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NECESSARY CONDITIONS FOR FIRST, SECOND AND THIRD DEGREE STOCHASTIC DOMINANCE EFFICIENCY OF ENTERPRISE MIXTURES*

by

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An Abstract

of

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Variations of a result by Dybvig and Ross provide necessary conditions for the FSD, SSD and TSD efficiency of enterprise mixtures. Several of their perfect market portfolio problem results are adapted to the enterprise mixture problem. The results imply that the sets of enterprise mixtures which satisfy the necessary conditions for FSD, SSD and TSD efficiency are unions of finite numbers of convex subsets. Linear programming can be used to determine whether a particular mixture and the (appropriately defined) convex subset to which it belongs satisfy these conditions. These ideas are illustrated by applying them to a simple example.

NECESSARY CONDITIONS FOR FIRST, SECOND AND THIRD DEGREE STOCHASTIC DOMINANCE EFFICIENCY OF ENTERPRISE MIXTURES

Expected income-variance (E,V) and expected income-absolute deviations (E,A) criteria have been used to analyze crop mixes, livestock production decisions and marketing strategies. These criteria have been criticized because they are not always consistent with the widely accepted expected utility approach to decision making under uncertainty. Stochastic dominance criteria, which are consistent with expected utility theory, have largely replaced E,V and E,A criteria for analyses involving discrete alternatives. However, stochastic dominance criteria are not commonly used for mixture problems as there is no widely known simple way to find all members of the relevant efficient sets.

Although stochastic dominance methodology for enterprise mixes is not yet fully developed, several techniques which exploit sufficient conditions for stochastic dominance efficiency are available. Tauer has shown that unique Target MOTAD solutions are second degree stochastic dominance (SSD) efficient. This technique, which was also independently developed by Watts, Held and Helmers, provides a way of identifying some, but usually not all, members of the set of SSD efficient mixtures. Similarly, Porter's mean-target semivariance provides a way of identifying some, but usually not all, members of the set of third degree stochastic dominance (TSD) efficient mixtures.

Dybvig and Ross present necessary conditions for SSD efficiency of portfolios. Their results can be modified to provide necessary conditions for SSD efficiency of enterprise mixtures. These conditions and necessary conditions for FSD and TSD efficiency are presented in this paper. Several of Dybvig and Ross' portfolio problem results are also adapted to the enterprise mixture problem. The adapted results help characterize the sets of mixtures which satisfy the necessary conditions for FSD, SSD and TSD efficiency. These ideas are illustrated by applying them to a simple example.

ASSUMPTIONS

Three alternative classes of utility functions are assumed. The first is the class of strictly increasing functions. The second class includes all strictly increasing, weakly concave functions. For the third class, nonnegative third derivatives are also required. These classes are similar those commonly assumed for FSD, SSD and TSD, respectively.[1/]

Tauer's assumptions about the joint probability distribution of the alternatives and the form of the constraints are adopted here as well. That is, the joint probability distribution of the outcomes associated with the various activities is assumed to be discrete and linear resource constraints are assumed. In this paper, p is a column vector of probabilities associated with s states of nature. x is a column vector of n activity levels. C is a matrix of net returns associated with the activities for the various states of nature. C_{ij} is the net return per unit of activity j when the ith state of nature occurs. y is a vector of (total) net returns for the various states of nature. Thus

(1) $y - Cx = 0.$

A is a matrix of resource or technical requirements and b is a vector of resource levels. The constraints on activity levels are

(2) $Ax \leqq b$ and

(3) $x \geqq 0.$

NECESSARY CONDITIONS FOR STOCHASTIC DOMINANCE EFFICIENCY

Dybvig and Ross present a general result and three special results which are relevant for this paper. The general result is that a mixture, x^0, is stochastically efficient if, and only if, there exists a vector, z^o, which satisfies the following conditions:[2/]

(4) $z^{o'}Cx^o \geqq z^{o'}Cx$ for all x vectors which satisfy (2) and (3),

(5) $z_i^o/p_i \geqq z_j^o/p_j$ if $Cx_i^o < Cx_j^o$ (for all i,j) and

(6) $z^o > 0$.

Necessary Conditions for FSD Efficiency

Conditions (4) and (6) are necessary for FSD efficiency. Condition (6) reflects the fact that, for strictly increasing utility functions, marginal utility (when defined) is positive. Jointly, conditions (4) and (6) reflect the fact that an enterprise mixture is not FSD efficient if a larger net return could be obtained under one state of nature without reducing the net return under any other state of nature.[3/]

Necessary Conditions for SSD Efficiency

To be SSD efficient, a mixture must satisfy the necessary conditions, (4) and (6), for FSD efficiency. It must also satisfy condition (5) which ensures that marginal utility is a nonincreasing function of income.

Necessary Conditions for TSD Efficiency

Conditions (4), (5), and (6) are necessary for TSD efficiency. However, condition (5) can be replaced by a stronger necessary condition:

$$(5') \quad \frac{z_k^o/p_k - z_j^o/p_j}{y_k^o - y_j^o} \underset{\not\geq}{\leq} \frac{z_j^o/p_j - z_i^o/p_i}{y_j^o - y_i^o} \underset{\not\geq}{\leq} 0 \quad \text{if} \quad Cx_k^o < Cx_j^o < Cx_i^o$$

Condition (5') ensures that marginal utility is a nonincreasing function of income and that the rate of <u>decrease</u> in marginal utility is nonincreasing (i.e., the third derivative of the utility function is nonnegative).

CHARACTERISTICS OF SETS WHICH SATISFY THE NECESSARY CONDITIONS

Three of Dybvig and Ross' special results help characterize the sets which satisfy the necessary conditions for FSD, SSD, or TSD efficiency. Although two of these special results are for perfect market portfolio problems, they can be adapted to enterprise mixture problems.

The first special result is that a vector, z^o, either satisfies condition (4) for all feasible x^o vectors or no feasible x^o vector. A similar statement can be made about vectors which satisfy both conditions (4) and (6). A second result is that the set of x^o vectors which satisfy conditions (4) through (6) is the union of a finite number of closed convex subsets. This means that the set of stochastically efficient mixtures is closed but, in general, not convex. Each of the individual closed convex subsets includes those feasible x vectors for which the elements of the associated y vectors share a common set of rankings. Dybvig and Ross also show that the set of stochastically efficient mixtures is connected.

Dybvig and Ross' third special result applies without modification to the enterprise mixture problem. Their first and second special results do not apply directly since the feasible set for the enterprise mixture problem is not always a hyperplane as is the case for the perfect market portfolio problem. Dybvig and Ross' footnote 2 suggests that when inequality constraints are present the first and second of the special results mentioned above are valid for interior mixtures. For enterprise mixture problems, strictly interior mixtures are unlikely to be efficient but other "interior" mixtures may be.

The feasible set for most enterprise mixture problems is a polyhedron. A polyhedron can be thought of as the union of its interior, the "interiors" of its faces, the "interiors" of its edges, and its vertices (or corner points).

The polyhedron's interior is the (open) set of those feasible mixtures which do not also lie on a face, edge or vertex. The "interior" of a face is the open set of feasible mixtures on the face which do not also lie on an edge or vertex. The "interior" of an edge is defined in an analogous way.

The first special result is obviously true for the interior of the polyhedron. Of greater practical significance is the fact that a similar result is valid for the "interiors" of faces and edges. That is, a vector, z^o, either satisfies condition (4) for all "interior" vectors, x^o, of a given face or edge or it does not satisfy condition (4) for any "interior" x^o vector of the face or edge. If an interior vector of an edge, a face or the polyhedron satisfies condition (4), then not only all interior vectors but also the "boundaries" satisfy condition (4). The converse is not always true. However, if all of the "boundaries" satisfy condition (4), then the interior must as well.

FSD Candidate Subsets

The modification needed in Dybvig and Ross' second special result depends on the degree of stochastic dominance being considered. For FSD, each of the candidates for the individual closed convex subsets is a vertex, an edge, a face or the polyhedron itself.[4/] Thus, the set of vectors which satisfy the necessary conditions is the union of a finite number of closed convex sets.

SSD Candidate Subsets

For SSD, each of the subset candidates is the intersection of a vertex, an edge, a face or the polyhedron itself and a set of x vectors for which the elements of the associated vectors share a common set of (weak) rankings. This means that each subset of vectors which satisfy the necessary conditions for SSD efficiency must be a convex subset (perhaps improper) of a subset of vectors which satisfy the necessary conditions for FSD efficiency.

TSD Candidate Subsets

The approach for specifying candidate subsets for TSD is both similar to, and different from, that employed for FSD and SSD. Just as the SSD criterion candidates are (typically proper) subsets of FSD criterion candidates, the TSD criterion candidates are subsets of those for SSD.

If a subset satisfies the necessary conditions for FSD or SSD efficiency, a single (but not necessarily unique) z^o vector applies to the entire subset. Conditions (4), (5') and (6) suggest that defining subset candidates for TSD efficiency in the same way would cause many of the subset candidates to be individual enterprise mixtures. Fortunately, an alternative approach exists. It involves defining each candidate subset as those mixtures which share the same set of basis variables when a linear programming formulation of the necessary conditions for TSD efficiency is solved.

A LINEAR PROGRAMMING FORMULATION

Linear programming can be used to determine whether any particular enterprise mixture, x^o, and the subset candidate to which it belongs satisfy the necessary conditions for FSD, SSD or TSD efficiency. Simply compute the y^o vector associated with x^o and then

(7) maximize $p'(y-y^o)$

subject to

(8) $G^o y \geq G^o y^o$

as well as (1), (2) and (3). If the optimal value of the objective function is zero, the necessary conditions are satisfied. If the optimal value is positive, the necessary conditions are not satisfied.

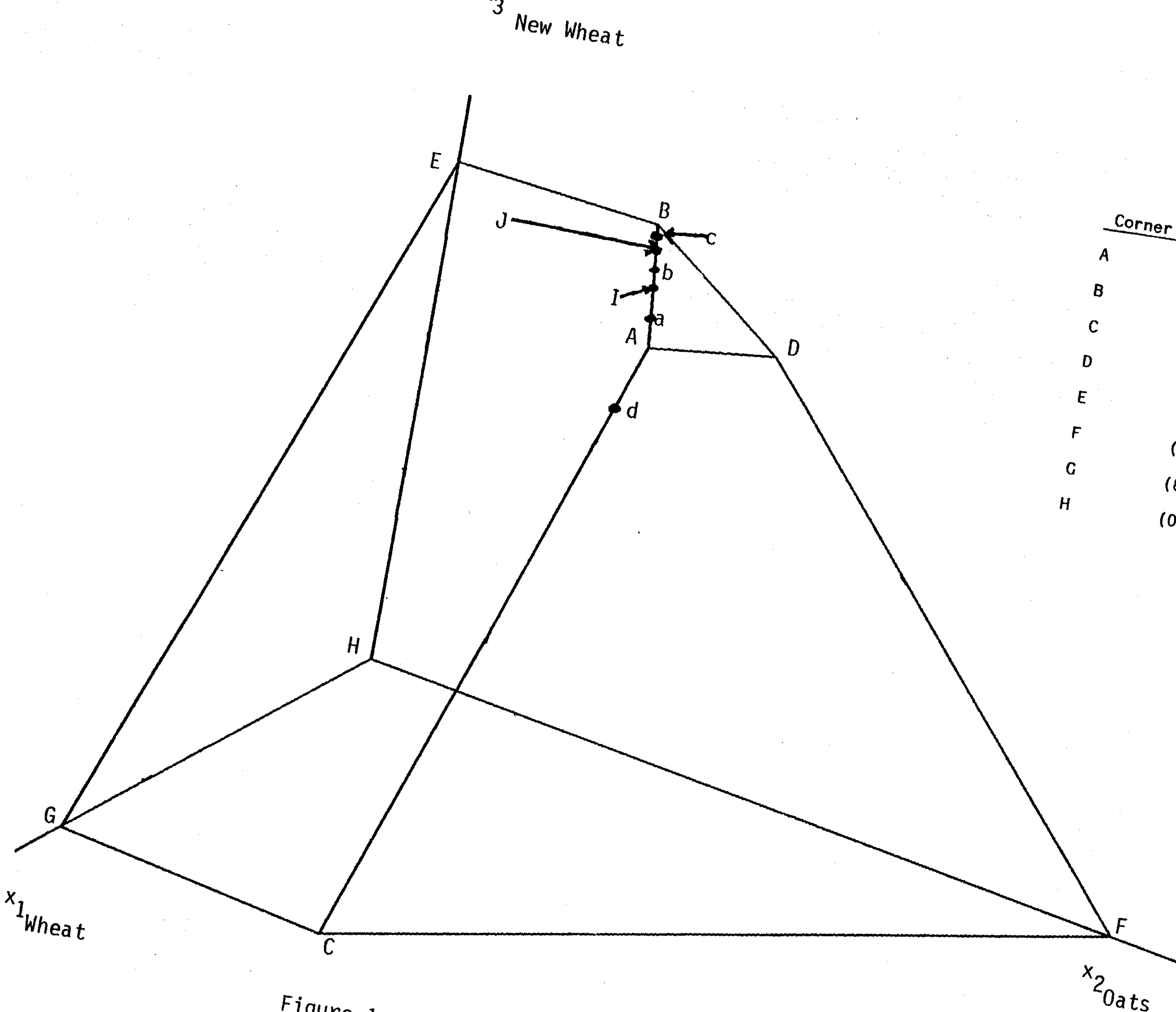
The specification of G^0 depends upon the criterion chosen. For FSD, G^0 is merely an s by s identify matrix. For SSD and TSD, the description of G^0 is simplified by assuming that the states of nature have been permuted so that $y_1^0 > y_2^0 > \ldots > y_s^0$.[5/] For these two criteria, G^0 is an s-1 by s matrix. For SSD, G_{ij}^0 is zero if $j \leq i$; otherwise, it equals p_j. For TSD, G_{ij}^0 is zero if $j \leq i$; it equals $p_j(y_i^0 - y_j^0)$ when $j > i$.

The constraints (8) ensure that the graph of the appropriate cumulative function associated with the optimal x vector lies at least as far to the right as the cumulative function associated with x^0 does.[6/] For the FSD version of the test, optimal income for each state of nature must be at least as great as that associated with x^0. For SSD (and TSD), income substitution is allowed among states of nature but only "SSD (or TSD) equal or better" substitution is permitted.

AN EXAMPLE

Three alternative ways of using these ideas are illustrated with data from Anderson, Dillon and Hardaker. Each of their five states of nature (observations) is assumed to be equally likely. Figure 1 shows the set of feasible x vectors. Upper case letters A through H identify the vertices of the feasible set. Faces ABEGC, ABD and ACFD lie on the planes defined by the wheat acreage, labor and land constraints. The other three faces lie on planes defined by nonnegativity constraints. The capital constraint is superfluous.

The ideas presented in earlier sections permit approximating stochastic dominance efficient sets. Consider the set of mixtures which satisfies the necessary conditions for FSD efficiency. Mixture A (uniquely) maximizes expected income subject to the resource and negativity constraints (2) and (3). Therefore, it must be FSD (as well as SSD and TSD) efficient. All other



Corner Point Coordinates

A	(1.33, 4.00, 6.67)
B	(0 , 3.20, 8.00)
C	(8.00, 4.00, 0)
D	(0 , 5.33, 6.67)
E	(0 , 0 , 8.00)
F	(0 , 12.00, 0)
G	(8.00, 0 , 0)
H	(0 , 0 , 0)

Figure 1. Feasible crop mixes.

mixtures which satisfy the necessary conditions must be "connected" to it directly or indirectly.[7/] Of the three edges connected to A, only AB and AC satisfy the necessary conditions for FSD efficiency. Each face of the feasible set has at least one inefficient edge. Therefore, Dybvig and Ross' "all or none" result means that none of the faces is FSD efficient. The interior of the feasible set must be FSD inefficient for the same reason.[8/]

A second way to employ the ideas in this paper involves starting with any connected set known to satisfy the appropriate necessary conditions. All mixtures on edge AC and those on line segment AJ are (unique) Target MOTAD solutions. Thus, they satisfy the necessary (and sufficient) conditions for SSD efficiency. It is theoretically possible that other mixtures satisfy the necessary (and perhaps the sufficient) conditions for SSD efficiency. Line segment JB is an appropriate candidate as it is the intersection of an edge and the set of x vectors for which elements of the associated y vectors share the same rankings (i.e., $y_2 \geqq y_1 \geqq y_5 \geqq y_4 \geqq y_3$). Its "interior" mixtures are found to be SSD inefficient by testing mixture c. Other candidate subsets could be examined. However, the FSD results guarantee that all would be found SSD inefficient. Thus, for this example, the set of SSD efficient mixtures is the set of Target MOTAD solutions.

A third way of employing the ideas is simply to determine whether selected mixtures satisfy the necessary conditions for stochastic dominance efficiency. This is illustrated by applying the TSD variant of the test to mixtures a, b and d. Of these three mixtures, only a satisfies the necessary conditions.[9/]

CONCLUDING REMARKS

Necessary conditions for FSD, SSD and TSD efficiency of enterprise mixtures were discussed. A linear programming formulation which permits determining whether a given enterprise mixture satisfies those conditions was presented. These ideas are illustrated by applying them to example data from Anderson, Dillon and Hardaker. For this example, the necessary conditions for SSD and TSD (and perhaps FSD) are also sufficient. This is not always true.

As contrasted with the outcomes commonly realized when the FSD criterion is applied to discrete alternatives, the FSD efficient set for the Anderson, Dillon and Hardaker example is a very small subset of the feasible mixtures. The relative effectiveness of SSD and TSD also differs from that ordinarily observed. Typically, the SSD criterion eliminates a significant fraction of the FSD efficient alternatives but the TSD criterion eliminates only a small proportion of the SSD efficient alternatives. The opposite results were observed in this paper.

Although the tests discussed here can be useful, the most appropriate role for them is not yet known. Complementary and/or competitive methods exist. Target MOTAD and mean-target semivariance have been mentioned above. Bawa, Lindenberg and Rafsky (BLR) suggest an approach which can be used to approximate stochastic dominance efficient sets for (imperfect market) portfolio problems. This approach could be extended to deal with the enterprise mixture problem. It is likely that to be cost-effective, any extension would have to deal explicitly with resource and nonnegativity constraints. The tests discussed in this paper also do this. Thus, these tests and the BLR algorithm may be complementary.

FOOTNOTES

1/ The classes of functions associated with FSD, SSD and TSD, respectively, have been variously described. As in Zentner et al., it is common to describe these classes in terms of the derivatives of various orders of the functions belonging to them. Fishburn and others have presented broader definitions. The classes considered in this paper are slightly larger than the former but slightly smaller than the latter.

2/ These conditions are specialized versions of those in Dybvig and Ross' Theorem 1. The only property they assumed for the set of feasible y vectors is convexity. Clearly, this assumption is valid for this paper.

3/ The argument Dybvig and Ross used to justify (their version of) condition (4) involved concave programming theory. That particular argument is not valid for FSD but the simpler one suggested here is.

4/ Even though the argument in the previous paragraph was in terms of "interiors" (which are not closed sets), the "all or none" result guarantees that the set of vectors which satisfy conditions (4) and (6) is the union of closed convex sets. Each candidate subset is closed and convex because it is the intersection of two closed, convex sets.

5/ This formulation assumes that the elements of y^0 are unique (i.e., no "ties"). Explicit examination of cases involving ties is unnecessary since ties are associated with more than one candidate subset and are efficient if any one of these candidate subsets is efficient.

6/ In this formulation, the alternative (cumulative) probability levels are fixed and the associated income levels are variable.

7/ A more extensive discussion of search strategies is presented elsewhere (McCamley and Kliebenstein, 1985).

8/ Inspection of the matrix, C, provides another way of determining the FSD efficiency status of the "interior" of face ABEGC and the interior of the feasible set. Face ABEGC lies on the plane defined by the wheat acreage constraint. Since the second column of C is strictly positive, each "interior" mixture on ABEGC is dominated by one (on edge AB or edge AC) which includes more oat acreage. The fact that all elements of C are positive means that all FSD efficient mixtures must be associated with faces, edges or vertices of the feasible set.

9/ A previous application of mean-target semivariance determined that all crop mixes on AJ except those involving less than (approximately) .58 hectares of (old) wheat satisfy sufficient conditions for TSD efficiency (McCamley and Kliebenstein, 1984). Since this set is identical to the set which satisfies the necessary conditions presented in this paper, it is the TSD efficient set.

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