0$ and $\partial\pi_{iN}/\partial P_{\ell N} > 0$ for neighboring ℓ when evaluated at P^r_N for $N > \bar{N}$. Hence small price increases from P^r_{iN} by all firms increase the profit of all firms by an amount that can be shown to be bounded away from zero independently of i. Hence there exists $\{P^c_{iN}\}$ satisfying (4.17) and (4.18) such that for $N > \bar{N}$ the right hand side of (4.16) is bounded away from zero independently of i. Now if $\max_i \pi^*_{iN}(P^c_N)$ is bounded then the left hand side of (4.16) is bounded above independently of i and N. This would imply that β^* can be chosen sufficiently close to one that (4.16) will hold for all i and N, and the theorem would be proved. Since $\{\max_i P^c_{iN}\}$ is eventually bounded strictly above by $C'(\infty)$ the output stream associated with $\{\pi^*_{iN}(P^c_N)\}$ must be bounded for each firm or else average cost would approach $C'(\infty)$ but price would not exceed $(\max_i P^C_{iN})$ and infinite losses would result. So $\{\pi^*_{iN}(P^C_N)\}$ is bounded above independently of i. Q.E.D.

<u>Theorem 4.3</u> If $s > 1$ then $\{M_N\}$ converges to competition.

<u>Proof</u>: If $C'(\infty) \geqslant k$ then $P^r_\infty = k$ and, since no price exceeding k is possible, convergence to competition is trivial. If $C'(\infty) < k$ then manipulate (4.1) to write

$$(4.19) \qquad [\pi^*_{iN}(P^C_N)/\pi_{iN}(P^C_N)] \leqslant [(1/(1-\beta)] - [\beta\nu_{iN}/\pi_{iN}(P^C_N)]$$

Since $\nu_{iN} \geqslant 0$, the right hand side is bounded above independently of i. Once again, renumber firms in each M_N so that j is the highest priced firm. The sum of firms' outputs cannot exceed $N^s f(0)Z$. That the highest priced firm will produce no more than average output implies

$$(4.20) \qquad D_{jN}(P^C_N) \leqslant N^s f(0)Z/N$$

Hence

$$(4.21) \qquad \pi_{jN}(P^C_N) < kD_{jN}(P^C_N) \leqslant kN^s f(0)Z/N$$

Let $\{P^C_N\}$ be any sequence of sustainable collusive price vectors that does not converge to competition. Then for a subsequence (since Theorem 3.1 implies $P^r_\infty = C'(\infty)$ in this case),

$$(4.22) \qquad P^C_{jN} - C'(\infty) > K$$

for some $K > 0$. Following reasoning that is now familiar, choose δ such that $\lambda(\delta) < K/2$. Then by charging $P^*_{jN} = P^C_{jN} - 2\lambda(\delta)$ for the subsequence, firm j undercuts all firms in $\Gamma_N(\delta)$ and guarantees profits of at least

(4.23) $\pi_{jN}(P^*_{jN}, \bar{P}^C_{jN}) \geq [P^*_{jN}-C'(\infty)]\ N^S 2\delta f(P^*_{jN}+\lambda(\delta)) > [K-2\lambda(\delta)]\ N^S 2\delta f(P^*_{jN}+\lambda(\delta))$

The first inequality follows because all consumers within δ of firm j will buy at least $f(P^*_{jN}+\lambda(\delta))$ and because average cost cannot exceed $C'(\infty)$. The last inequality is from (4.22) and the choices of δ and P^*_{jN}. Now $\pi^*_{jN}(P^C_N)$ cannot be less than $\pi_{iN}(P^*_{jN}, P^C_{jN})$ by definition, so (4.21) and (4.23) imply

(4.24) $\pi^*_{jN}(P^C_{jN})/\pi_{jN}(P^C_N) > N[K-2\lambda(\delta)]2\delta f(P^*_{jN}+\lambda(\delta))/kf(0)Z$

The right hand side of (4.24) is unbounded for small δ. Since the right hand side of (4.19) is bounded, (4.24) implies that (4.19) cannot hold for all N in the subsequence, contradicting the definition of $\{P^C_N\}$. Q.E.D.

5. Concluding Remarks

This paper has investigated the extent to which a class of large dynamic markets for differentiated products are characterized by approximately marginal cost pricing. In section 3 it was shown that all static Nash equilibria converge to competition, just as is true for the Cournot markets considered by Ruffin [8]. In section 3 it was shown that convergence to competition is avoidable when dynamic subgame perfect Nash equilibria are considered if and only if the limiting static Nash price is less than the supremum of marginal cost and demand is replicated at precisely the same rate as the number of firms. It is interesting to note that, despite the difference in the models, this result is exactly analagous to the result derived for symmetric Cournot models by Lambson [5].

These results should extend with only minor modifications to more general models. For example, more general downward-sloping demand functions which preserve the boundedness of revenue should pose no problem. The linear specification of λ simplified the exposition in two ways: it allowed an explicit solution of $\partial\gamma_N^x/\partial P_{iN}$ and it implied that an undercutter would capture the undercut firm's entire market. (D'Aspremont, Gabszewicz and Thisse [2] pointed out that the second property needn't hold if $\lambda(\cdot)$ is a convex function.) Neither of these convenient properties is indispensable in proving the theorems, however, and extending the results to convex λ should be a straightforward exercise.

Appendix

Lemma: If either $s \leqslant 1$, or else $s > 1$ and $C''(\infty) < k$, then there exists $\bar{N}$ such that for all $N > \bar{N}$ all firms are in competition on both sides of their markets.

Proof: Renumber firms so $P^r_{jN} \geq P^r_{iN}$ for each N and all i. If $P^r_{jN} < k - \epsilon$ for for some $\epsilon > 0$ and all N then all firms will be in competition for all N larger than some $\bar{N}$. To see that $P^r_{jN} < k - \epsilon$ for all N and some $\epsilon > 0$ assume otherwise. Then there is a subsequence such that for any $\delta > 0$ and all N exceeding some N_δ,

(A.1) $\quad k - P^r_{jN} < \delta$

Let $\Gamma_N(\delta)$ be the set of firms on both sides of j satisfying $|j-i| < \delta$. By (3.4)

(A.2) $\quad P^r_{jN} \leq P^r_{iN} + \lambda(\delta)$

for $i \epsilon \Gamma(\delta)$. By (A.1) and (A.2)

(A.3) $\quad P^r_{iN} > k - \delta - \lambda(\delta)$

for $i \epsilon \Gamma_N(\delta)$. Now $P_{jN} < P^r_{iN} - \lambda(\delta)$ will undercut $i \epsilon \Gamma(\delta)$ so, by (A.3), if firm j charges

(A.4) $\quad P^e_j \equiv k - \delta - 2\lambda(\delta)$

it will undercut all $i \epsilon \Gamma_N(\delta)$ for $N > N_\delta$ and be able to service at least an interval of length 2δ. This contradicts the definition of P^r_{jN} as follows: Given P^r_N firm j will sell to an interval no greater than Z/N, so since profit cannot exceed revenue and no consumer serviced by firm j will buy more than $f(P^r_{jN})$, (A.5) $\pi_{jN}(P^r_N) \leq P^r_{jN}[N^s f(P^r_{jN})Z/N]$

If $s \leq 1$ then $\pi_{jN}(P^r_N)$ approaches zero for the subsequence (since P^r_{jN} approaches k). But choose δ so $P^e_j > C'(0) + \eta$ for some $\eta > 0$. Then (for $N > N_\delta$) if firm j charges P^e_j and chooses output q^e_{jN} so $q^e_{jN} > u$ for some $u > 0$, $C'(q^e_{jN}) \leq C'(0)+\eta$, and q^e_{jN} doesn't exceed demand (which is bounded strictly away from zero for $N > N_\delta$), then

(A.6) $\pi_{jN}(P^e_j, \bar{P}^r_{jN}) \geq [P^e_j - (C'(0) + \eta)]u$

so $\pi_{jN}(P^e_j, \bar{P}^r_{jN})$ is bounded strictly away from zero for $N > N_\delta$. Then (A.5) and (A.6) imply that $\{\pi_{jN}(P^e_j,\bar{P}^r_{jN})/\pi_{jN}(P^r_N)\}$ is unbounded for a subsequence, contradicting the definition of P^r_N. If $s > 1$ and $C'(\infty) < k$ choose δ so $P^e_j > C'(\infty)$. Then (for $N > N_\delta$) if firm j charges P^e_j and satisfies demand, since average cost cannot exceed $C'(\infty)$ and each serviced consumer within δ of firm j will buy at least $f(P^e_j + \lambda(\delta))$,

(A.7) $\pi_{jN}(P^e_j, \bar{P}^r_{jN}) \geq [P^e_j - C'(\infty)]N^s f(P^e_j + \lambda(\delta))2\delta$

Then (A.5) and (A.7) imply that $\{\pi_{jN}(P^e_j, \bar{P}^r_{jN})/\pi_{jN}(P^r_N)\}$ is unbounded for a subsequence, contradicting the optimality of P^r_{jN}. Q.E.D.

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