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ABSTRACT

An Introduction to Durable Investment Analysis*

by
Lindon J. Robison**

This paper suggests that the study of resource allocation problems has been narrowly focused on either nondurable assets divisible in use and acquisition or on durables lumpy in acquisition and use. A new theorem is presented in this paper which provides a tool needed to analyze a much broader set of problems, including the analysis of durables lumpy in acquisition but divisible in use. It also allows analysts to examine changes in the value of durables resulting from changes in variable input prices, etc.

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**Lindon J. Robison is an Assistant Professor of Agricultural Economics at Michigan State University.

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AN INTRODUCTION TO DURABLE INVESTMENT ANALYSIS

Introduction

Economists are concerned with the allocation of resources among competing activities or investments. One distinction that can be made between investments is durability. Nondurable assets are used up--change form--in a single production period. Examples are feed, seed, fertilizer, and fuel all bought and used up in a single growing season where the growing season is the relevant period. Durable assets, on the other hand, provide services beyond a single period. Tractors, combines, cars, trucks, buildings, and breeding stock exemplify durable assets. However, feed, seed, fertilizer, and fuel could also be considered durable assets if the time period were short enough. The distinction between durable and nondurable assets depends in a rather arbitrary way on the time period of interest to the analyst.

The distinction between durable and nondurable assets is interesting because it implies that owners of durables carry an inventory of services that may be extracted from the durable. Hence, elements of inventory analysis should be included in the analysis of durable assets. But other distinctions are also of interest. Is the asset lumpy or divisible in acquisition? Is the asset lumpy or divisible in use? Can disinvestment in the asset occur or is it irreversible? While the distinction between durable and nondurable assets is somewhat arbitrary, the distinction between whether assets are lumpy or divisible in acquisition is not. Gasoline purchased at the gas station is divisible in acquisition. The inventory of available services supplied by a tractor is <u>not</u> divisible in acquisition. You either acquire the tractor or you don't. On the other hand, tractors are available in different sizes and different states of repair, providing some level of divisibility in acquisition, but for the most part, we continue to consider them lumpy in acquisition.

It is interesting to note that, although many durables are lumpy in acquisition, for the most part, almost all are divisible in use; that is, the services the asset supplies can be extracted in nearly divisible quantities. A passenger car is a lumpy asset; yet, its services can be extracted in a nearly completely divisible manner. Cars may deliver passenger miles per hour from a rate of 0 to 55 miles per hour (assuming speed laws are observed). Services from tractors, combines, and buildings also deliver services in divisible quantities. In contrast, light bulbs, for the most part, have a fixed service extraction rate exemplifying a much smaller class of assets whose services are lumpy in use.

Finally, an investment in fertilizer already applied to the land is not recoverable--irreversible. Farm machinery, for the most part, is a reversible investment. This distinction between assets is particularly important when considering disinvestment decisions.

Given this classification of assets--durable versus nondurable, lumpy versus divisible in acquisition and use, and reversible versus nonreversible--16 distinct investment types are possible as listed in Table 1. Given the large number of possible investment types, it is interesting that our classroom treatment of investments appears to focus on investments identified in Table 1 as Types 1 and 16. In production economics, the models usually static (timeless). As a result, they focus on nondurable assets, assets that are acquired and used up in a single period. When durables are introduced, their costs do not usually vary with output and intertemporal links are ignored. Furthermore, to apply that wonderful "marginal analysis" tool, our production models assume infinitely divisible assets--divisible in acquisition as well as use. And, because nothing is left unused, the assets are irreversible. This asset is Type 16 (nondurable, divisible in acquisition and use, and irreversible).

Now consider the investment models used in the farm management literature. Exposure to the real world of decision makers forces farm management scientists to recognize that the investments of most interest are durable. This focus on durables comes at a cost, however, and the cost appears to be the assumption of divisibility in use, which precludes application of marginal principles.

If the durable assets are lumpy in acquisition and in use, then the only possible application of the marginalist's tool is in determining the life of the durable--which has been solved repeatedly (see Perrin). It ignores, though, the possible application of marginal analysis tools to derive the optimal amount of services to extract from durables, i.e., lumpy in acquisition but divisible in use. The investment in question--durable, lumpy in acquisition and use, and possibly reversible--is Type 1 in Table 1. The resulting investment approach that ignores the divisible nature of durable services is not much more revealing than the static approach that ignores lumpiness in acquisition and the durable nature of investments.

From my observations, the investment analyses of most interest and most often faced are investment Types 3 and 4. It is difficult to conceive of durables whose level of services cannot be altered--making it divisible in use. Yet with our inattention to divisibility in use, we have ignored many important questions. Consider the kinds of questions we cannot answer, given our present analytic models. Suppose we produce transportation services from a passenger car, a durable (lumpy in acquisition and divisible in use), and nondurable inputs (gas, maintenance, etc.). What is the value of the car to the firm and how will this value change in response to increases in the price of gas? The answer to this question requires both the discounting techniques of the farm managers as well as the marginal-maximizing tools of the production economist. The procedure for solving such questions follows.

First, define the time period. Then, solve for the optimal amount of services to extract from the durable. Then, value those services. Then, in an iterative manner, solve for the optimal life of the durable and find the present value of its services. Describing the complete solution for answering these questions is beyond the scope of this paper but is discussed in detail in Robison and Abkin. Instead, the remainder of this paper focuses on only one of these questions; namely, how to value services from the durable, given that its services are divisible in use.

Valuing the Durable's Services

Historically, assigning the value of production to inputs has been of interest to economists. Early studies were interested in determining what share labor could claim of an output it produced in combination with capital and other inputs. Farm management experts have long dealt with the issue of how to identify the returns to farmland so as to establish the maximum bid price that could be offered. A similar approach has been applied to farm machinery and other durables.

The two efforts to identify returns to factors of production resulted in two different methodological approaches. As a theoretical tool, economists applied Euler's theorem to partition the output among the inputs. A practical approach adopted by farm managers was the residual approach,'which assigned to the durables what was left of returns after paying for variable expenses.

Neither Euler's theorem nor the residual approach ultimately solved the problem of how to assign the output to its inputs. Euler's theorem, which said each input's share of the output was the marginal value product of the last unit used times the total number of units used, was applicable only for linearly homogeneous production functions; and the residual farm management approach has no basis for assigning net surplus from production to the durable any more than they could defend assigning it to the nondurable inputs.

Faced with an apparent impasse on how to value durable asset services, hence durables, this paper proves a theorem, the Product Exhaustion Theorem, which states that, by specifying relationships between inputs, the output of any continuous production process can be divided among the inputs in such a way that the output is just exhausted.

Critical to the theoretical development is the understanding that, although in many cases durables are lumpy in acquisition, they provide services that are extracted in completely divisible amounts. Therefore, being compatible with traditional marginal economic analysis, we can ask: how many units of service should we extract from the durable? Then, knowing the level of services to extract, we face the question: what is the total value of services extracted from the durable? The following theorem will allow us to answer such a question.

The Product Exhaustion Theorem

Theorem 1: Let $f(x,y)$ be a continuous function with derivatives $f_x(x,y)$ and $f_y(x,y)$. If the relationship between x and y is specified, say $y = m(x)$ or $x = m^{-1}(y)$ for all x and y, then:

$$\int_x f_x(x,m(x))dx + \int_y f_y(m^{-1}(y), y)dy = f(x,y)$$

Proof: After substituting $m(x)$ for y, express the function $f(x,y)$ as $f(x,m(x))$. The derivative of $f(x,m(x))$ with respect to x can be written as:

$$df = f_x(x,m(x))dx + f_y(x,m(x))m_x(x)dx$$

The anti-derivative of df, which by the second fundamental law of calculus equals f, can be written as:

$$f(x,m(x)) = \int_x f_x(x,m(x))dx + \int_{m(x)} f_y(x,m(x))m_x(x)dx$$

Then, by using the change-of-variable technique, we substitute dy for $m_x(x)dx$, and y for $m(x)$ in the second integral to obtain the desired result. The change-of-variable technique then allows us to substitute in the above expression y for $m(x)$; $m^{-1}(y)$ for x, and dy for $m_x(x)dx$. We write the result as:

$$f(x,y) = \int_x f_x(x,m(x))dx + \int_y f_y(m^{-1}(x),y)dx$$

which proves our theorem. This theorem can also be proven for three inputs, which by mathematical induction proves it for n inputs (see Robison).

At this point, an interpretation of the product exhaustion theorem may prove helpful. The interpretation is aided by Figure 1, which illustrates isoquants q_1, q_2, . . ., etc. The isoquants represent constant levels of output obtained from various combinations of inputs x and y used in the process $f(x,y)$. The isoquants are connected by ray OA, which may be considered an expansion path.

Now, consider output levels q_1 and q_2 obtained with inputs (x_1,y_1) and (x_2,y_2), respectively, at points B and C along the expansion path OA (Figure 1). The increase in output from q_1 to q_2, moving from point B to point C, can be achieved by increasing the input x by an amount (x_2-x_1) and by increasing input y by an amount y_2-y_1. The increase in output that results, q_2-q_1, is a result of increases in both x and y. Were only x increased, output would have only increased by q_o-q_1, the remaining increase, q_2-q_o, being attributed to the increase in y equal to y_2-y_1. Output q_o-q_1, then, approximates the contribution to output of the input x at output level q_1. A similar contribution could be obtained for increases in the input y by the amount y_2-y_1. These measures only approximate the contributions of x and y, however, since, along the expansion path, x and y are changing simultaneously. Output increases measured at smaller and smaller increases in x and y would, though, improve the accuracy of our measure. Moreover, for infinitely small changes in x and y, completely accurate measures of the output contributions of x and y are obtained and the results are the same as those obtained with the product exhaustion theorem.

Now, after having established a methodology for assigning the output to the inputs, we are ready to ask the next question: what is the proper relationship to define between the inputs? This question, of course, is an economic one which depends on the marginal value products of the inputs and their marginal costs. At a point in time, this relationship depends only on those costs that vary with use as opposed to costs that vary with time. Economists prescribe that the relationship among the inputs should be such that the ratios of their marginal value products divided by their prices be equal for all inputs at any point in time.

Euler's Theorem

Euler's theorem obtains a result similar in nature to the product exhaustion theorem. It also partitions output among inputs in a way that just exhausts the product. Its limitation is that it applies only to linear homogeneous functions and does not relate the output share to the expansion followed. We now illustrate the product exhaustion formula, obtain Euler's Theorem as a special case, and then provide other examples of the product exhaustion theorem using a homogeneous function not of degree one.

To begin, a linear homogeneous function measured over input variables x and y has the property:

(1) $tf(x,y) = f(tx,ty)$

Interpreted graphically, this definition implies that, for a t percentage increase in both x and y measured along any given linear expansion path, output q will increase by the same percent. It also means that the partial derivatives of f with respect to x and y (f_x and f_y) measured along the expansion path are constants, thus allowing us to describe the output attributed to x and y in a special way. Since f_x and f_y are constants, we obtain from the product exhaustion theorem the result:

(2) $f_x \int_x dx + f_y \int_y dy = xf_x + yf_y = f(x,y)$

That $xf_x + yf_y$ equals f(x,y) is Euler's well-known result; that it can be obtained as a special case of the product exhaustion theorem is not so well known.

We now illustrate the product exhaustion theorem using the linear homogeneous function:

(3) $f(x,y) = x^{\alpha} y^{1-\alpha}$

and the relationship between x and y along a linear expansion path as $y = kx$. The unconstrained partial derivative of f with respect to x can be written as:

(4) $\partial f / \partial x = \alpha x^{\alpha-1} y^{1-\alpha}$

and substituting kx for y, where kx is the value of y along the expansion path given x, we write:

(5) $f_x = \alpha k^{1-\alpha}$

a constant. Similarly, we could obtain an expression for the constant partial derivative for y along the expansion path as:

(6) $f_y = (1-\alpha) / k$

As already pointed out, multiplying the constant partial derivatives by the inputs used or integrating returns the same result:

(7) $\alpha k^{1-\alpha} \int_x dx + ((1-\alpha) / k) \int_y dy = xf_x + yf_y$

a result which can be graphically portrayed in Figure 2. Graphically, the output attributed to x is equal to the rectangle in Figure 2, where the horizontal length equals the input level x and the vertical length equals the constant marginal product of x measured anywhere along the expansion path OA in Figure 1.

Euler's Theorem holds in this case because the average product equals the marginal product. As a result, multiplying the marginal product of the last unit of production times the total units of inputs used is equivalent to multiplying the average product of x times the total units of inputs used (the area of the rectangle in Figure 2)--which obtains exactly that portion of output attributed to x. But whenever the marginal product is not constant, the output attributed to an input cannot be found by merely multiplying the marginal product of the last unit it produced by the total units of inputs used in production. To illustrate, consider two examples where Euler's Theorem does not hold: a nonlinear homogeneous function. By definition, a homogeneous function of degree h has the property that:

(8) $t^h f(x,y) = f(tx,ty)$

As an example, consider the nonlinear homogeneous function:

(9) $f(x,y) = x^{\alpha} y^{\beta}$

where $\alpha + \beta \neq 1$.

Again, assuming a linear expansion path of the form y = kx allows us to express the derivative of f with respect to x measured along the expansion path as:

(10) $f_x = \alpha k^{\beta} x^{\alpha+\beta-1}$

Obviously, if plus equals one, the function is linearly homogeneous and our earlier results hold. However, when the sum of plus does not equal one, f_x measured along the expansion path is no longer constant. In Figure 3, the output attributed to x is represented as the area under the curve f_x, which, if $\alpha + \beta > 1$, increases with x (or if $\alpha + \beta < 1$ decreases with x). To measure the area under f_x now requires we integrate f_x

over x. To measure the area under the curve by multiplying f_x of the last unit of x used by the units of x used would underestimate the contributions of x if $\alpha+\beta<1$ and overstate their contributions if $\alpha+\beta>1$.

Obviously, Euler's Theorem no longer partitions output among the inputs so as to exhaust the product. To illustrate:

(11) $x f_x + y f_y = \alpha k^{\beta} x^{\alpha+\beta} + \beta y^{\alpha+\beta} k^{-\alpha} = (\alpha+\beta)\, x^{\alpha} y^{\beta} = (\alpha+\beta)\, f(x,y)$

where the product is just exhausted only in the case where + equals one.

Using the results of the Product Exhaustion Theorem, we write:

(12) $\int_x f_x + \int_x f_y = \alpha K^{\beta} x^{\alpha+\beta}/(\alpha+\beta) + \beta\, y^{\alpha+\beta} K^{-\alpha}/(\alpha+\beta) = f(x,y)$

A similar example using a nonhomogeneous function could be constructed.

Applying the Product Exhaustion Theorem (PET)

Having obtained a measurement procedure for ascribing to an input its contribution to output, we are now in a position to answer questions not previously answerable. To best illustrate the power of the new theorem, we compare and contrast the questions answerable with our usual static analysis with those answerable with the Product Exhaustion Theorem results.

Consider a two-input production process f defined over inputs x and y which can be purchased at input prices p_x and p_y. If p is the output price, how much of either x or y should be employed to maximize profit? For this question, our existing static tools are well adapted: acquire until the last unit's cost just equals the value of its marginal product. With diminishing marginal products, we are assured that if the last unit of input earns enough revenue to cover its costs, earlier units' returns must exceed their cost. However, whether or not surplus returns exceed nonvariable inventory costs is not yet determined.

Consider now a question not answered by our usual marginal analysis. Suppose a production process employs (1) an input y that can be purchased and used in divisible units, and (2) services x from a durable. If the durable has a capacity to deliver up to $\bar{x}$ level of services, how much is the durable worth? How will changes in the price of the variable input alter the durable's value? Alternatively, what is the most the firm can pay to acquire the durable? Or, what is the minimum output required to justify purchasing the durable? To answer these questions requires we answer the question ignored before: if the value of the last unit of service from the durable equals its variable cost, what is the value of all previous units of services extracted from the durable? That question is now answerable. But that is not all. Having obtained an expression for the value of services from a durable, we can differentiate to determine how the value of durable services would change if input or output prices were altered; if the production process changed; if the price of the output changed; etc.

We derive such analytic results by letting the expansion path relationship between x and y be written as:

(13) $y = m(x,p_x,p_y)$

which recognizes explicitly the role of input prices on the expansion path. Moreover, let the derived demand for x be written as:

(14) $\bar{x} = h(p_x,p_y,p)$ where

$d\bar{x}/dp_x < 0$ and $d\bar{x}/dp_y \gtrless 0$

depending on whether x and y are substitutes or complements. Then, the value of durable services V(x) can be written as:

(15) $V(\bar{x}) = \int_0^{\bar{x}} pf_x[x,m(x,p_x,p_y)]\ dx$

To find the impact of an increase in p_x on $V(\bar{x})$, we differentiate the above expression to obtain:*

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*The formula used to find the first-order conditions of an integral can be written as:

$$\frac{d}{dx}\int_p^q f(s,x)\ ds = \int_p^s \frac{\partial}{\partial x}\,[f(s,x)ds] + f(q,x)\frac{dq}{dx} - f(p,x)\frac{dp}{dx}$$

(16) $$dV(\bar{x})/dp_x = pf_x[\bar{x},m(\bar{x},p_x,p_y)]\ \partial\bar{x}/\partial p_x + \int_0^{\bar{x}} pf_{xy}[x,m(x,p_x,p_y)]\ (\partial m/\partial p_x)\,dx$$

The first expression on the right-hand side of the above equation gives the impact on the change in the value of services attributed to the durable of a change in the level of services extracted. In this case, the change is negative since $\partial\bar{x}/\partial p_x$ is negative. The second expression gives the change in the value of services resulting from following a different expansion path. In this case, with two variables, f_{xy} of necessity is positive, and the slope of the isoquant increases with increases in the variable price of durable services. Thus, the sign of $dV(\bar{x})/dp_x$ is indeterminate.

Another analytic result could be obtained for changes in $V(\bar{x})$ associated with output price changes, i.e., changes in p. The result is:

(17) $$dV(\bar{x})/dp = pf_x[\bar{x},m(\bar{x},p_x,p_y)]\ \partial\bar{x}/\partial p + \int_0^{\bar{x}} pf_x[x,m(x,p_x,p_y)]\ dx > 0$$

since $\bar{x}/\ p$ exceeds zero. Other variants of the problem are described in Robison.

Summary

This paper has suggested that the most relevant class of investment problems--investments in durables lumpy in acquisition and divisible in use--has been ignored, for the most part, by agricultural economists. To analyze such an investment type requires that an analytic method for valuing services from a lumpy durable be obtained. Such a method has been presented in the form of the "Product Exhaustion Theorem." With this theorem, important investment questions can be answered, such as: how will the value of durables change in response to nondurable price changes, etc.

Table 1 -- Classification of Investment Models

Investment Type	Durable	Non-Durable	Lumpy in Acquisition	Divisible in Acquisition	Lumpy in Use	Divisible in Use	Reversible	Irreversible
1	X		X		X		X	
2	X		X		X			X
3	X		X			X	X	
4	X		X			X		X
5	X			X	X		X	
6	X			X	X			X
7	X			X		X	X	
8	X			X		X		X
9		X	X		X		X	
10		X	X		X			X
11		X	X			X	X	
12		X	X			X		X
13		X		X	X		X	
14		X		X	X			X
15		X		X		X	X	
16		X		X		X		X

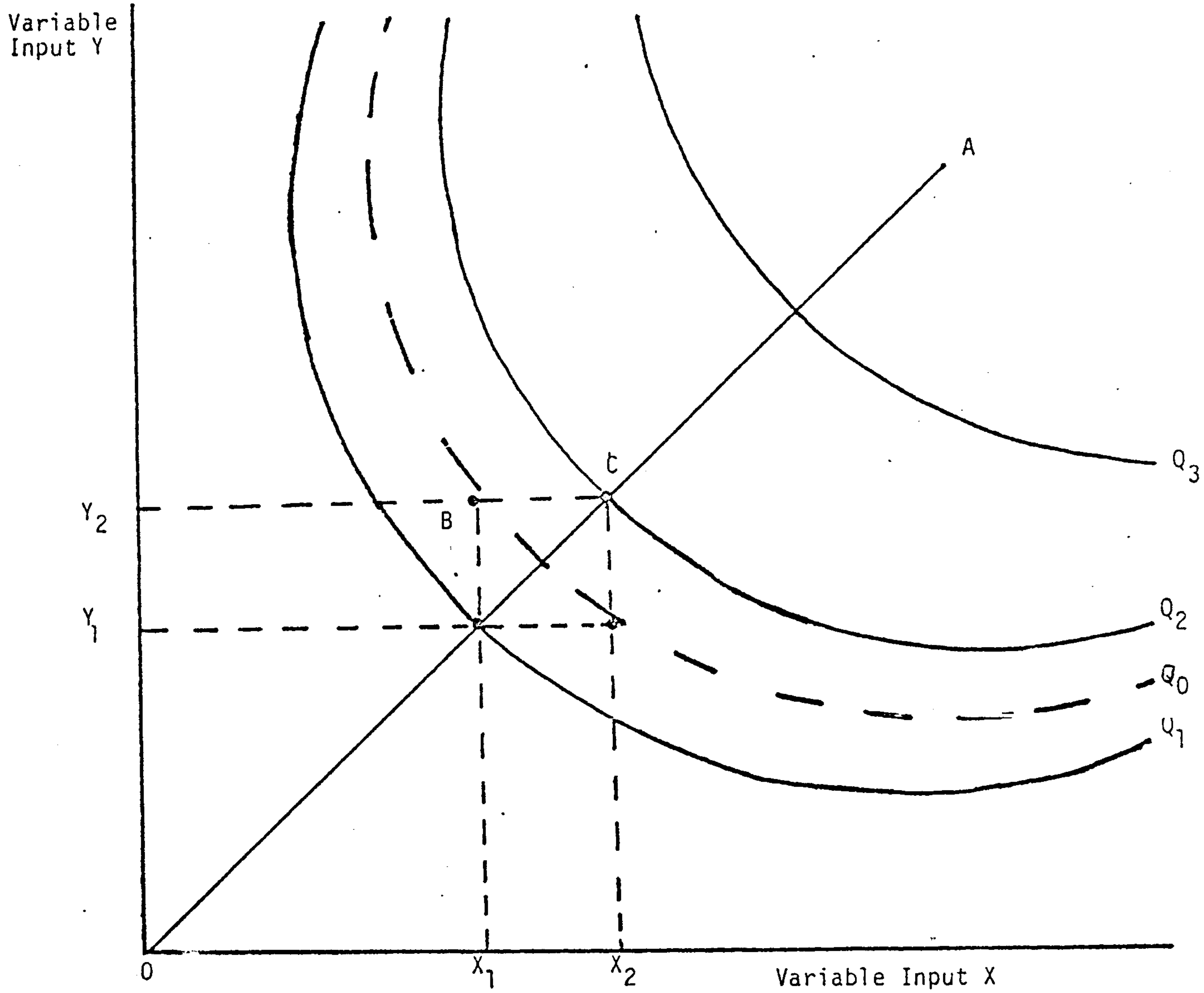

Figure 1

A Graphical Presentation of the Product Exhaustion Theorem

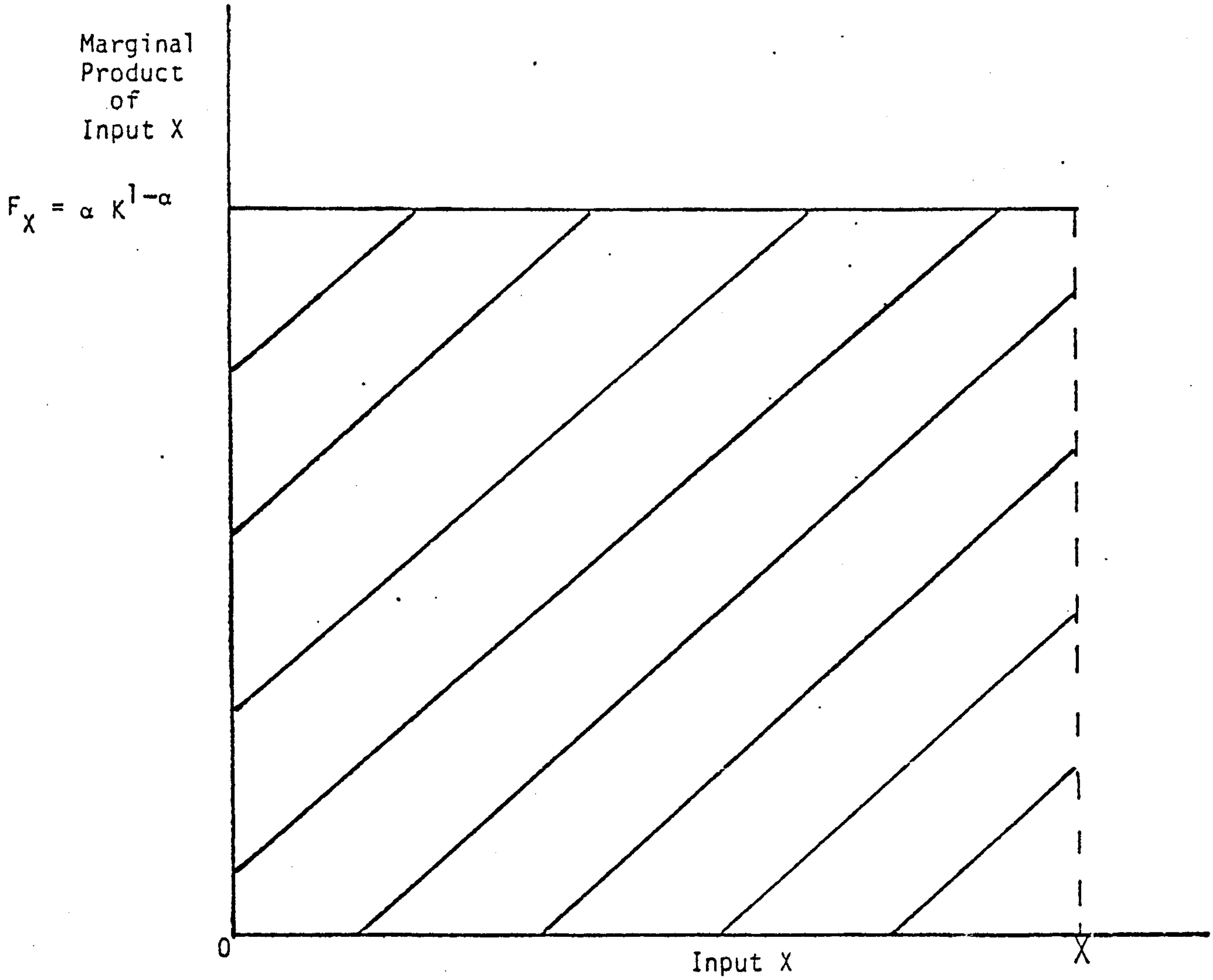

Figure 2

Output Attributed to X for a Linear Homogeneous Function
(The Product of X and F_X)

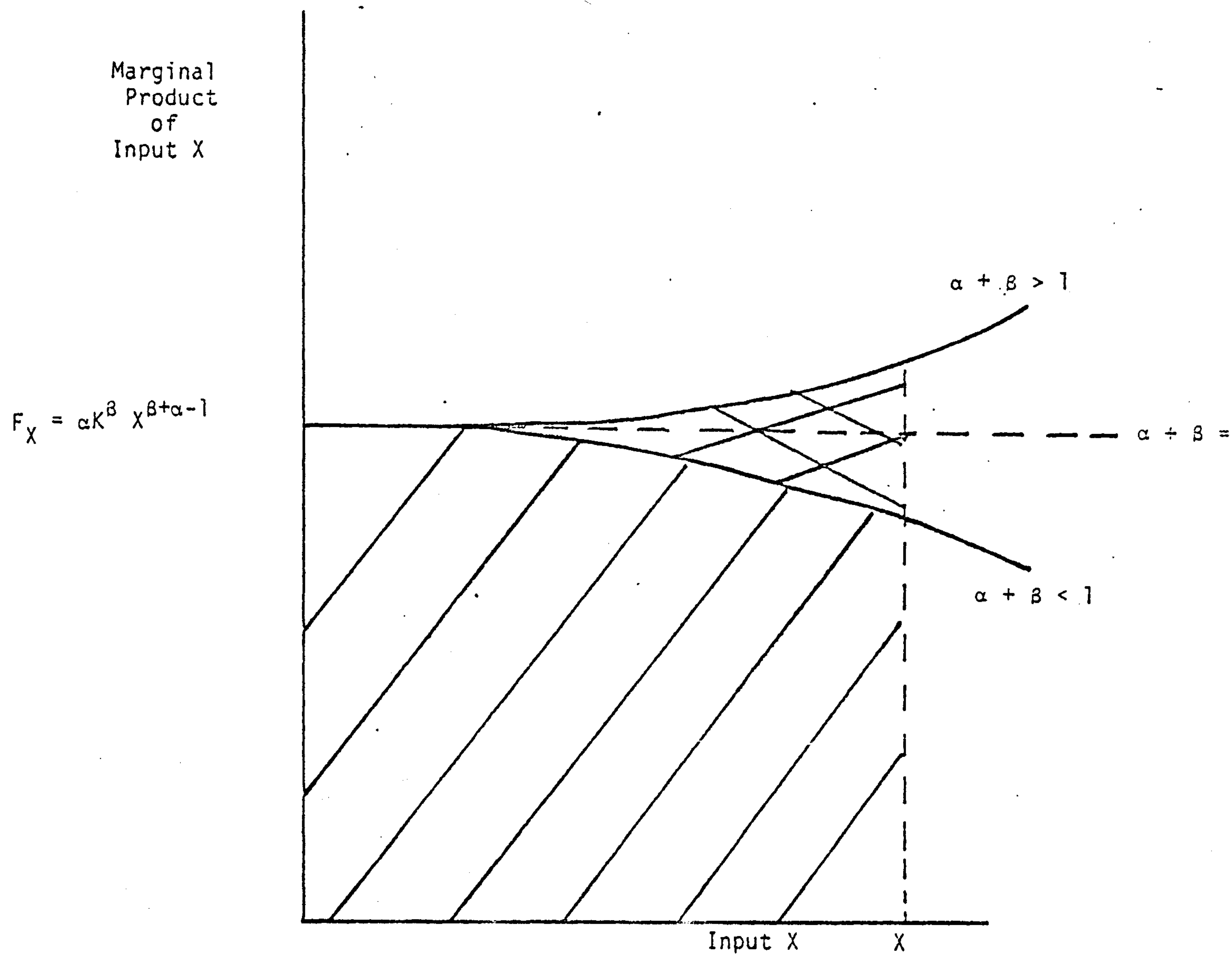

Figure 3

Output Attributed to X Measured for a Non-Linear Homogeneous Function

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