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DISTINGUISHING RANDOM AND DETERMINISTIC SYSTEMS

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ABSTRACT

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This paper discusses tests on time series for the presence of low dimensional deterministic chaos. Empirical applications on U.S. business cycle data are reviewed. Two tasks are necessary to test a time series for (low dimensional) deterministic chaos: (1) dimension must be calculated and shown to be low relative to comparable (psuedo) random numbers; (2) Lyapunov exponents and other measures of local instability must be calculated and show local instability relative to a useful standard. The evidence for low dimensional deterministic chaos in U.S. postwar II GNP data is weak (English). Brock: University of Wisconsin, Madison, Wisconsin. Journal of Economic Literature Classification Numbers: 023, 211, 213.

DISTINGUISHING RANDOM AND DETERMINISTIC SYSTEMS

by

W. A. Brock

1. Introduction

Recently there has been a lot of interest in nonlinear deterministic economic models that generate highly irregular trajectories (for example, Benhabib and Day [1983], Dana and Malgrange [1986], Grandmont [1983], Day [1982], Stutzer [1980], Day and Shafer [1983], Deneckere and Pelikan [1984], Boldrin and Montrucchio [1985].) Intense exploration of low dimensional deterministic dynamical systems models has been going on in physics and chemistry, Swinney [1983], ecology and biology, May [1976], population dynamics Brillinger et al [1980] and so on.

The literature cited above relies heavily on mathematical literature on "chaos" and nonlinear dynamics. See Collet and Eckmann [1980], Guckenheimer and Holmes [1983] and Grandmont's introduction to this volume for overviews of the relevant mathematical literature and economic literature respectively.

The main reason for this recent explosion of interest in nonlinear dynamics by the applied sciences is that the trajectories generated by some nonlinear difference equations look completely random to the naked eye. A particularly dramatic example was given by Sakai and Tokumaru [1980]. They show that most trajectories of the difference equation $x_{t+1} = F(x_t)$, x_o given where

$$(1) \qquad F(x) \equiv x/a,\ x\varepsilon[0,a],\ F(x) \equiv (1-x)/(1-a),\ x\varepsilon[a,1],\ 0 < a < 1$$

generate the same autocorrelation coefficients as the first-order AR process

(2) $v_{t+1} = (2a-1)\ v_t + u_{t+1},\ \{u_t\}$ i.i.d.

Bunow and Weiss [1979] present chaotic first order difference equations that generate spectra indistinguishable from that generated by a sequence of i.i.d. random variables.

Brock and Chamberlain [1984] show that given any spectral measure G there is a deterministic overlapping generations economy whose equilibrium trajectory generates an empirical spectrum that approximates G. Hence linear time series methods (spectral analysis and autocovariance functions) may not be able to observationally distinguish between deterministic and random systems. We come to the subject of this article.

The subject of this article is to discuss tests that are potentially capable of distinguishing between certainty and uncertainty. We will argue that such tests can be usefully applied to economic data.

This paper is organized as follows. Section 1 contains the introduction. The second section gives tests for deterministic chaos that can be applied to a time series of data. Section 3 reports applications of the tests to postwar U.S. business cycle data and to the Wölfer sunspot series. In Sections 4, 5 we attempt to relate and integrate the "new methods" of data analysis discussed here with received practice in economics. It is a "first cut" and should be treated as such. Section 6 is a short summary. Finally an expanded version of this paper is available in Brock [1985].

2. Tests for Chaos

In order to demonstrate that an apparently random time series $\{a_t\}$ is actually deterministic chaos the researcher must show (1) the dimension is low and (2) there is a positive Lyapunov exponent. To explain we need

Definition 2.1 (Takens [1983]). The time series of real numbers $\{a_t\}_{t=1}^{\infty}$ has a smoothly deterministic explanation if there exists a system (h,F,x_o) such that $h: R^n \to R$, $F: R^n \to R^n$ are smooth and

(2.1) $a_t = h(x_t),\ x_t = F(x_{t-1}),\ t=1,2,\ldots;\ x_o$ given.

There is an analogue of this definition for continuous time systems but, for space reasons, we concentrate on discrete time systems in this paper. The following definition will be needed

Definition 2.2: An attractor Λ of the deterministic dynamical system F is a compact set with a neighborhood U such that for almost every (in the sense of Lebesque measure on R^n) initial condition $x_o \varepsilon U$ we have $x_t(x_o) \to \Lambda,\ t \to \infty$. That is the limit set of $\{x_t(x_o)\}_{t=1}^{\infty}$ is the attractor Λ. The basin of attraction of an attractor is the closure of the set of initial conditions whose trajectories approach the attractor as time tends to $+\infty$. A chaotic attractor is one for which typical orbits on the attractor locally spread apart exponentially. To put it another way typical orbits on the attractor have a positive Lyapunov exponent.

Definition 2.3 (Guckenheimer and Holmes [1983, pp. 283-284]). Let $F: R^n \to R^n$ define a discrete dynamical system. Fix $x \varepsilon R^n$. Suppose that there are subspaces $V_i^{(1)} \supset V_i^{(2)} \supset \ldots \supset V_i^{(n)}$ in the tangent space at $F^i(x)$ and numbers $\mu_1 > \mu_2 > \ldots > \mu_n$ with the properties that

a) $DF(V_i^{(j)}) = V_{i+1}^{(j)}$

b) $\dim V_i^{(j)} = n+1-j$

c) $\lim_{N\to\infty} (1/N) \ln ||D_x F^N(v)|| = \mu_j$ for all $v \varepsilon V_o^{(j)} - V_o^{(j+1)}$.

Then the μ_j are called the Lyapunov exponents of F. Lyapunov exponents are a generalization to general attractors of eigenvalues of $D_{\bar{x}}F$ at a fixed point $\bar{x}$. For later use we need the Oseledec multiplicative ergodic theorem.

Theorem 2.1: (Oseledec [1968], Guckenheimer and Holmes [1983, p. 284], Benettin et al [1976]). Let F be C^1 and let DF be Hölder continuous for some exponent θ. Let F and its attractor Λ possess an ergodic invariant (Guckenheimer and Holmes [1983, p. 280]) measure μ. Then there is a μ-measurable set $\Lambda_1 \subseteq \Lambda$ such that $\mu(\Lambda_1) = \mu(\Lambda)$ such that for all $x \in \Lambda_1$, Lyapunov exponents exist.

Remark: Since dim $V^{(1)} = n+1-1 = n$ therefore, as pointed out by Bennettin et. al [1976, p. 2339], if one chooses the vector v in (c) "at random" one may expect to find $\mu_j = \mu_1$.

Example 2.1: The tent maps (1.1) have as invariant measure $\mu(dx) = j(x)dx$, $j(x) = 1$, $x\varepsilon[0,1]$. The logistic map $x_{t+1} = 4x_t(1-x_t)$ has invariant measure $\mu(dx) = j(x)dx$, $j(x) = 1/(\pi(x(1-x))^{1/2})$. The largest Lyapunov exponent for these maps is given by

$$(2.2) \qquad \lambda = \int_0^1 \ln|F'(x)|\mu(dx)$$

so that for the tent map $F(x) = 2x$, $x_t\varepsilon[0,1/2]$, $F(x) = 2(1-x), x\varepsilon[\frac{1}{2}, 1]$, we get $\lambda = \ln 2$.

With this preparation we may explain the two tasks that the researcher must do to test for deterministic chaos in time series data: (1) show that the dimension of the time series is low, (2) show that the largest Lyapunov exponent is positive.

Task I: Calculating Dimension of a Time Series

In order to motivate notions of dimension suppose that $\{a_t\}_{t=1}^{\infty}$ has a deterministic explanation by the system (h,F,x_o) and look at the m-history starting at t:

$$(2.3) \qquad a_t^m \equiv (a_t, a_{t+1}, \ldots, a_{t+m-1}) = (h(x_t)), \ldots, h(F^{m-1}(x_t)) \equiv J_m(x_t).$$

Hence $J_m: \Lambda \subseteq R^n \to R^m$. In order to grasp what is coming look at examples (1.1). Here $\Lambda = [0,1]$ and $j(x) = 1$ for the tent maps. Thus $J_m: [0,1] \to R^m$, x_t is distributed according to μ which is uniform on [0,1] so that the dimension of $J_m(\Lambda) \equiv \{J_m(x), x\varepsilon\Lambda\}$ is <u>one</u> for any sensible notion of dimension. This is so because $J_m(\Lambda)$ is a one dimensional arc embedded in R^m provided that J_m is a smooth map. Furthermore Takens [1980] has shown that generically J_m is 1-1 from Λ to $J_m(\Lambda)$ if $m \geq 2n+1$. The same reasoning applies whatever the dimensions of Λ. Calculate the dimension D_m of $J_m(\Lambda)$ for all "embedding dimensions" m and find $\lim_{m\to\infty} D_m$.

We now face the practical problem of calculating the dimension of $J_m(\Lambda)$ for each m from a finite data set $\{a_t\}_{t=1}^{N}$. After much experimentation discussed in Brock [1985] the natural science community seems to have settled on the Grassberger-Procaccia [1983] correlation dimension α_m as the most useful dimension measure. It is defined for $\varepsilon > 0$ by

$$(2.4) \qquad \alpha_m \equiv \lim_{\varepsilon\to 0} \ell n\, C_m(\varepsilon)/\ell n\, \varepsilon$$

(2.5) $\quad C_m(\varepsilon) \equiv \#\{(i,j)/||a_i^m - a_j^m|| < \varepsilon,\ 1 < i < N_m,\ 1 < j < N_m\}/N_m^2$

(2.6) $\quad N_m \equiv N-(m-1).$

Here #A denotes the cardinality of set A and ℓn is the natural logarithm.

Natural scientists (e.g. Swinney [1985] and references) construct Grassberg-Procaccia dimension plots of $\ell n\ C_m(\varepsilon)$ against $\ell n\ \varepsilon$ and attempt to measure the slopes of $\hat{\alpha}_m$ of these G-P plots for each embedding dimension m. After constructing these plots they look to see if $\hat{\alpha}_m$ levels off to some $\hat{\alpha}$ as $m \to \infty$.

This procedure requires skill and judgment for two reasons. First since $\ell n\ C_m(\varepsilon) = 0$ for large ε you must "estimate" the slope of the G-P plot over a range of ε of moderate to small size. Second Brock and Dechert [1985] prove that if there is noise in the data of positive variance even if the variance is small then for each embedding dimension m, $\alpha_m = m$ almost surely. Therefore the slopes $\hat{\alpha}_m$ of the dimension plots must be estimated over ε's larger than the scale $\underline{\varepsilon}$ of any noise that is present in the data set.

In practice a range of ε's over which the slope $\hat{\alpha}_m$ of the G-P plot appears to be "stable" is chosen by "eyeballing". The noise level $\underline{\varepsilon}$ is "estimated" by hunting for an $\underline{\varepsilon}$ small enough so that $\hat{\alpha}_m \sim m$ when $\hat{\alpha}_m$ is estimated over ε in $(0,\underline{\varepsilon}]$ (Ben-Mizrachi et. al [1984]). This procedure is applied to data in Section 3 of this article.

Task II: Calculating the Largest Lyapunov Exponent

Calculation of the largest Lyapunov exponent λ is based upon the formulae (a-c) of Definition 2.3 and the Oseledec Theorem 2.1. We briefly explain the Wolf et. al [1984] algorithm which we use in our empirical work.

For each embedding dimension we use the time series $\{a_t\}_{t=1}^{N}$ to form a a time series $\{a_t^m\}_{t=1}^{N_m}$ of m-histories. Start the algorithm by locating the nearest neighbor $a_{t_1}^m \neq a_1^m$ to the initial m-history a_1^m. Let $d_1^{(1)} = ||a_{t_1}^m - a_1^m||$. Note that $d_1^{(1)}$ is the smallest positive distance $||a_t^m - a_1^m||$. Select an evolution time q and set $d_2^{(1)} = ||a_{t_1+q}^m - a_{1+q}^m||$ and store $g_1(q) = d_2^{(1)}/d_1^{(1)}$. This ends the first iteration. We are now ready to enter the main program loop.

Ideally, in order to start the second iteration we would like to find a new m-history $a_{t_2}^m$ near a_{1+q}^m whose angle $\theta(a_{t_2}^m - a_{1+q}^m, a_{t_1+q}^m - a_{1+q}^m)$ is close to zero. In this way we mimic the definition 2.3(c) of Lyapunov exponent as closely as possible with $a_{t_1}^m - a_1^m$ playing the role of v. Definition 2.3(b) shows that except for hairline cases $\lim(1/N)\, \ell n||D_x F^N(v)|| = \mu_1$, the largest Lyapunov exponent, because the set $V_o^{(1)} - V_o^{(2)}$ has full Lebesque measure.

Motivated by this strategy we choose t_2 to minimize the penalty function

$$(2.7) \qquad p(a_t^m - a_{1+q}^m,\ a_{t_1+q}^m - a_{1+q}^m) \equiv ||a_t^m - a_{1+q}^m|| + w|\theta(a_t^m - a_{1+q}^m,\ a_{t_1+q}^m - a_{1+q}^m)|$$

subject to the nondegeneracy requirement $a_t^m \neq a_{1+q}^m$. Here w is a penalty weight on the deviation $|\theta|$ from zero. Store

$$(2.8) \qquad g_2(q) \equiv d_2^{(2)}/d_1^{(2)},\ d_1^{(2)} \equiv ||a_{t_2}^m - a_{1+q}^m||,\ d_2^{(2)} \equiv ||a_{t_2+q}^m - a_{1+2q}^m||.$$

This ends iteration two. Continue in this manner.

For iteration k store

$$(2.9) \qquad g_k(q) \equiv d_2^{(k)}/d_1^{(k)},\ d_1^{(k)} \equiv ||a_{t_k}^m - a_{1+(k-1)q}^m||, d_2^{(k)} \equiv ||a_{t_k+q}^m - a_{1+kq}^m||,$$

where t_k minimizes

$$p(a_t^m - a_{1+(k-1)q}^m,\ a_{t_{k-1}+q}^m - a_{1+(k-1)q}^m)$$

subject to $a_t^m \neq a_{1+(k-1)q}^m$. Continue until k=K where K solves $\max \{k \mid 1+kq \leq N_m\}$. Set

$$(2.10)\qquad \hat{\lambda}_q \equiv \frac{1}{K} \sum_{k=1}^{K} [\ell n(d_2^{(k)}/d_1^{(k)})/q]$$

Brock and Dechert [1985] locate sufficient conditions on the system (h,F,x_o) that deterministically explains the data under the null hypothesis of a deterministic explanation that enables them to prove that an idealized form of the above algorithm converges to the largest Lyapunov exponent λ for almost all starting vectors. See the Wolf, et. al paper for a discussion of numerical experience with this algorithm.

3. Empirical Application of These Ideas

Empirical calculation of the Grassberger-Procaccia [1983] correlation dimension α and the largest Lyapunov exponent λ is the procedure typically used in natural science to test for the presence of chaos in time series data (Swinney [1985], Wolf et al [1984]). Economists must deal with time series much shorter than the 10,000-30,000 observations typically used in natural science work and, furthermore, economic time series are probably noisier. The problem is especially acute in business cycle analysis.

Brock and Sayers [1985] test for nonlinearities in quarterly data on U.S. real gnp and U.S. real gross private domestic investment by (a) calculating the Grassberger-Procaccia [1983] correlation dimension and estimating the largest Lyapunov exponent for various embedding dimensions; (b) calculating measures of asymmetry such as, Blatt [1978], and measures of skewness and kurtosis. For lack of space, we only describe some of the results for U.S. quarterly data from 1947:1 to 1985:1 (1972 = 100), and for Wolfer's sunspot numbers 1749–1924.

Let x_t = real GNP at quarter t and y_t = real gross private investment at quarter t. The data was detrended by the following OLS regressions:

(3.1) $\log x_t = 2.681 + .003671t + ex_t$

(3.2) $\log y_t = 1.851 + .003765t + ey_t.$

In view of the well known result that autoregressive models of order two (AR(2) fit detrended U.S. real GNP well we fit two AR(2) models:

(3.3a) $ex_t = 1.36 \quad ex_{t-1} \quad -.42 \quad ex_{t-2} + \delta x_t, \; R^2 = .933$

(3.3b) $\sigma_{ex} = .0167, \; \sigma_{\delta x} = .0043, \; \sigma_{ex}/\sigma_{\delta x} = 3.88, \; sk_{ex} = -.244, \; sk_{\delta x} = -.086,$

$k_{ex} = 2.19, \; k_{\delta x} = 4.05.$

(3.4a) $ey_t = 1.12 \quad ey_{t-1} \quad -.31 \quad ey_{t-2} + \delta y_t, \; R^2 = .760$

(3.4b) $\sigma_{ey} = .051, \; \sigma_{\delta y} = .025, \; \sigma_{ey}/\sigma_{\delta y} = 2.03, \; sk_{ey} = -.46, \; sk_{\delta y} = -.57,$

$k_{ey} = 2.78, \; k_{\delta y} = 5.14.$

Have σ_A, sk_A, k_A denote standard deviation of A, skewness of A, kurtosis of A. Standard errors are not reported because the second coefficient changes a lot when we fit AR(3) and AR(4) to this data.

For $\{ex_t\}$, $\{ey_t\}$, $\{\delta x_t\}$, $\{\delta y_t\}$ we calculated, for embedding dimension d,

(3.5) $C(\varepsilon) \equiv \frac{1}{n_d{}^2} \#\{(i,j) \mid \|a_i^d - a_j^d\| < \varepsilon\}, \; C^*(\varepsilon) \equiv \frac{1}{n_d^*} \#\{(i,j) \mid i \neq j, \|a_i^d - a_j^d\| < \varepsilon\}$

(3.6) $\alpha(\varepsilon) \equiv \ln C(\varepsilon)/\ln\varepsilon, \quad \alpha^*(\varepsilon) \equiv \ln C^*(\varepsilon)/\ln\varepsilon,$

$$(3.7) \qquad SC(\varepsilon_k,\varepsilon_{k-1}) \equiv (\ln C(\varepsilon_k) - \ln C(\varepsilon_{k-1}))/(\ln\varepsilon_k - \ln\varepsilon_{k-1})$$

$$(3.8) \qquad SC^*(\varepsilon_k,\varepsilon_{k-1}) \equiv (\ln C^*(\varepsilon_k) - \ln C^*(\varepsilon_{k-1}))/(\ln\varepsilon_k - \ln\varepsilon_{k-1})$$

and estimated the largest Lyapunov exponent λ. Here $n_d^* \equiv n_d^2 - n_d$, $|S|$ denotes the cardinality of set S; $n_d = n-(d-1)$ is the number of d-histories $a^d = (a_t^d,\ldots,a_{t+d-1})$ constructed from the sample of length n, and $1 \leq i,j \leq n_d$.

Three difficulties emerged. First a standard of comparison was needed so that meaning could be attached to a "large" or "small" dimension or Lyapunov exponent. The same measures of dimension and Lyapunov exponent were calculated for normal psuedo random numbers of the same mean and standard deviation for each series.

Second, the utility of the measure $\alpha(\varepsilon)$ (suppressing obvious subscripts for ease in typing) is based on "the power law conjecture".

$$(3.9) \qquad C(\varepsilon) \cong K(\varepsilon)\varepsilon^{\alpha(\varepsilon)}, \ \alpha(\varepsilon) \to \alpha, K(\varepsilon) \to K, \ \varepsilon \to 0.$$

so that

$$(3.10) \qquad \ln C(\varepsilon)/\ln\varepsilon = \ln K(\varepsilon)/\ln\varepsilon + \alpha(\varepsilon) \to \alpha, \ \varepsilon \to 0.$$

Have the subscript "∞" on any symbol denotes the limit as n_d or n_d^* goes to infinity. With a time series of length 10,000, $\ln C_{d,n_d}(\varepsilon)/\ln\varepsilon$ may approximate $\ln C_{d,\infty}(\varepsilon)/\ln\varepsilon$ even for small ε but a time series of length 100-200 is a different matter. For example let $\underline{\rho} = \min \{||a_i^d - a_j^d|| > 0, 1 \leq j \leq n_d\} > 0$, be the smallest positive distance. For $\varepsilon < \underline{\rho}$, $C_{d,n_d}(\varepsilon) = 1/n_d$, $\alpha_{d,n_d}(\varepsilon) = -\ln(n_d)/\ln\varepsilon \to 0$, $\varepsilon \to 0$. Note also that the slope estimator

$SC_{d,n_d}(\varepsilon_k,\varepsilon_{k-1}) = 0$ for ε_k, $\varepsilon_{k-1} < \underline{\rho}$. This is true even if $\{a_t\}_{t=1}^n$ is a sequence of random numbers for which the theoretical value of α and SC is d. Hence α and SC are poor estimators of the underlying dimension especially for small data sets.

Since α^*, SC* give values much closer to the theoretical value of d for random numbers therefore we calculated the measures α^*, SC* as well. These measures attempt to remove the distortion in a possible power law caused by the fact $C_{d,n_d}(\varepsilon) = 1/n_d$ on $[0, \underline{\rho})$.

There is the further problem of how to report the results of dimension calculations for the small data sets used in business cycle analysis. That is to say what constitutes a "confidence interval" and "rejection region" under a given null hypothesis for an "estimator" where we know very little about the sampling distribution and where generating an empirical distribution under various null hypotheses is expensive? We shall handle this problem as we have seen it handled in the natural science literature that we have seen to date. That is to try to report our findings in such a way that the reader may make her own judgment as to the significance or stability of any estimate. Due to lack of space we can only report the highlights of the results of Brock and Sayers [1985] here.

Third, even less is known about the small sample properties of our estimate of the largest Lyapunov exponent. Hence we calculated a measure of <u>cumulative</u> spreading between initially close trajectories for (a) our data set, (b) random numbers of the same mean and variance as our data set, and (c) a time series generated by the tent map (1.1) with a = 1/2.

In a nutshell our results indicate the following: there is not enough information in the 1947:1-1985:1 data set for the Grassberger-Procaccia type dimension measures to reject the null hypothesis that detrended real GNP and real GPDI are generated by an AR(2) process as estimated in (3.3) and (3.4). To put it another way we need more data to establish that U.S. real GNP and real GPDI are generated (in the main) by a low dimension chaotic deterministic dynamical system.

It may be helpful to future researchers in this area to describe the chain of thought that lead us to this conclusion. First we calculated measures of dimension for $\{ex_t\}$, $\{ey_t\}$ and got evidence of low dimension as revealed in the tables below.

Second, our colleague Don Hester suggested that AR(2) models like (3.3) and (3.4) might generate a low dimension estimate if they fit the data well. Although a theorem of Brock and Dechert [1985] assures us that the dimension of an infinite data set generated by (3.3) or (3.4) must be infinity a data set of length 153 may be a long way from infinity especially if the rate of convergence of dimension estimates is slow.

It is easy to prove that if, say, $\{ex_t\}$ has a smoothly deterministic explanation

$$ex_t = h(z_t), \; z_t = F(z_{t-1}), \; z_o \text{ given,}$$

then

$$\delta x_t = h(F^2(z_{t-2})) - 1.36\; h(F(z_{t-2})) + .42\; h(z_{t-2}) \equiv M(z_{t-2}).$$

Hence the dimension of $\{\delta x_t\}$ should be the same as the dimension of $\{ex_t\}$ if $\{ex_t\}$ has a deterministic explanation. In fact Table 1 below shows evidence that the dimension of $\{\delta x_t\}$ is slightly (?) smaller than the "large" dimension of a sequence $\{\tilde{\delta} x_t\}$ of the same length of normal psuedo random numbers with the same mean and variance. This is evidence against the chaotic low dimensional deterministic dynamical system hypothesis.

Third, faced with this mixed evidence we generated sequences $\{\tilde{\delta}x_t\}$, $\{\tilde{\delta}y_t\}$ of normal psuedo random numbers with the same mean and variance as $\{\delta x_t\}$, $\{\delta y_t\}$ and generated two simulated AR(2) time series $\{\tilde{e}x_t\}$, $\{\tilde{e}y_t\}$ of the same length as $\{ex_t\}$, $\{ey_t\}$ using the fitted values of the AR(2) models in (3.3) and (3.4). Grassberger-Procaccia measures of dimension were calculated for $\{\tilde{e}x_t\}$ and $\{\tilde{e}y_t\}$ and compared with those calculated for $\{ex_t\}$ and $\{ey_t\}$. There appears to be no significant difference to the naked eye. See the table for real GNP. A qualitatively similar result holds for real GPDI (Brock and Sayers [1985]).

Fourth, the same battery of tests were applied to the Wölfer sunspot series (Anderson [1971] and Priestley [1981]) which displays asymmetries in that the average gradient of rise from trough to peak is greater than the average gradient of fall from peak to trough across "cycles" (Priestley [1981, 882]). Furthermore, an AR(2) model fits the sunspot series quite well although a better fit is obtained with a bilinear model according to the Akaike criterion (Priestley [1981, p. 884]). The story is similar to that of real GNP and real GDPI. See Tables 2, 3.

Fifth, the same procedure was applied to the Lyapunov exponent calculations. Look at Table 3 which contains calculations for Wolfer's sunspot numbers. We calculated $\hat{\lambda}_q$ from (2.10) for q=1,2,...,9 as well as the mean $\bar{g}$ and the standard deviation σ_g of $\{g_k(q)\}_{k=1}^{K}$. Notice that if $\{a_t\}$ is a sequence of i.i.d. random numbers then from (2.9) and (2.10) $\ln(d_2^{(k)}/d_1^{(k)})/q \to 0$, $q \to \infty$. Hence $\hat{\lambda}_q \to 0$, $q \to \infty$ in this case. But, in contrast, if there is deterministic chaos present in the data we should see a tendency of $\bar{g}(q)$ to grow with q. This is so because $\bar{g}(q)$ is a measure of spreading of nearby trajectories.

Look at Table 3 where three typical runs with psuedo random numbers were done in order to get a perspective. Notice the absence of any tendency of $\overline{g}(q)$ to grow with q and the tendency of $\hat{\lambda}_q$ to fall with q for each of the three runs. Now turn to the same calculations with the sunspot numbers. If there are instabilities i.e., deterministic chaos present in the sunspot data we should see a tendency of $\overline{g}(q)$ to grow with q and $\overline{\lambda}_q$ should not fall with q. Our own naked eye sees little difference from the three runs of random numbers.

Conclusion: dimension tests and Lyapunov exponent calculations have a hard time rejecting a linear model where there is a lot of variation of the data "within the regression plane" relative to the variation of the data "around the regression plane," when the data set is small. Brock and Dechert [1985] formalize this empirical finding into a theorem. Our findings reported above should not be construed as negative to the attempt to find evidence of significant nonlinearities in economic data.

This is so for several reasons. First we used aggregate data. Hence many nonlinearities at the microlevel may have been "washed out". Second the data set 1947:1-1985:1 is a period where severity of recessions has fallen (Zarnowitz and Moore [1984, pp. 12-15]) and growth cycles (Moore and Zarnowitz [1984, Table 8]) look quite symmetric throughout this period. Therefore our results square with DeLong and Summers [1984] who find little evidence of significant skewness of growth rates of GNP and industrial production for the postwar period. The failure to find large kurtosis agrees with Blanchard and Watson [1984]. This does not contradict Blatt because he looks at different series over different periods for example pig iron production data from Burns and Mitchell (Blatt [1981, p. 231)]).

Third, longer data sets may enable us to reject an AR(2) model in favor of a model with significant nonlinearities for U.S. GNP and U.S. GPDI. In any event, DeLong and Summers [1984] agree with Neftci [1984] that there is evidence of asymmetry in the postwar U.S. unemployment rate even though they disagree elsewhere. Hence that series and well known "cyclic" cattle and hog sales series may be a good place to look for significant nonlinearities in economic time series. At the risk of repetition we make a few more remarks about Lyapunov exponent calculations.

Lyapunov Exponents

Due to lack of space we report Lyapunov exponent calculations for the sunspot series only since there seems to be more consensus that asymmetries and hence, nonlinearities, are present in that series (Priestley [1981, p. 882] and references). See Brock and Sayers [1985] for more extensive calculations and for calculations for U.S. real GNP and real U.S. gross private domestic investment.

In interpreting the table the reader is cautioned that at the time of this writing we have little knowledge of the sampling distribution of these estimates. Nevertheless we think the reader will agree that there is little evidence of a tendency of $\bar{g}$ to grow with q -- a tendency that indicates the presence of systematic local spreading of trajectories in the data rather than psuedo random noise. Notice from the table that psuedo random noise generates a positive Lyapunov exponent. In contrast to the situation depicted in table 3 we ran the same algorithm on data generated by the tent map $F(x) = 2x$, $x\varepsilon[0,1/2]$, $F(x) = 2-2x$, $x\varepsilon[1/2,1]$ and got $d_2/d_1 = 2^q$ out to the third or fourth decimal place as well as $\hat{\lambda}_q \sim \ln 2$. Even if noise was added $\bar{g}_q$ rose with q.

Obviously much more work must be done to reject the null hypothesis of the presence of nonlinear chaos in sunspot data. See Brock and Sayers [1985] for a far more extensive discussion.

4. Panel Data

The methods discussed above can be adapted to test whether a panel $\{a_{it}\}_{t=0}^{T_i}$, $i=1,2,\ldots,I$ has a low dimensional deterministic explanation. To set the stage for what follows let us discuss the simplest problem first.

Definition 1 We say that the panel $\{a_{it}\}$ has a deterministic explanation if there exists a dynamical system (h,F) where $h:R^n \to R, F:R^n \to R^n$ and a distribution of initial conditions $\{x_{io}\}$ such that

$$a_{it} = h(x_{it}), \tag{1}$$

$$x_{i,t+1} = F(x_{it}), \quad t=0,1,\ldots,T_i-1. \tag{2}$$

Obviously for this definition to imply restrictions on the data the dimension n must be small relative to T_i. Also note that testing a panel for a low dimensional deterministic explanation is somewhat analogous to looking for low dimensional nonlinear "factors" or testing for the presence of a low dimensional or nonlinear unobservable index (cf. Sargent and Sims [1977] for linear unobservable index models).

Furthermore, testing for a low dimensional deterministic explanation has little to do with testing for strange attractors when we are working with panel data rather than time series data. In the case of time series data one needs the presence of a dynamical system with a nondegenerate invariant measure in order to make headway. This forces one to consider attractors that are not points or periodic m-cycles. In the case of panel data we can posit a nondegenerate distribution of initial conditions. Nevertheless the methods of Takens [1980], [1983], Ben-Mizrachi et al [1984], Scheinkman [1985], and their references for calculating various notions of "dimension" may be applied to testing panel data for a low dimensional deterministic explanation. Likewise the methods discussed in section 3 may be used to calculate measures of local expansion or contraction of trajectories in panel data.

Before we get into details we want to point out the relevance of the panel data problem to economic time series data. We believe that there is a consensus (cf. Zarnowitz [1984] and his references) that most economic time series are "blatantly nonstationary" (Brillinger and Hatanaka [1969, p. 133]) but that in certain cases the nonstationarity is of a slowly altering nature. In these cases one could use judgment or tests for nonstationarity to split the time series into panels where one hopes that approximate stationarity of the underlying dynamics holds within each "panel". The panel data analysis to be presented below may be useful in such cases.

The idea of our test is simple. If x is low dimensional, for example one dimensional, then the image of the domain of F under the map $H_m(\cdot)$:

(3) $$H_m(x) \equiv (h(x),\ h(F(x)),\dots,h(F^N(x)))$$

is no more than one dimensional no matter how large is N. That is to say if $x \in R^n$ then for all m = 1,2,... we must have $\dim H_m(R^n) < \dim R^n$ for any sensible notion of dimension if h,F are smooth. Smoothness is needed to prevent "space filling curve" type pathology.

Examine the sequences $a_i^1 \equiv a_{io}$, $a_i^2 \equiv (a_{io}, a_{i1})$, $a_i^3 \equiv (a_{io}, a_{i1}, a_{i2})$,... The set $\{a_i^m\}_{i=1}^I$ must lie on the image of an n dimensional manifold lying in m dimensional space. This will be revealed by calculating the dimension of

$\{a^m\}_{i=1}^I$. However, this may require a lot of data. A method of forming the inputs into the dimension calculation at each "embedding dimension m" that is less wasteful of the panel data is to use not only $(a_{io},\ldots,a_{im})$ but also $(a_{i1},\ldots,a_{i,m+1}),\ldots,(a_{it},\ldots,a_{i,m+t})$ for $t \leq T_i$ for each i. Turn now to a more realistic problem involving panel data.

The case where (h,F) may depend upon i

Clearly we must restrict the dependence of h and F upon i or we could manufacture a deterministic explanation for any panel. To keep things specific suppose that only F can depend upon i and F depends upon i through the one dimensional parameter α_i. That is to say $F = F(x,\alpha)$.

Definition 2 We say that $\{a_{it}\}$ has a one dimensional characteristic deterministic explanation if everything is as in Definition 1 with $F(x_{it})$ replaced by $F(x_{it},\alpha_i)$.

We may apply exactly the same reasoning as above to test for a one dimensional characteristic deterministic explanation of $\{a_{it}\}$. That is to say the map:

$$H_m(x,\alpha) \equiv (h(x), h(F(x,a),\ldots,h(F^m(x,\alpha)))$$

sends $R^n \times R$ into R^{m+1}. Indeed one could probably generalize Takens [1980] to even show that H_m is 1-1 for m large enough. In this case reconstruction of the underlying dynamics may be possible. We must leave this to future research. In any event a low dimension of $\{a_i^m\}$ that saturates with m indicates a low dimensional set of characteristics and a low dimensional dynamical system drives the data.

5. Tests for Deterministic Explanations of Business Cycles.

We doubt that anyone believes that business cycles are generated by a low dimensional *deterministic* dynamical system. What is at issue is whether there is evidence consistent with the hypothesis that a low dimensional nonlinear deterministic dynamical system explains a "substantial" portion of the variance in a collection of macroeconomic time series. Let us expand upon this point.

Most modern macro-money-finance models, especially the recursive setups used by rational expectationists, imply the existence of a low dimensional "state" variable, call it x, such that if $\{a_{it}\}$ is a collection of economic time series we must have

(1) $a_{it} = h_i(x_t)$, all i,t.

Lucas [1975] and Michener [1984] are examples. The state variable x_t is typically generated by a stochastically stable process $x_{t+1} = F(x_t, \nu_{t+1})$. Such models are usually "log linearized" by the rational expectationists. Then linear time series methods are used to test hypotheses suggested by rational expectations theory. Sargent [1981] is typical.

If one is persuaded by the evidence for nonlinearity in economic and financial time series discussed by Blatt [1981], Neftci [1984], Hinich and Patterson [1985], Zarnowitz [1984], et al. then it is natural to believe that the basically (locally) log linear theories discussed above are missing important elements of the business cycle. As Zarnowitz [1984] points out, the leading rational expectations models emphasized *exogenous* instability rather than *endogenous* instability as stressed by the older literature and Keynesian models. Indeed one may look upon efforts such as Benhabib and Day [1982], Day [1982], Day and Shafer [1985] and Grandmont [1983] as attempts to formalize the earlier literature.

For the purposes of our discussion here one may view a major part of Grandmont's [1983] work as an attempt to show that his model is potentially capable, like the rational expectations models of Lucas [1975] are capable, of generating high pairwise coherences between relevant macroeconomic time series at the relevant business cycle frequencies (Sargent [1979, p. 256]). For a contrary view see Sim's comments [1984] on Grandmont's [1983] paper.

How might one test for such a possibility in macro data? To put it another way how might one test for the presence of a low dimensional state variable governed by chaotic dynamics that drives a cross section of time series macro data? Let us adapt the approach to panel data developed in section 4.

Suppose $\{x_t\}$ in (1) follows

(2) $$x_t = F(x_{t-1}),\ x_o \text{ given}$$

where F is chaotic. Of course the methods of sections 2 and 3 can be applied to each time series $\{h_i(x_t)\}_{t=0}^{\infty}$ to test for a deterministic explanation but this "wastes" the restrictions implied by a low dimensional state variable. Besides, a single macroeconomic time series is too short for reliable dimension calculations. A test is contained in the following proposition.

<u>Proposition</u> Suppose that $\{a_{it}\}$ has a deterministic explanation with a common low dimensional state variable. That is there exist smooth maps $h_i\colon\ R^m \to R$, $i=1,2,\ldots,I$ and $F\colon\ R^m \to R^m$, $x_0 \in R^m$ such that

(3) $$a_{it} = h_i(x_t),\ x_{t+1} = F(x_t),\ x_0 \text{ given}$$

Then, except for hairline cases we must have: There is i, such that there is N such that for $n > N$, $t < s$

(4) $\quad a^n_{it} \sim a^n_{is}$ implies $a_{j,t+n+\tau} \sim a_{j,s+n+\tau}$,

for <u>all</u> $j=1,2,\ldots,I$; and $\tau = 0,1,2,\ldots$

<u>Proof:</u> Adapt Takens [1980].

Notice that this test is similar to the Guckenheimer [1982] test except that close n-histories, $a_{it} \sim a_{is}$ for <u>some</u> i implies that the "future," $\{a_{j,s+n+\tau}\}_{t>0}$ must unfold exactly like the "past" for <u>all</u> of the I time series $j=1,2,\ldots,I$. This is perfect predictability with a vengeance and it is likely that any set of macroeconomic time series will vigorously reject this hypothesis.[1/]

However it does suggest a setup that captures the spirit of (4), but not the letter, that is more relevant to the real world of noisy economic time series. The idea is to rephrase (4) in terms related to Granger causality tests (Granger and Newbold [1977, pp. 224-226] and their uses in macroeconomics.

To explain let

(5) $\quad W_n(\delta) \equiv \{(t,s) \mid t < s \text{ and there is } i \text{ such that } ||a^n_{it} - a^n_{is}|| < \delta\}.$

Then there must be N such that for $n > N$ and δ small and for each j over $W_n(\delta)$, a_{jt}, $a_{j,t+n+\tau}$ predicts $a_{j,s+n+\tau}$ and no other information such as $\{a_{j,s+n+\tau-k}\}_{k>1}$ helps predict $a_{j,s+n+\tau}$. I.e. test that other series such as the "natural" series $\{a_{j,s+n+\tau-k}\}_{k>1}$, is <u>not</u> Granger causally prior to

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[1/] We say this for two reasons, one practical, one theoretical. The practical matter is that trying to measure the activities of several tens of millions of human beings is bound to create noisy data. People are more complicated than particles. The theoretical proposition is that frictionless markets must generate locally unpredictable price changes else systematic profit opportunities exist. See Sims [1984] for a recent formalization of this argument.

$a_{j,s+n+\tau}$ over $W_n(\delta)$. That is to say, given $a_{j,t+n+\tau}$, the incremental predictive content over $W_n(\delta)$ for $a_{j,s+n+\tau}$ is nil for all other information including the natural information $\{a_{j,s+n+\tau-k}\}_{k>1}$.

Put this way it seems that all we have to do now is apply ordinary least squares regression analysis as in econometrics textbooks (e.g. Johnston [1984]) and casuality tests as in, for example, Litterman and Weiss [1985]. There are two major problems that are not resolved in this paper that must be resolved before our "test" can be applied. First the requirement that $(t,s) \in W_n(\delta)$ makes t,s random variables which may cause "sample selection" bias. Second if the true data $\{\bar{a}_{it}\}$ satisfies (4) but the observer sees the noisy data $a_{it} \equiv \bar{a}_{it} + \varepsilon_{it}$ than we have an errors in variables problem. To sum up we have an errors in variables problem with "sample selection" bias. It is well known (e.g., Johnston [1984, p. 428]) that OLS estimates of α_j, β_j are inconsistent in the face of errors in variables. Furthermore the estimates of α_j, β_j that you get out of various approaches to fix the errors in variables problems may depend dramatically on the assumptions you make about the error structure. For an enlightening discussion and application see Goldberger [1984].

This is as far as we can go on applying and adapting the dimension analysis methods explicated in this paper to business cycle analysis. The only point that we want to make is that dimension analysis suggests interesting approaches to business cycle analysis that dovetail neatly with received methods.

5. Summary

In this paper we briefly reviewed methods devised by the natural sciences to test for the presence of low dimensional nonlinear chaos in time series data. The tests consist of two parts. First calculate some notion of dimension and show that it is small. Second calculate an estimate of the largest Lyapunov exponent and show that it is positive.

In this paper we amended and adapted these tests to maximize the information available in the short data sets available in business cycle analysis. Our findings indicate that there is not enough information available in U.S. real GNP, real gross private domestic investment, and Wölfer's sunspot series for the two part test discussed here to reject the null hypothesis that the series under scrutiny were generated by an AR(2) process.

The final section of the paper indicates how to extend the methods reported here to the case of panel data and a cross section of time series. Exploitation of this extra information should enable sharper tests for nonlinear chaos.

Finally a more extensive discussion of these methods is available in Brock [1985] and a more extensive discussion of empirical applications to business cycles and labor statistics is in Brock and Sayers [1985] and Sayers [1985]. Scheinkman's [1985] paper in this volume applies these methods to the stock market.

A list of references that I have found useful in my literature search as well as references to other studies that I have found is in the reference section.

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Table 1

Dimension Calculations for Detrended U.S. Real GNP

U.S. Real GNP

1947:1-1985:1

ex_t, n_d=134

ε	S_{20}	α_{20}	α^*_{20}	SC_{20}	SC^*_{20}
$.9$	9582	5.96	6.02	2.11	1.97
$.9^2$	7674	4.03	3.99	2.45	2.65
$.9^3$	5936	3.50	3.55	3.17	3.26
$.9^4$	4248	3.42	3.48	3.35	3.46
$.9^5$	2990	3.40	3.48	3.41	3.59
$.9^6$	2090	3.40	3.50	3.14	3.39
$.9^7$	1502	3.36	3.48	2.87	3.19
$.9^8$	1112	3.30	3.44	3.13	3.65
$.9^9$	800	3.28	3.47	2.38	2.95
$.9^{10}$	622	3.19	3.41		

$\tilde{e}x_t$, n_d=134

ε	S_{20}	α_{20}	α^*_{20}	SC_{20}	SC^*_{20}
$.9$	7246	8.61	8.72	1.54	1.56
$.9^2$	6166	5.07	5.14	1.59	1.63
$.9^3$	5216	3.91	3.97	2.19	2.25
$.9^4$	4144	3.48	3.54	3.13	3.25
$.9^5$	2982	3.41	3.48	2.62	2.75
$.9^6$	2266	3.27	3.36	2.32	2.49
$.9^7$	1774	3.14	3.23	3.54	3.88
$.9^8$	1224	3.19	3.32	2.91	3.32
$.9^9$	902	3.15	3.32	2.61	3.13
$.9^{10}$	686	3.10	3.30		

δx_t, n=132

ε	S_{20}	α_{20}	α^*_{20}	SC_{20}	SC^*_{20}
$.9$	5636	11.00	10.87	6.03	6.21
$.9^2$	2992	8.50	8.54	6.02	7.30
$.9^3$	1590	7.67	8.12	7.16	7.24
$.9^4$	750	7.54	7.90	6.68	8.98
$.9^5$	372	7.36	8.12	4.25	7.76
$.9^6$	238	6.84	8.06	2.55	7.13
$.9^7$	182	6.23	7.93	2.49	17.39
$.9^8$	140	5.76	9.11	.56	∞
$.9^9$	132	5.14	∞	0	∞
$.9^{10}$	132	4.62	∞	0	∞

$\tilde{\delta}x_t$, n=132

ε	S_{20}	α_{20}	α^*_{20}	SC_{20}	SC^*_{20}
$.9$	4966	11.91	12.10	6.93	7.22
$.9^2$	2392	9.42	9.66	8.16	8.95
$.9^3$	1012	9.00	9.42	9.05	11.65
$.9^4$	390	9.02	9.98	7.13	15.20
$.9^5$	184	8.64	11.02	2.46	12.19
$.9^6$	142	7.61	11.79	.69	∞
$.9^7$	132	—	∞	0	∞
$.9^8$	132	—	∞	0	∞
$.9^9$	132	—	∞	0	∞
$.9^{10}$	132	—	∞	0	∞

Embedding dimension d = 20, n_d=132, 134, $S_{20} = \#\{(i,j) \mid \|a_i^d - a_j^d\| < \varepsilon\}$

Table 2

Dimension Calculations for Wolfer Sunspot Numbers

Wölfer Sunspot Numbers: 1749-1924
From Anderson [1971]

x_t, n_d=157

ε	S_{20}	α_{20}	α^*_{20}	SC_{20}	SC^*_{20}
$.9$	2627	21.25	21.77	2.98	3.21
$.9^2$	1919	12.12	12.49	2.93	4.52
$.9^3$	1409	9.05	9.83	2.98	2.16
$.9^4$	1029	7.54	7.91	4.61	5.75
$.9^5$	633	6.95	7.48	3.18	4.51
$.9^6$	453	6.32	6.99	2.81	4.72
$.9^7$	337	5.82	6.66	2.00	4.17
$.9^8$	273	5.34	6.35	1.18	3.07
$.9^9$	241	4.88	5.99	.77	2.89
$.9^{10}$	223	4.47	5.62		

δx_t, n_d=155

ε	S_{20}	α_{20}	α^*_{20}	SC_{20}	SC^*_{20}
$.9$	1543	26.06	27.00	5.00	5.74
$.9^2$	551	11.94	16.37	4.79	6.16
$.9^3$	313	10.30	12.97	5.37	8.72
$.9^4$	197	9.12	11.91	4.39	12.58
$.9^5$	179	7.75	12.04	.91	5.31
$.9^6$	159	6.80	10.92	1.13	17.01
$.9^7$	155	5.98	11.79	.24	∞
$.9^8$					

$x_t \equiv s_t - \bar{s}$, $\quad x_t = 1.336\ x_{t-1} - .65\ x_{t-2}$, $R^2 = .802$, $n = 174$, $n_d = 155, 157$

Table 3

Lyapunov Exponents and Other Instability Measures for Wolfer's Sunpot Numbers

Sunspot numbers

q	K	$\hat{\lambda}$	g	σ_g
1	100	.64	3.22	5.67
2	50	.57	8.22	22.81
3	45	.38	5.96	7.67
4	35	.35	7.78	8.68
5	25	.28	8.52	10.71
6	24	.21	6.45	8.20
7	21	.25	7.93	5.45
8	18	.15	4.31	2.55
9	16	.16	5.78	4.54

3 runs on random numbers to compare with sunspot numbers

q	K	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	σ_{g1}	σ_{g2}	σ_{g3}
1	100	1.01	.89	1.01	4.06	4.51	3.95	4.26	10.57	3.77
2	50	.64	.68	.57	5.88	6.32	4.67	6.80	6.64	5.49
3	45	.47	.47	.49	5.84	6.07	8.07	6.01	6.22	10.31
4	35	.36	.33	.35	6.13	6.49	6.55	4.92	10.90	6.01
5	25	.28	.24	.30	5.79	5.21	6.09	3.89	5.02	4.35
6	24	.29	.19	.25	10.66	4.19	7.53	12.52	2.95	9.05
7	21	.19	.18	.23	5.95	6.79	8.16	8.62	9.49	8.24
8	18	.17	.15	.21	5.50	17.87	7.42	4.37	60.42	6.12
9	16	.17	.15	.16	7.07	4.88	8.94	6.16	2.88	13.12

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