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ESTIMATING ELASTICITIES OF SUBSTITUTION IN A MODEL OF PARTIAL STATIC EQUILIBRIUM: AN APPLICATION TO U.S. AGRICULTURE, 1947-1974

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Estimating Elasticities of Substitution in a Model of Partial Static Equilibrium: An Application to U.S. Agriculture, 1947-1974

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I. Introduction

During the 1970's the neoclassical cost function gained substantial popularity as a tool for estimating the structure of production -- especially for estimating substitution possibilities. This surge of popularity can be attributed to the widespread application of duality theory to economic analysis and the concomitant development of flexible functional forms.[1] An important assumption which underlies most cost function applications is that all inputs are in static equilibrium. In many instances, however, the assumption of full static equilibrium is suspect, and hence so are the empirical results.[2]

Two basic approaches can be followed to relax the assumption of full static equilibrium. First, costs of adjustment can be recognized explicitly, and the firm can be assumed to be continuously in dynamic equilibrium rather than static equilibrium. The theoretical foundations for models of dynamic equilibrium with explicit costs of adjustment were provided by Eisner and Strotz (1963), Lucas (1967), and others. Berndt, Morrison-White, and Watkins (1979) provide a brief review of empirical applications based on this approach, which they refer to as third generation dynamic models. Second, the firm can be assumed to be in static equilibrium with respect to a subset of inputs (rather than all inputs) conditional on the observed levels of the remaining inputs. It is convenient to refer to this framework as one of partial static equilibrium. The inputs which are in partial static equilibrium are referred to as variable inputs and the remaining inputs are designated as fixed or quasi-fixed inputs.

The specification of dynamic equilibrium is theoretically attractive and leads to elegant models. However, these models are difficult to implement empirically. Furthermore, departures from full static equilibrium may result for reasons other than internal costs of adjustment. For example, regulatory restrictions may hinder capital mobility. In such cases dynamic equilibrium will be an inappropriate specification.

The specification of partial static equilibrium covers the case of dynamic equilibrium as well as other departures from full static equilibrium. Even if dynamic equilibrium were an appropriate specification, the partial static equilibrium specification might be preferred since explicit modelling of the adjustment process can be avoided. The theoretical basis for the partial static equilibrium cost function (hereafter referred to as the variable cost function) can be found in discussions of the variable profit function, of which it is a special case. Diewert (1974) attributes the notion of a variable profit function to Samuelson (1953-4) and early discussion of its properties to Gorman (1968) and a 1970 unpublished version of McFadden (1978). The first empirical application of a variable profit function appears to be Lau and Yotopoulos (1971).

Lau (1976) provides a general theoretical treatment of variable profit functions. Both static equilibrium cost functions and variable cost functions can be treated as special cases of the variable profit function. Lau makes clear that, under quite general regularity conditions, estimates of the structure of production can be obtained from either cost function specification.

Furthermore, knowledge of the structure of production allows one to infer measures such as elasticities of substitution conditional on the levels of any subset of inputs. This point appears to have been overlooked by Mork (1978) who assumed that full static equilibrium was valid but specified a variable cost function in order to obtain "short-run" substitution and price elasticities.[3]

The first objective of the present paper is to derive specific procedures for estimating elasticities of substitution when the partial static equilibrium formulation is appropriate.[4] The second objective is to apply the procedures to U.S. agriculture using the translog variable cost function.

Our application to U.S. agriculture is relevant because it is widely believed that the U.S. farm sector has been in disequilibrium throughout the postwar period.[5] The principal source of alleged disequilibrium is the lack of mobility of self-employed farm labor. Although the number of self-employed farmers has declined continually in the postwar period, it is believed that the exodus has not been rapid enough to achieve the cost-minimizing mix of farm inputs. Thus we treat self-employed farm labor as a quasi-fixed factor in the cost function for the farm sector. We treat land as a fixed factor for the farm sector, since there is little latitude in the amount of land held by the entire sector. A case could be made for treating agricultural structures and equipment as quasi-fixed. However, since the stocks of structures and equipment have grown steadily for most of the postwar period, we believe it is appropriate to treat them as variable rather than quasi-fixed factors.

All of the data are taken from Brown (1977), who constructed new estimates of the entire range of inputs for the farm sector. The data permit specification of a large number of variable factors, but we limit our application to three aggregates: services from hired labor, services from structures and equipment, and all other purchased inputs -- including fertilizer, feed, seed, and energy -- which are referred to as materials.

II. Methodology

We begin by specifying a general production function for the case of a single output and multiple inputs:[6]

(1) $$Y = F(X_1, \ldots, X_n, t),$$

where the inclusion of time (t) allows the structure of production to vary over time. If the production function has convex isoquants, and if for any level of output the cost minimizing input mix is employed, then there exists a total cost function which is dual to (1):

(2) $$CT = G(Y, P_1, \ldots, P_n, t),$$

where the P_i are the prices of the X_i and $CT = \sum_{i=1}^{n} P_i X_i$ is total cost. If the cost minimizing output mix is not employed, (2) is not valid. However, if cost is minimized with respect to a subset of the factor inputs conditional on the level of output and the remaining inputs, then there exists a variable cost function which is dual to (1):

(3) $$CV = H(Y, P_1, \ldots, P_\ell, Z_1, \ldots, Z_m, t),$$

where the Z_i represent the subset of the X_i which are not necessarily in static equilibrium, $\ell + m = n$, and $CV = \sum_{i=1}^{\ell} P_i X_i$.

Uzawa (1962) has shown that the elasticities of substitution defined by Allen (1938) can be computed from the partial derivatives of the cost function. The full static equilibrium elasticities of substitution can be computed from the total cost function:

$$\sigma_{ij} = \frac{CT \cdot CT_{ij}}{CT_i \cdot CT_j} , \tag{4}$$

where $CT_i = \partial CT/\partial P_i$, etc. The partial static equilibrium elasticities of substitution can be computed from the variable cost function:

$$\sigma^P_{ij} = \frac{CV \cdot CV_{ij}}{CV_i \cdot CV_j} , \tag{5}$$

where $CV_i = \partial CV/\partial P_i$, etc.

The partial static equilibrium elasticities of substitution are valid only for the levels of the fixed factors at which they are evaluated. Furthermore, they do not provide any information as to the substitution possibilities among the fixed factors or between the fixed and variable factors. However, this information can be obtained from the variable cost function, as we demonstrate explicitly below.

If all factors are at their full static equilibrium levels, total cost can be written as the sum of the variable cost function and expenditure for the Z_i:

$$CT = H(Y, P, Z^*) + \Sigma P_i Z^*_i = I(Y,P,Z^*) , \tag{6}$$

where Z^*_i indicates the equilibrium level of Z_i. Our task is to compute the σ_{ij} from (6). In doing so we make use of the full static equilibrium condition for a quasi-fixed factor.

(7) $$\partial CV/\partial Z_i^* = -P_i .$$

Let us define b_i as a binary variable which is zero if i is a variable factor and unity otherwise. The first partial derivatives of CT are:

(8) $$\frac{\partial CT}{\partial P_i} = \frac{\partial CV}{\partial P_i} + \left\{ \sum_k \frac{\partial CV}{\partial Z_k^*} \frac{\partial Z_k^*}{\partial P_i} + \sum_k P_k \frac{\partial Z_k^*}{\partial P_i} \right\} + b_i Z_i^* , \qquad \forall_i ,$$

but the bracketed terms sum to zero by (7). The second partial derivatives of CT are:

(9) $$\frac{\partial^2 CT}{\partial P_i \partial P_j} = \frac{\partial^2 CV}{\partial P_i \partial P_j} + \sum_k \frac{\partial^2 CV}{\partial P_i \partial Z_k^*} \frac{\partial Z_k^*}{\partial P_j} + \sum_k \frac{\partial Z_k^*}{\partial P_j} \left(\frac{\partial^2 CV}{\partial Z_k^* \partial P_j} + \sum_\ell \frac{\partial^2 CV}{\partial Z_k^* \partial Z_\ell^*} \frac{\partial Z_\ell^*}{\partial P_j} \right)$$
$$+ b_i \frac{\partial Z_i^*}{\partial P_j} + b_j \frac{\partial Z_j^*}{\partial P_i}$$
$$+ \left\{ \sum_k \frac{\partial CV}{\partial Z_k^*} \frac{\partial^2 Z_k^*}{\partial P_i \partial P_j} + \sum_k P_k \frac{\partial^2 Z_k^*}{\partial P_i \partial P_j} \right\} , \qquad \forall_{i,j} ,$$

where again the terms in curly brackets sum to zero by (7).

Equations (8) and (9) contain partial derivatives of CV, which can be evaluated from an estimated variable cost function. Evaluation of (8) and (9) also requires estimates of the $\partial Z_i^*/\partial P_j$. For many of the functional forms commonly used for cost functions in empirical work, including the translog, no closed form expression for the Z_i^* are available. Thus one must compute these derivatives indirectly via numerical methods.

The first order condition which define the optimal levels of the fixed factors given in equation (6) can be rewritten as the implicit functions:

(10) $$\frac{\partial CT}{\partial Z_i^*} = \frac{\partial I(Y,P,Z^*)}{\partial Z_i^*} = I_i = 0, \qquad \forall_i .$$

These functions can be solved for the Z_i^* as functions of Y and P, and thence for the $\partial Z_i^*/\partial P_j$.

Define:

(11) $$B = \begin{bmatrix} \frac{\partial I_1}{\partial Z_1^*} & \frac{\partial I_1}{\partial Z_2^*} & \cdots & \frac{\partial I_1}{\partial Z_m^*} \\ \vdots & & & \\ \frac{\partial I_m}{\partial Z_1^*} & \frac{\partial I_m}{\partial Z_2^*} & \cdots & \frac{\partial I_m}{\partial Z_m^*} \end{bmatrix}$$

(12) $$\underline{a}_i' = \frac{\partial I_1}{\partial P_i} \;\; \frac{\partial I_2}{\partial P_i} \;\; \cdots \;\; \frac{\partial I_m}{\partial P_i} .$$

The total differential of I_i can be written:

(13) $$dI_i = 0 = \frac{\partial I_i}{\partial Y} dY + \sum_k \frac{\partial I_i}{\partial P_k} dP_k + \sum_j \frac{\partial I_i}{\partial Z_j^*} dZ_j^* ,$$

Setting $dY = dP_j = 0$, $\forall_j \neq i$, and dividing both sides by dP_i we have

(14) $$0 = \underline{a}_i + B \begin{bmatrix} \frac{\partial Z_1^*}{\partial P_i} \\ \vdots \\ \frac{\partial Z_m^*}{\partial P_i} \end{bmatrix} , \qquad \forall_i .$$

It follows that:

$$(15) \qquad \begin{bmatrix} \frac{\partial Z_1^*}{\partial P_i} \\ \vdots \\ \frac{\partial Z_m^*}{\partial P_i} \end{bmatrix} = - B^{-1} \underline{a}_i \quad ,$$

which can be evaluated for each i. We note that for a single fixed factor (15) simplifies to:

$$(16) \qquad \frac{\partial Z^*}{\partial P_i} = - \frac{\partial I_{Z*}/\partial P_i}{\partial I_{Z*}/\partial Z^*} \quad .$$

With these results, evaluation of equation (9) is straightforward for any functional form selected for the variable cost function CV. Before proceeding to the application of these results to a specific example, however, we note the following simplified forms of equations (8) and (9):[7]

$$(8') \qquad \frac{\partial CT}{\partial P_i} = \begin{cases} \frac{\partial CV}{\partial P_i} & i \varepsilon VF \\ Z_i^* & i \varepsilon FF \end{cases}$$

$$(9') \qquad \frac{\partial^2 CT}{\partial P_i \partial P_j} = \begin{cases} \frac{\partial^2 CV}{\partial P_i dP_j} + \sum_k \frac{\partial Z_k^*}{\partial P_j} \frac{\partial 2_{CV}}{\partial Z_k^* \partial P_i} & i,j \varepsilon VF \\ \sum_k \frac{\partial Z_k^*}{\partial P_i} \frac{\partial^2 CV}{\partial Z_k^* \partial P_j} & i \varepsilon FF,\ j \varepsilon VF \\ \frac{\partial Z_i^*}{\partial P_j} & i,j \varepsilon FF \end{cases}$$

where VF is the set of variable factors and FF is the set of fixed factors. Note also that if i is a fixed factor, the x^{th} element of $\underline{a}_i$ is

$$a_{ik} = \frac{\partial I_k}{\partial P_i} = \begin{cases} 1 \text{ if } k = i \\ 0 \text{ if } k \neq i \end{cases} \tag{17}$$

Hence, from equation (15)

$$\frac{\partial Z_k^*}{\partial P_i} = \begin{cases} \left[-B^{-1}\right]_{ik} & \text{for } i\varepsilon FF \\ \left[-B^{-1}\ \underline{a}_i\right]_k & \text{for } i\varepsilon VF \end{cases} \tag{18}$$

As before, all derivatives in (8'), (9'), (17) and (18) are evaluated at the point Z=Z*.

The results in (9') and (18) can also be used to show that the Le Chatelier principle holds if the variable cost function is convex in the fixed factors at Z=Z*, or equivalently, if the second order conditions necessary for Z* to minimize CT are met. The Le Chatelier principle requires that the own price response of variable factors decreases in absolute value with the number of factors which are quasi-fixed; hence,

$$\frac{\partial^2 CV}{\partial P_i^2} \geq \frac{\partial^2 CT}{\partial P_i^2} \qquad i\varepsilon VF.$$

From equation (9'), we see that for variable factors

$$\frac{\partial^2 CT}{\partial P_i^2} = \frac{\partial^2 CV}{\partial P_i^2} + \sum_k \frac{\partial Z_k^*}{\partial P_i} \frac{\partial^2 CV}{\partial Z_k^* \partial P_i} . \qquad i\varepsilon VF$$

However, from equations (12) and (18),

$$\sum_k \frac{\partial Z_k^*}{\partial P_i} \frac{\partial^2 CV}{\partial Z_k^* \partial P_i} = a_i' \, [-B^{-1}] a_i \ ,$$

which is negative semi-definite provided that B is positive semi-definite. Hence, if B is positive semi-definite, the Le Chatelier principle holds.[8]

The requirement that CV be convex everywhere in Z, including at Z=Z*, guarantees that B $\left(\equiv \frac{\partial^2 CT}{\partial \underline{Z}^{*2}} = \frac{\partial^2 CV}{\partial \underline{Z}^{*2}} \right)$ is positive semi-definite, and thus that the Le Chatelier principle holds. It might also be noted that the second-order conditions for Z* to yield a proper minimum of CT require that $\frac{\partial^2 CT}{\partial \underline{Z}^{*2}}$ be positive definite. Thus, operationally, in solving for the value(s) of Z which minimize CT, verification that the second-order conditions hold will ensure that CV is convex in Z* and that the Le Chatelier principle holds.

To implement the method described above, we specify a translog form for the variable cost function with additive error term ε_c:

$$(19) \qquad \begin{aligned} \ln CV = {} & \alpha_0 + \alpha_Y \ln Y + \sum_i^{\ell} \alpha_i \ln P_i + \sum_i^{m} \beta_i \ln Z_i + \frac{1}{2} \gamma_{YY} (\ln Y)^2 \\ & + \frac{1}{2} \sum_i^{\ell} \sum_j^{\ell} \gamma_{ij} \ln P_i \ln P_j + \frac{1}{2} \sum_i^{m} \sum_j^{m} \delta_{ij} \ln Z_i \ln Z_j \\ & + \sum_i^{\ell} \rho_{Y_i} \ln Y \ln P_i + \sum_i^{\ell} \sum_j^{m} \rho_{ij} \ln P_i \ln Z_j \\ & + \sum_i^{m} \Pi_i \ln Y \ln Z_i + \phi_t t + \frac{1}{2} \phi_{tt} t^2 \\ & + \phi_{tY} t \ln Y + \sum_i \phi_{tP_i} t \ln P_i \\ & + \sum_i^{m} \phi_{tZ_i} t \ln Z_i + \varepsilon_c . \end{aligned}$$

From Shephard's Lemma (1953), we know that in partial static equilibrium the derivatives $\partial \ell n\ CV/\partial \ell n\ P_i$ are equal to the shares of these factors in variable cost, $S_i = P_i X_i / \sum_{i=1}^{\ell} P_i X_i$. Adding disturbance terms ε_i to reflect errors in optimization yields:

$$S_i = \alpha_i + \rho_{Y_i}\ \ell n\ Y + \sum_j^{\ell} \gamma_{ij}\ \ell n\ P_j + \sum_j^{m} \rho_{ij}\ \ell n\ Z_j + \phi_{tP_i} t + \varepsilon_i,\ \forall_i \tag{20}$$

The set of equations, (19) and (20), can be used to estimate the parameters of CV, from which the elasticities of substitution can be derived.

III. Data

Estimation of the variable cost function for the U.S. farm sector requires time series data on the levels of output and the fixed inputs, the prices of the variable factors, and the level of variable cost. The data, taken from Brown (1977), are described below.

The three variable inputs which we distinguish are hired labor services, capital services, and materials. A translog index of hired labor was constructed from data provided by Gollop and Jorgenson (1980).[8] The implicit wage rate for this index was then used as the price index for hired labor services. The capital services index was based on estimated capital stocks of farm equipment, structures, and inventories. A price index for farm capital services was then estimated using the assumption that on average over the postwar period the rate of return in farming has been the same as in the U.S. corporate sector. Translog quantity and price indexes were derived for materials from detailed data on fertilizer and liming materials, feed, seed, livestock, electricity, petroleum products, and fourteen miscellaneous categories. The level of variable cost was then calculated as the sum of compensation for hired labor, annualized capital costs, and expenditures on other purchased inputs.

The two fixed inputs which we distinguish are self-employed labor services and land. A translog index of self-employed labor was developed from data on hours worked provided by Gollop and Jorgenson (1980). The weights used are based on the assumption that relative wage rates for self-employed workers are the same as for hired workers with the same personal characteristics. An index of land in the farm sector was created from farm acreage by states, obtained from various U.S.D.A. publications. The weights for the index are based on estimates of the value of land per acre from the U.S.D.A. publications Farm Real Estate Historical Series (for years prior to 1963) and Farm Real Estate Market (for 1965 to 1974).

Finally, a measure of farm output was constructed from basic data published annually by the U.S. Department of Agriculture in Agricultural Statistics. A translog index was used to aggregate twelve distinct classes of livestock and nine major classifications of crops into a single measure.

For comparison purposes we also estimate the translog variable cost function with self-employed labor treated as a variable rather than quasi-fixed factor. This requires a price index and cost estimate for self-employed labor. We estimate the cost of self-employed labor as total farm income minus all other input costs. This figure is then divided by the quantity index to obtain the appropriate price index.

IV. Estimates of Partial Static Equilibrium Substitution Possibilities from the Translog Variable Cost Function

The parameters of the variable cost function were estimated by performing multivariate regression on equations (19) and (20). Efficient estimates were obtained using a modification of Zellner's (1962) method. Since the cost shares in (20) sum to unity, the estimated covariance matrix is singular, and one of the share equations must be deleted at the second stage of the estimation procedure. The estimates obtained are independent of which equation is deleted, and the estimates are asymptotically equivalent to maximum likelihood estimates.

Without loss of generality symmetry was imposed on the γ_{ij} and δ_{ij}. We also required the theoretical restriction of homogeneity of degree one in input prices to hold, using the following linear restrictions:

$$\text{(21)} \qquad \sum_i^{\ell} \alpha_i = 1$$

$$\sum_i^{\ell} \gamma_{ij} = \sum_i^{\ell} \gamma_{ji} = \sum_i^{\ell} \rho_{Yi} = \sum_i^{\ell} \rho_{ij} = \sum_i^{\ell} \phi_{tP_i} = 0 \qquad \forall_j .$$

Finally, constant returns to scale was imposed on the underlying structure of production by requiring (see Lau (1978)):

$$\text{(22)} \qquad \alpha_Y + \sum_i^{m} \beta_i = 1; \; \rho_{Yi} + \sum_j^{m} \rho_{ij} = 0 \qquad \forall_i ;$$

$$\gamma_{YY} + \sum_i^{m} \Pi_i = 0; \quad \Pi_j + \sum_i^{m} \delta_{ij} = 0 \qquad \forall_i ;$$

$$\phi_{tY} + \sum_i^{m} \phi_{tZ_i} = 0.$$

The variable cost function (19) has thirty-six parameters; with the imposition of symmetry, linear homogeneity in factor prices, and constant returns to scale there are twenty-one independent parameters to be estimated. The full set of parameter estimates is presented in Table 1. The fitted variable cost function satisfies at every sample point the regularity conditions that it be nondecreasing and concave in prices of variable factors and nonincreasing and convex in the levels of the fixed factors. R^2 statistics are .997, .770, .809, and .947 for the cost function and shares of capital, hired labor, and materials, respectively.[9]

The estimated partial static equilibrium elasticities of substitution among the variable factors are presented in the first three columns of Table 2. These estimates indicate that hired labor is highly substitutable for materials and moderately substitutable for capital. Capital and materials are estimated to be poor substitutes. The only significant trend in the estimated elasticities of substitution is the decline for capital and labor. Table 2 also contains the own-price elasticities for the three variable inputs. These elasticities are computed as the product of the variable cost share and the Allen own-elasticity of substitution. Demand for all three variable factors is estimated to be price-inelastic, but the elasticity for hired labor is substantially higher than for capital or materials.

V. Estimates of Static Equilibrium Substitution Possibilities from the Translog Variable Cost Function

Let us denote the elasticity of variable cost with respect to the $i^{\underline{th}}$ variable input price (the predicted value of S_i in equation (20)) by θ_i. Making use of this notation we can express the derivatives which we need

Table 1

Parameter Estimates for Translog
Variable Cost Function
(standard errors in parentheses)

PARAM-ETERS	ESTI-MATES	PARAM-ETERS	ESTI-MATES	PARAM-ETERS	ESTI-MATES
α_0	10.1827 (.0069)	γ_{HM}	.0165 (.0110)	ρ_{HF}	.0001 (.0304)
α_Y	-.3224 (.2255)	γ_{MM}	.1600 (.0138)	ρ_{MF}	-.0812 (.0151)
α_K	.4233 (.0023)	δ_{AA}	-26.1954 (8.692)	Π_A	31.4549 (8.795)
α_H	.1421 (.0026)	δ_{AF}	-5.2595 (2.863)	Π_F	4.8143 (2.683)
α_M	.4345 (.0013)	δ_{FF}	.4452 (1.780)	ϕ_T	4.0973 (.6505)
β_A	1.1109 (.2148)	ρ_{YK}	-.1833 (.0808)	ϕ_{TT}	-250.211 (75.48)
β_F	.2115 (.0853)	ρ_{YH}	.1572 (.0861)	ϕ_{TY}	96.8287 (25.96)
γ_{YY}	-36.2692 (9.429)	ρ_{YM}	.0261 (.0442)	ϕ_{TPK}	.7175 (.2502)
γ_{KK}	.2207 (.0209)	ρ_{KA}	.1022 (.0790)	ϕ_{TPH}	-.6997 (.2698)
γ_{KH}	-.0440 (.0202)	ρ_{HA}	-.1573 (.0860)	ϕ_{TPM}	-.0178 (.1270)
γ_{KM}	-.1766 (.0123)	ρ_{MA}	.0551 (.0407)	ϕ_{TZA}	-86.4174 (24.06)
γ_{HH}	.0275 (.0239)	ρ_{KF}	.0811 (.0279)	ϕ_{TZF}	-10.4113 (9.460)

K = capital (structures and equipment)
H = hired labor
M = materials

A = land (acreage)
F = self-employed (family) labor
Y = aggregate output index

Table 2

Substitution and Own-Price Elasticities
Estimated Under the Assumption of
Partial Static Equilibrium

YEAR	σ_{KH}	σ_{KM}	σ_{HM}	η_{KK}	η_{HH}	η_{MM}
1947	.424	-.108	1.197	-.040	-.662	-.199
1948	.471	-.094	1.224	-.056	-.664	-.198
1949	.422	-.094	1.203	-.043	-.664	-.199
1950	.398	-.075	1.207	-.043	-.666	-.198
1951	.414	-.054	1.226	-.053	-.667	-.200
1952	.394	-.063	1.213	-.046	-.667	-.199
1953	.372	-.063	1.208	-.041	-.667	-.197
1954	.356	-.046	1.215	-.043	-.668	-.197
1955	.355	-.044	1.215	-.043	-.668	-.197
1956	.329	-.048	1.209	-.036	-.668	-.194
1957	.332	-.023	1.224	-.045	-.668	-.196
1958	.353	-.021	1.231	-.050	-.668	-.198
1959	.324	-.024	1.223	-.044	-.668	-.195
1960	.324	-.022	1.224	-.044	-.668	-.196
1961	.318	.003	1.242	-.051	-.668	-.197
1962	.305	.011	1.246	-.052	-.668	-.197
1963	.296	.016	1.248	-.052	-.667	-.197
1964	.270	.025	1.252	-.051	-.666	-.195
1965	.286	.034	1.266	-.056	-.665	-.198
1966	.250	.051	1.278	-.056	-.662	-.197
1967	.268	.040	1.268	-.055	-.665	-.197
1968	.268	.039	1.267	-.055	-.665	-.197
1969	.271	.056	1.297	-.060	-.661	-.199
1970	.220	.062	1.285	-.056	-.660	-.196
1971	.232	.044	1.264	-.052	-.664	-.194
1972	.219	.054	1.274	-.054	-.661	-.195
1973	.224	.073	1.307	-.059	-.656	-.198
1974	.087	.084	1.292	-.050	-.650	-.188

to evaluate (8') and (9') in terms of the derivatives of ℓn CV as follows:

$$\text{(23)} \qquad \frac{\partial CV}{\partial P_i} = \begin{cases} \dfrac{\hat{CV}\theta_i}{P_i} & i\varepsilon VF \\ 0 & i\varepsilon FF \end{cases},$$

$$\text{(24)} \qquad \frac{\partial CV}{\partial Z_k^*} = \frac{\hat{CV}}{Z_k^*} \; \frac{\partial\, \ell n\; CV}{\partial\, \ell n\; Z_k^*}$$

$$\text{(25)} \qquad \frac{\partial^2 CV}{\partial P_i \partial P_j} = \begin{cases} \dfrac{\hat{CV}}{P_i P_j}\left(\theta_i\theta_j + \dfrac{\partial^2\, \ell n\; CV}{\partial \ell n P_i \partial \ell n P_j} - \omega_{ij}\theta_i\right) & \omega_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases} \quad i,j\varepsilon VF, \\ 0 & i \text{ or } j\varepsilon FF \end{cases}$$

$$\text{(26)} \qquad \frac{\partial^2 CV}{\partial P_i \partial Z_k^*} = \begin{cases} \dfrac{\hat{CV}}{P_i Z_k^*}\left(\theta_i \dfrac{\partial\, \ell n\; CV}{\partial\, Z_k} + \dfrac{\partial^2\, \ell n CV}{\partial\, \ell n\; P_i \partial\, \ell n Z_k^*}\right) & i\varepsilon VF \\ 0 & i\varepsilon FF \end{cases}$$

where VF is the set of variable factors, FF is the set of fixed factors, and $\hat{CV}$ is the fitted value of ℓn CV exponentiated.

The steps required to obtain the estimated full static equilibrium elasticities are:

(a) Numerically solve (10) for the optimal levels of the quasi-fixed factors in each year; (b) evaluate the variable cost function at the optimal levels of the quasi-fixed factors for each year; (c) evaluate the derivatives (18), (23), (24), (25), and (26); (d) evaluate the derivatives (8') and (9'); and finally (e) compute the elasticities of substitution from (4). The full static equilibrium own price elasticities are then computed by multiplying the full static equilibrium Allen own-elasticities of substitution by the shares of total cost estimated at the optimal levels of the quasi-fixed factors.

This operational procedure suggests the following intuitive explanation of how inferences about full static equilibrium are being drawn from estimates of the variable cost function. The estimated variable cost function enables us to plot out a number of partial static equilibrium average total cost functions. Using equation (7), we obtain the points of tangency between these curves and the full static equilibrium average total cost curve. These points enable us to construct the full static equilibrium cost function.

In our application family labor is considered to be a quasi-fixed factor, while land is a fixed factor.[10] Thus we first solve for the optimal level of family labor in each year.[11] These estimates are presented in Table 3 along with the actual values for family labor. The estimates indicate that there was a very large surplus of family farm labor in the early postwar years. In subsequent years both actual and optimal levels of family labor declined, but the ratio of optimal to actual increased.

The estimated optimal levels of farm family labor from Table 3 can be used to obtain estimated static equilibrium elasticities of substitution, which are presented in Table 4. Comparing the estimates in Tables 2 and 4, we find little difference between the estimated substitution possibilities among K, H, and M in partial and full static equilibrium. The estimated elasticities of substitution between family labor and the variable factors indicate that family labor is substitutable with hired labor and materials but complementary with capital.

Table 5 contains the estimated price elasticities of demand for the inputs in static equilibrium. As in Table 2 we find a very price-inelastic

Table 3

Actual (Z_F) and Cost-Minimizing (Z_F^*) Levels of the Quasi-Fixed Factor Family Labor

Year	Z_F	Z_F^*
1947	2.224	.765
1948	2.070	.408
1949	2.014	.591
1950	1.969	.788
1951	1.864	.675
1952	1.840	.600
1953	1.807	.666
1954	1.779	.809
1955	1.728	.710
1956	1.607	.776
1957	1.477	.821
1958	1.361	.543
1959	1.361	.554
1960	1.308	.505
1961	1.191	.560
1962	1.159	.588
1963	1.066	.497
1964	1.014	.637
1965	.998	.506
1966	.934	.649
1967	1.000	.508
1968	.995	.457
1969	.970	.458
1970	.935	.578
1971	.886	.324
1972	.893	.328
1973	.897	.277
1974	.856	.668

Table 4

Full Static Equilibrium Elasticities of Substitution
Estimated from the Translog Variable Cost Function

Year	σ_{KH}	σ_{KM}	σ_{HM}	σ_{KF}	σ_{HF}	σ_{MF}
1947	.331	-.189	1.275	-.106	.368	.646
1948	.301	-.121	1.251	-.256	.262	.563
1949	.294	-.155	1.263	-.193	.313	.600
1950	.327	-.122	1.286	-.097	.353	.639
1951	.336	-.059	1.284	-.147	.298	.605
1952	.276	-.091	1.268	-.208	.285	.580
1953	.267	-.102	1.275	-.174	.311	.598
1954	.292	-.078	1.298	-.090	.378	.634
1955	.268	-.073	1.289	-.139	.322	.610
1956	.262	-.093	1.301	-.077	.366	.642
1957	.296	-.059	1.326	-.067	.388	.671
1958	.265	-.024	1.293	-.161	.289	.590
1959	.223	-.038	1.288	-.182	.290	.581
1960	.210	-.031	1.280	-.224	.263	.559
1961	.249	.026	1.299	-.188	.263	.567
1962	.247	.040	1.305	-.186	.248	.566
1963	.215	.050	1.292	-.248	.208	.529
1964	.242	.061	1.319	-.168	.248	.571
1965	.220	.081	1.306	-.238	.195	.529
1966	.236	.100	1.342	-.166	.226	.567
1967	.196	.089	1.304	-.262	.178	.513
1968	.975	.084	1.297	-.281	.173	.503
1969	.189	.117	1.321	-.257	.163	.511
1970	.177	.115	1.337	-.209	.201	.538
1971	.068	.070	1.278	-.362	.143	.459
1972	.056	.086	1.285	-.358	.137	.457
1973	.033	.118	1.296	-.369	.112	.441
1974	.064	.140	1.357	-.218	.197	.523

Table 5

Full Static Equilibrium Own-Price Elasticities
Estimated from the Translog Variable Cost Function

Year	η_{KK}	η_{HH}	η_{MM}	η_{FF}
1947	.043	-.676	-.279	-.304
1948	.034	-.670	-.251	-.231
1949	.048	-.673	-.261	-.268
1950	.016	-.677	-.277	-.294
1951	-.007	-.675	-.268	-.258
1952	.025	-.674	-.256	-.248
1953.	.029	-.675	-.261	-.266
1954	.004	-.678	-.275	-.291
1955	.013	-.677	-.266	-.274
1956	.015	-.680	-.277	-.303
1957	-.016	-.681	-.290	-.316
1958	-.006	-.675	-.261	-.251
1959	.010	-.675	-.255	-.251
1960	.011	-.674	-.248	-.232
1961	-.024	-.673	-.255	-.224
1962	-.029	-.672	-.256	-.220
1963	-.028	-.670	-.244	-.189
1964	-.038	-.671	-.259	-.221
1965	-.043	-.668	-.247	-.179
1966	-.053	-.666	-.260	-.204
1967	-.044	-.667	-.242	-.165
1968	-.039	-.667	-.238	-.161
1969	-.056	-.662	-.244	-.153
1970	-.052	-.662	-.250	-.184
1971	-.020	-.665	-.218	-.135
1972	-.027	-.663	-.219	-.130
1973	-.043	-.657	-.218	-.108
1974	-.050	-.652	-.244	-.181

demand for capital. The demand for hired labor is the most elastic of all the inputs, and the estimated elasticities are virtually the same in partial and full static equilibrium. The demand for materials is found to be somewhat more price-elastic in full static equilibrium than in partial static equilibrium. The own-price elasticity for family labor is estimated to be fairly small and declining over time.

The large discrepancies between actual and optimal levels of family labor in Table 3 tends to confirm the specification of family labor as a quasi-fixed factor rather than a variable factor. It would be of interest to know, however, how much different our estimates of substitution possibilities in the farm sector would be if family labor were specified as a variable factor. It is straightforward to investigate this question by estimating a translog variable cost function with family labor moved from the quasi-fixed to the variable category. We estimate such a cost function, with land still treated as a fixed factor, and present the estimated elasticities of substitution in Table 6.

The estimated elasticities of substitution in Table 6 are quite different from those in Table 4. Treating family labor as a variable input causes the following changes in the estimated elasticities of substitution: (a) Family labor and hired labor are found to be highly complementary (in recent years) rather than substitutable. (b) Family labor and materials are found to be much less substitutable in recent years. (c) Family labor and capital are found to be complementary only in recent years, rather than in all years. (d) Capital and materials are found to be more complementary in early years and more substitutable in recent years. (e) Substitution possibilities between capital and hired labor are esti-

Table 6

Full Static Equilibrium Elasticities of Substitution Estimated from the Translog Variable Cost Function with Family Labor Treated as a Variable Factor

Year	σ_{KH}	σ_{KM}	σ_{HM}	σ_{KF}	σ_{HF}	σ_{MF}
1947	.377	-.464	1.424	.342	.200	.589
1948	.472	-.311	1.465	.407	.119	.522
1949	.406	-.350	1.426	.351	.142	.574
1950	.397	-.287	1.420	.337	.094	.578
1951	.445	-.181	1.438	.364	.011	.541
1952	.425	-.177	1.413	.329	.014	.559
1953	.398	-.181	1.403	.306	.009	.575
1954	.391	-.140	1.405	.300	-.042	.574
1955	.393	-.119	1.403	.295	-.064	.571
1956	.361	-.141	1.399	.279	-.056	.588
1957	.378	-.091	1.419	.304	-.120	.571
1958	.415	-.046	1.430	.322	-.167	.544
1959	.394	-.027	1.408	.282	-.196	.557
1960	.425	.015	1.392	.259	-.236	.537
1961	.463	.091	1.397	.250	-.375	.492
1962	.476	.129	1.386	.216	-.458	.471
1963	.498	.172	1.376	.175	-.572	.434
1964	.491	.185	1.372	.153	-.622	.433
1965	.525	.231	1.370	.111	-.794	.366
1966	.519	.259	1.374	.074	-.959	.342
1967	.548	.280	1.340	-.038	-1.052	.286
1968	.544	.283	1.336	-.056	-1.077	.286
1969	.560	.318	1.353	-.095	-1.374	.196
1970	.533	.320	1.342	-.140	-1.401	.237
1971	.537	.320	1.327	-.192	-1.406	.229
1972	.538	.343	1.328	-.273	-1.662	.173
1973	.560	.387	1.340	-.454	-2.375	.027
1974	.514	.395	1.317	-.662	-2.650	.007

Table 7

Full Static Equilibrium Own-Price Elasticities
Estimated from the Translog Variable Cost Function
with Family Labor Treated as a Variable Factor

Year	η_{KK}	η_{HH}	η_{MM}	η_{FF}
1947	-.047	-.546	-.290	-.268
1948	-.118	-.543	-.275	-.264
1949	-.076	-.543	-.292	-.266
1950	-.080	-.538	-.298	-.264
1951	-.117	-.533	-.295	-.257
1952	-.102	-.535	-.301	-.256
1953	-.089	-.533	-.305	-.257
1954	-.095	-.528	-.306	-.253
1955	-.098	-.526	-.307	-.251
1956	-.082	-.524	-.308	-.254
1957	-.105	-.516	-.308	-.250
1958	-.125	-.515	-.306	-.243
1959	-.115	-.515	-.309	-.238
1960	-.123	-.521	-.310	-.224
1961	-.143	-.518	-.310	-.197
1962	-.148	-.519	-.310	-.174
1963	-.154	-.521	-.310	-.139
1964	-.153	-.519	-.310	-.129
1965	-.163	-.521	-.310	-.072
1966	-.165	-.514	-.310	-.037
1967	-.164	-.529	-.309	-.039
1968	-.164	-.529	-.309	-.047
1969	-.170	-.522	-.309	-.129
1970	-.166	-.518	-.307	-.125
1971	-.164	-.524	-.305	-.150
1972	-.166	-.520	-.304	-.229
1973	-.172	-.513	-.305	-.464
1974	-.166	-.508	-.294	-.539

mated to have increased over time rather than decreased. The only estimated elasticitiy of substitution which did not change much was that between hired labor and materials.

The estimated elasticities of substitution in Table 6 do not appear to be as plausible as those in Table 4. Much of the implausibility can be attributed to the fact that the curvature conditions are violated for the last eight years of the sample. Even for the earlier years, however, the results based on family labor as a quasi-fixed factor appear more plausible. For example, substitutability between family labor and hired labor with complementarity between family labor and capital seems much more plausible than the converse.

The estimated own-price elasticities in Table 7 are quite similar to those in Table 5. For the three inputs which are treated as variable in both cost functions, the difference between the two sets of estimates are quite small. The family labor estimates are similar until the early 1960's, after which they differ substantially.

VI. Concluding Remarks

In this paper we have presented an empirical framework for estimating substitution possibilities in situations where full static equilibrium is not a tenable assumption. Economists have long questioned whether it is appropriate to assume that capital stocks are in full static equilibrium. Our framework can be used to investigate such cases. Furthermore, this formulation is not limited to treating one or more capital stocks as quasi-fixed inputs. Any inputs which are thought to be in disequilibrium can be treated as quasi-fixed, as we have shown by treating family labor as a quasi-fixed input in the U.S. farm sector.

Although our example dealt with only one quasi-fixed factor, application of the method developed to models with more quasi-fixed factors introduces no methodological complications. The principal practical complication is that the cost function must be minimized with respect to all quasi-fixed factors simultaneously.

In our empirical application three sets of elasticities of substitution were presented for the farm sector over the 1947-74 period. The first two sets were estimated under the assumption that family labor is a quasi-fixed factor. The first set portrays substitution possibilities among the variable inputs conditional on the observed level of family labor. The second set portrays substitution possibilities which would prevail among family labor and the variable inputs if family labor were at its optimal level. These two sets of elasticities of substitution are quite similar. The third set of elasticities of substitution is based on the assumption that the observed levels of family labor were in fact optimal. These elasticities differ substantially from the other two sets. We conclude that the specification of particular inputs as variable or quasi-fixed may have important consequences in the estimation of substitution possibilities.

Footnotes

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[1]Diewert (1974) and Fuss and McFadden (1978, Volume 1) each provides extensive discussion of both topics.

[2]Full static equilibrium and partial static equilibrium are often referred to as "long run" and "short run" equilibrium. We avoid the latter terminology because movement from partial to full static equilibrium requires input adjustments that may not take place with the passage of time.

[3]Mork (1978, p. 1) erroneously states: "Estimation of the short run demand response requires a short run estimation model." In the paper he displays some valid relationships between "short-run" and "long-run" elasticities in the case of full static equilibrium. However, if full static equilibrium were the appropriate formulation, then the long-run elasticities should be estimated directly and the short-run elasticities inferred indirectly using his formula -- rather than vice versa.

[4]Caves, Christensen, and Swanson (1980) have recently demonstrated how to estimate shifts in the production structure (productivity growth) when the partial static equilibrium approach is appropriate.

[5]Tweeten (1969) provides discussion and further references.

[6]We do not develop the theory for the multiproduct case since our emphasis is on substitution possibilities among factor inputs. Generalization of our methodology to the multiproduct case is straightforward.

[7]These simplifications result from applying (14) and from the fact that any derivatives of CV with respect to prices of fixed factors are equal to zero. These results were derived by Lau (1976, p. 150).

[8]Note that this result (also presented in Lau (1976)) is necessarily true only in the neighborhood of the full static equilibrium. That is, the relationship given in (9') holds only when all derivatives are evaluated at $Z=Z^*$. Thus, while $\partial^2 CT/\partial Z^{*2} \geq 0$ guarantees that $\left.\partial^2 CV/\partial P_i^2\right|_{Z=Z^*} \geq \partial^2 CT/\partial P_i^2$, it may not be true that $\left.\partial^2 CV/\partial P_i^2\right|_{Z=Z_0} \geq \partial^2 CT/\partial P_i^2$.

[9]The translog index number formula can be written:

$$\ell n\ (Y_t/Y_{t-1}) = \Sigma\ \bar{W}_i\ \ell n\ (Y_{ti}/Y_{t,i-1}),$$

where $\bar{W}_i = (W_{it} + W_{i,t-1})/2$, and $W_i = P_iY_i/\Sigma P_iY_i$. This formula was suggested by Fisher (1922), advocated by Tornqvist (1936), Theil (1965), and Kloek (1966), and has been used extensively by Christensen and Jorgenson (1973) and others. Diewert (1976) showed that this index is exact for a translog function.

[10]The R^2 statistic for each equation is defined as $R^2 = 1 - \Sigma\ e_t^2/\Sigma(y_t-\bar{y})^2$, where e_t is the residual and y_t the value of the dependent variable in period t. Of course, for estimation procedures other than ordinary least squares, R^2 cannot in general be interpretated as the proportion of variance explained, since the residuals are no longer orthogonal to the regressors. However, it still provides a useful indicator of goodness-of-fit.

[11]In the development up to this point, it has been assumed that CT refers to total costs. Treating land as fixed rather than quasi-fixed implies that CT includes only non-land costs, and that land is left at its actual value in performing the calculations necessary to construct the static equilibrium elasticities. The quasi-fixed factors (family labor in this application) are set at those values which minimize CT.

[12]Due to the nonlinear way in which Z enters the translog variable cost function, no closed form solution for Z* results from the first order conditions given in equation (7). However, a variety of computer algorithms are available for minimizing a function such as CT. We found the algorithm given in Berndt, et al. (1976), modified to use the actual second derivatives $\frac{\partial^2 CT}{\partial Z^2}$ rather than the approximation they suggest, to be easy to use and reliable. This algorithm also performed well in experiments with two fixed factors. However, a numerical zero-finding routine, used to solve for those Z values which satisfied the first-order conditions for the two-fixed-factor case, produced results which seemed very reasonable, but which corresponded to a saddle point rather than a minimum. This highlights the importance of selecting a reliable optimization algorithm and, of course, checking the second order condition.

References

R.G.D. Allen, Mathematical Analysis for Economists, St. Martin's Press, New York, 1938.

E. R. Berndt, C. J. Morrison-White, and G. C. Watkins, "Energy Substitution and Capital Utilization in a Dynamic Context," Economics Department, University of British Columbia, November 1979.

E. R. Berndt, B. H. Hall, R. E. Hall, and J. A. Hausman, "Estimation and and Inference in Nonlinear Statistical Models," Annals of Economic and Social Measurement, 3/4, 1976, pp. 653-65.

R. S. Brown, Productivity, Returns, and the Structure of Production in U.S. Agriculture, 1947-1974, Economics Department, University of Wisconsin-Madison, unpublished doctoral dissertation, 1977.

D. W. Caves, L. R. Christensen, and J. A. Swanson, "Productivity Growth, Scale Economies, and Capacity Utilization in U.S. Railroads, 1955-1974," Discussion Paper #8002, Social Systems Research Institute, University of Wisconsin-Madison, January 1980.

L. R. Christensen and D. W. Jorgenson, "Measuring the Performance of the Private Sector of the U.S. Economy, 1929-1969," in Measuring Economic and Social Performance, M. Moss, ed., Natural Bureau of Economic Research, 1973.

W. E. Diewert, "Exact and Superlative Index Numbers," Journal of Econometrics, Vol. 4, No. 2, 1976, pp. 115-45.

__________, "Applications of Duality Theory," in Frontiers of Quantitative Economics (D.A. Kendrick and M.D. Intriligator, eds.) Vol. II, North-Holland, Amsterdam, 1974, pp. 106-71.

R. Eisner and R. H. Strotz, "Determinants of Business Investment," in Impacts of Monetary Policy, Englewood Cliffs, N.J.: Prentice-Hall, 1963.

I. Fisher, The Making of Index Numbers, Hougton Mifflin Company, Boston, 1922.

M. Fuss and D. McFadden, Production Economics: A Dual Approach to Theory and Applications, 2 volumes, Amsterdam, North-Holland Publishing Company, 1978.

F. M. Gollop and D. W. Jorgenson, "U.S. Productivity Growth by Industry, 1947-1973," in New Developments in Productivity Measurement and Analysis (S. W. Kendrick and B. N. Vaccara, eds.), Studies in Income and Wealth, Vol. 44, University of Chicago Press for the National Bureau of Economic Research, 1980, pp. 17-124.

W. M. Gorman, "Measuring the Quantities of Fixed Factors," in Value, Capital, and Growth: Papers in Honour of Sir John Hicks, (J.N. Wolfe, ed.), Chicago, Aldine Publishing Co., 1968, pp. 141-72.

T. Kloek, Index cijfers: enige methodologisch aspecten, Pasmans, The Hague, 1966.

L. J. Lau, "A Characterization of the Normalized Restricted Profit Function," Journal of Economic Theory, Vol. 12, February 1976, pp. 131-63.

L. J. Lau, "Applications of Profit Functions," in Fuss and McFadden (1978, Volume 1), pp. 133-216.

L. J. Lau and P. A. Yotopoulos, "A Test of Relative Efficiency and an Application to Indian Agriculture," American Economic Review, Vol. 61, 1971, pp. 94-109.

R. E. Lucas, "Optimal Investment Policy and the Flexible Accelerator," International Economic Review, Vol. 8, No. 1, 1967, pp. 75-85.

D. McFadden, "Cost, Revenue, and Profit Functions," in Fuss and McFadden (1978, Volume 1), pp. 3-109.

K. A. Mork, "The Aggregate Demand for Primary Energy in the Short and Long Run for the U.S., 1949-1975," Report No. MIT-EL 78-007 WP, Energy Laboratory, Massachusetts Institute of Technology, 1978.

P. A. Samuelson, "Prices of Factors and Goods in General Equilibrium," Review of Economic Studies, Vol. 21, 1953-54, pp. 1-20.

R. W. Shephard, Cost and Production Functions, Princeton University Press, Princeton, 1953.

H. Theil, "The Information Approach to Demand Analysis," Econometrica, Vol. 33, 1965, pp. 67-87.

L. Tornqvist, "The Bank of Finland's Consumption Price Index," Bank of Finland Monthly Bulletin, No. 10, 1936, pp. 1-8.

L. G. Tweeten, "Theories Explaining the Persistence of Low Resource Returns in a Growing Farm Economy," American Journal of Agricultural Economics, Vol. 51, 1969, pp. 798-817.

U.S. Department of Agriculture, Agricultural Statistics, U.S. Government Printing Office, Washington, D.C. (annual).

__________, "Farm Real Estate Market Developments," Economic Research Service, 1965-1974.

__________, "Farm Real Estate Historical Series," Economic Research Service, Bulletin #570, 1973.

H. Uzawa, "Production Functions with Constant Elasticities of Substitution, Review of Economic Studies, Vol. 29, 1962, pp. 291-99.

A. Zellner, "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias," Journal of the American Statistical Association, Vol. 57, June 1962, pp. 348-368.