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Econometric Analysis of Risk Models - Some Explorations and Problems

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It is not my intention in this paper to present evidence that factor demands and output supplies are responsive to risk as defined beyond the first moment of profit. However, such risk response appears likely (there is mounting evidence that variances are determinants of output supply (Just, Behrman, Ryan)). The interest here is in the examination of some issues and methodologies which could be used to test for the existence of risk responsive behavior in these rather aggregative econometric models. The focus will be on risk aversion as it affects input and output choice. Attention will be focused on output and output price uncertainty.

Structural Econometric Production Systems

Very often, a researcher is posed with two sorts of problems regarding risk: 1) risk response is of little interest but production parameters such as elasticity of substitution and other constant output demand parameters are of interest; 2) risk response is of inherent or primary interest.

THE COST FUNCTION

Recently, Pope (1978) has shown that for certain classes of risk problems, one can consistently estimate parameters of the cost function and hence (via Shepard's Lemma) obtain constant-output input demand functions even though risk aversion is present.

Let $C = \sum_{i=1}^{N} c_i x_i$ denote the cost function, where x_i denotes input demands and c_i denote unit factor costs. Let $q = f(x_1 \ldots x_N)$ denote the production function, where q is output. Finally, let E denote the expectation operator, U utility, and P is the random output price.1/ Then, it can be shown that the dual problems

$$\text{(1)} \quad \text{a.} \ \max_{X} E[U(\pi)] \qquad\qquad \text{b.} \ \min C \ \text{ s.t. } q^* = f(x_1 \ldots x_N)$$

are mutually consistent, where π is profit and q^* denotes the solution to problem a., $q^* = f(x_1^* \ldots x_N^*)$. That is, the risk averse firm still minimizes costs, but chooses a lower optimal output level than the risk neutral firm (Sandmo).

If the problem above assumed production uncertainty using the common form, $q = f(x)\varepsilon$, $E(\varepsilon) = 1$, then the firm is still a cost minimizer. Finally, if both price and multiplicative production uncertainty are included in the problem, the firm remains a cost minimizer.

The implications of these results are emphasized by a concrete econometric example. Consider the translog cost system (Burgess) under output uncertainty:

$$\text{(2)} \quad \ln C = a_0 + \sum_i a_i \ln c_i + \frac{1}{2} \sum_i \sum_j B_{ij} \ln c_i \ln c_j + a_q \ln q + \sum_i d_i \ln q \ln c_i + \frac{1}{2} e \ln^2 q + e_0$$

$$\text{(3)} \quad S_i = a_i + \sum_j B_{ij} \ln c_j + d_i \ln q \ln c_i + e_i$$

$$e = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_N \end{bmatrix} \sim (0, \Sigma), \qquad i = 1 \ldots N\text{-}1$$

where the e's are errors and S_i are the cost shares, $c_i x_i/C$. By the results given above, the elasticity of substitution, and other factor demand parameters can be consistently estimated from (2) or (3) under risk neutral or risk averse behavior given the assumption made above. These results do not hold for the indirect profit function popularized by Lau.[2/]

The Primal Production System

Presumably, the primary interest of this session is to estimate risk response under risk aversion. The cost function will not provide this information in the cases discussed above. It appears to the author that the use of either the indirect profit or cost functions are cumbersome for the study of risk aversion. For this reason, the primal system will be considered.

Output Price Uncertainty

If only output price uncertainty is assumed, the primal system is easily amenable for simple classes of utility functions (the class of functions which are linear in the central moments of profit). Consider the utility function $E(U) = e - (\lambda/2)\ V$, $\lambda > 0$, where e and V are expected income and variance respectively and λ is a risk aversion parameter. Consider also the firm given in (1) a. The resulting econometric system can be made linear in parameters. The first order conditions are:

(4) $E(P)f_i - c_i - \lambda f\ f_i\ V(P) = 0 \qquad i=1\ .\ .\ .\ N,$

where $f_i = \partial f/\partial x_i$ and $V(P)$ = variance of price. If (4) is multiplied by $x_i/(E(P)f)$, it can be rewritten as:

$$(5)\quad \frac{\partial \ln f}{\partial \ln x_i}\left(1 - \lambda q \frac{V(P)}{E(P)}\right) = S_i, \qquad i=1 \ldots N$$

where $S_i = c_i\, x_i / E(P)q$ is the factor share evaluated at expected price.

For the Cobb-Douglas production function $\partial \ln f / \partial \ln x_i$ is a constant (the elasticity). Let a_i denote the elasticity. In this case, the econometric system becomes:

$$(6)\quad \ln q_t = a_0 + \sum_i a_i \ln x_{it} + e_{0t}$$

$$(6')\quad S_{it} = a_i + \tilde{a}_i Z_{it} + e_{it},$$

where the e's are random errors, $Z_{it} = q_t \frac{V(P_t)}{E(P_t)}$, and $\tilde{a}_i \equiv a_i \lambda$.

Note that (6') has accommodated the possibility of changing perceptions for expected price and/or variance of price (as in Just). These forecasts could be estimated by a distributed lag within the model--leading to nonlinear estimation techniques. Alternatively, the forecasts could be made optimally outside the model by Box-Jenkins forecasts (McCallum) and then inserted as data in (6').[3/] Finally, note that the risk aversion parameter λ can be identified. Indeed, this is the value of structural estimation.

Although more cumbersome, the approach above can be extended to the multiproduct case. However, the problem above must accommodate production uncertainty if it is to be generally applicable in agriculture.

Production Uncertainty

A major problem associated with the introduction of production uncertainty in economic models is that there is no universal agreement as to how random forces, such as weather, affect the probability distribution of output given input levels. By far the most used theoretical and empirical representation is

$$q_t = f(X_t)W_t \qquad E(W_t) = 1.$$

For this production function, the system corresponding to (6) is[4/];

$$(7) \quad \ln q_t = a_0 + \sum_i a_i \ln x_{it} + e_{0t}$$

$$(8) \quad S_{it} = a_i - \tilde{a}_i Z_{it} + e_{it} \qquad i=1 \ldots N,$$

where $S_{it} = (c_i x_{it}/P_t f_t)$, $Z_{it} = P_t f_t V(W_t)$, and $\tilde{a}_i = \lambda a_i$. Note that Z_{it} is now unobservable. Expected output, f_t, is included in Z_{it} and S_{it}. Hence, (8) is nonlinear in the parameters. This indicates the inherent problems when production uncertainty is introduced: The conditional moments, of production must be estimated within the model.

Since there are many possible alternatives for the estimation of (8), only four prominent possibilities are considered here due to space limitations.

a. Estimate (8) via nonlinear techniques either with $V(W_t)$ fixed as an unknown parameter or with a changing estimate of $V(W_t)$.

b. Insert q_t for f_t in (8) and estimate the system linearly.[5/]

c. The third possibility is to presume that $\ln x_{it}$ is not a function of e_{0t} (see Hoch for a justification) and estimate the production function and hence, $V(W_t)$. Insert these consistent estimates of f_t and $V(W_t)$ in (8) and estimate (8) in the next stages(s).

d. Predict f_t with the exogenous variables in (8); then estimate (8) in the second stage with f_t replaced by $\hat{f}_t$ (Kelejian).

In many situations, approach (c) would be preferable. However, the estimates of $\tilde{a}_i$ would be inconsistent unless one could argue that $\ln x_i$ was not a function of e_{it}.[6/] No attempt will be made here to evaluate or motivate each of the four approaches but they are suggested for further research. With the exception of (a), it appears that theoretical econometric problems are likely present.

Other Stochastic Specifications

Just and Pope have argued that alternative stochastic specifications of production uncertainty are superior to that given in (7). One possible simple form is written

$$q_t = f(x_t) + h(x_t)W_t \qquad E(W_t) = 0.$$

Assume f_t and h_t are Cobb-Douglas such that

$$q_t = a_0 \prod_{i=1}^{N} x_{it}^{a_i} + b_0 \prod_{i=1}^{N} x_{it}^{b_i} W_t.$$

Let $P_t = E(P_t) + \gamma_t$, $E(\gamma_t) = 0$ and $E(\gamma_t W_t) = 0$. The resulting nonlinear system of share equations are:

$$(9) \quad S_{it} = a_i - \tilde{a}_i f_t \frac{V(\gamma_t)}{E(P_t)} - \tilde{b}_i (h_t^2/f_t) [E(P_t) V(W_t) - \frac{V(\gamma_t W_t)}{E(P_t)}]$$

$$i=1 \ldots N$$

where $\tilde{a}_i = a_i \lambda$, $\tilde{b}_i = b_i \lambda$, and $S_{it} = c_i x_{it}/E(P_t) f_t$. Clearly (9) is highly nonlinear. However, modern computing programs can easily accommodate the system in (9). In the final analysis, only further empirical work can determine whether the restrictive multiplicative production disturbance or the more general disturbance associated with (9) is most appropriate.

A Numerical Example

In a recent journal article, Rosine and Helmberger estimated factor demand parameters via factor shares for the Cobb-Douglas production function of U.S. agriculture. Aggregation biases are clearly introduced in the model and such aggregation suredly diminishes the variability; however, the application may be instructive. In many situations one does have micro time series where the above equations may prove more fruitful. Since we do not want the example to be unduly complicated, equations (6') shall be estimated.7/ Further, nonneutral technical change is assumed such that the production function is given by

$$q_t = a_0 \prod_{i=1}^{N} x_i^{a_i + b_i \cdot t}$$

Corresponding to (6') are the share equations

$$S_{it} = a_i + \tilde{a}_i Z_{it} + b_i \cdot t + \tilde{b}_i Z_{it} \cdot t \qquad i=1 \ldots N \tag{10}$$

where $\tilde{b}_i = \lambda b_i$. Expectations and variances of prices are estimated by three period weighted moving averages.

Table 1 presents the results of the ordinary least squares regressions.8/ As anticipated, the aggregate U.S. data does not present compelling evidence of risk averse behavior. These results may improve, however, if an optimal lag structure were estimated and nonlinear techniques used with a more general stochastic model. If risk aversion is present, one expects the $\hat{\tilde{a}}_i$ and $\hat{a}_i$, $\hat{\tilde{b}}_i$ and $\hat{b}_i$ to be of opposite signs. This holds true for labor and operating inputs but the $\hat{\tilde{a}}_i$ coefficient has a positive coefficient in the capital equation. The important point from a welfare point of view is that risk may alter the absolute and relative well being of factors of production (e.g., Pope and Just, Batra).

REDUCED FORM ESTIMATION

It is almost impossible to start with a direct utility function and derive meaningful explicit reduced form factor demands. This has lead a number of researchers to approximate the reduced form with a first order Taylor's series approximation in the moments of profit. The implications of such a procedure are largely unexplored. As in demand theory, many unrealistic implications are drawn from this specification (Pope, 1977). It would appear that second order (log linear or linear) specifications need empirical investigation in order to test whether the separable specification is appropriate.

Consider the utility function associated with (4) within a simple Leontief production scheme where $\pi = \sum_i x_i \pi_i$ denotes farm profit per unit level of activity i. The optimal activity levels are given by

$$(11) \quad x_i = \frac{1}{\lambda} \sum_j V^{ij} E(\pi_j)$$

where V^{ij} is the i-jth element of the covariance matrix.

Even for the case where all covariances are zero (and hence, V^{ij} $(i \neq j) = 0$) we obtain

$$x_i = E(\pi_i)/V(\pi_i).$$

Here, clearly demands are not separable in $E(\pi)$ and $V(\pi)$. _A priori_ one expects

$$\frac{\partial x_i}{\partial V(\pi)\ \partial E(\pi)} \neq 0$$

but only empirical research using second order functions would vindicate or vitiate the linear approach.

Theoretical Restrictions

There are two reasons why the theoretical restrictions implied by the theory may be of interest: 1) to test the theory; 2) to gain degrees of freedom by the imposition of restrictions implied by the theory. In either case, where data are available on several outputs or inputs, theoretical restrictions may be a useful tool.

Without derivation, three cases are presented below for the utility functions of the form

$$(12)\quad E\,[U(\pi)] = E(\pi) + \sum_{t=2}^{T} a_t\, \sigma_t,$$

where the a_t are constants and $\sigma_t = E[\pi - E(\pi)]^t$. Note that the utility function associated with (4) as well as other empirically useful utility functions, is of this form.[9/]

Output Price Uncertainty

Additive ($P_k = E(P_k) + \gamma_k \equiv \bar{P}_k + \gamma_k$, $E(\gamma_k) = 0$)

$$\frac{\partial q_k}{\partial \bar{P}_k} \geq 0\ , \quad \frac{\partial q_k}{\partial \bar{P}_{k'}} = \frac{\partial q_{k'}}{\partial \bar{P}_k}$$

$$\frac{\partial x_i}{\partial \bar{P}_k} = -\frac{\partial q_k}{\partial c_i}\,, \quad \frac{\partial x_i}{\partial c_i} \leq 0$$

$$\frac{\partial x_i}{\partial c_{i'}} = \frac{\partial x_{i'}}{\partial c_i}\,.$$

Where k and k' represents distinct outputs, and x_i and c_i are the ith input and input cost respectively and production is implicitly joint.

Output Price (Multiplicative $P_k = \bar{P}_k \gamma_k$, $E(\gamma_k) = 1$ and $E(\gamma_k \gamma_{k'}) = 0$)

For convenience, only the case where T = 2 will be given

$$\frac{\partial q_k}{\partial \bar{P}_k} \quad (1 + 4a_2 \bar{P}_k \; q_k V(\gamma_k) \geq 0$$

$$\frac{\partial q_k}{\partial c_i} \quad (1 + 4a_2 \bar{P}_k \dot{q}_k V(\gamma_k)) = \frac{\partial x_i}{\partial \bar{P}_k}$$

$$\frac{\partial x_i}{\partial c_{i'}} = \frac{\partial x_{i'}}{\partial c_i}$$

$$\frac{\partial x_i}{\partial c_i} \leq 0.$$

Production and Price Uncertainty

Nearly all that can be said here is

$$\frac{\partial x_i^k}{\partial c_i} \leq 0$$

$$\frac{\partial x_i^k}{\partial c_{i'}} = \frac{\partial x_{i'}^k}{\partial c_i} \quad ,$$

where the superscript indicates the quantity of the input used in the production of good k (that is nonjoint production functions are now assumed).

Though this is a rather criptic presentation of the restrictions, it may be beneficial to raise the question as to the worth of the imposition or testing of the restrictions in particular cases. To the author's knowledge no such test has been conducted under risk (see Court and Woods for the riskless case). As seen above, particular assumptions lead easily to econometrically amenable restrictions. Finally, it should be mentioned that the theory allows one to estimate risk aversion measures. This shall not be discussed here but again the interested reader is referred to Pope (1977).

SUMMARY AND CONCLUSIONS

Though aggregation causes many problems for risk analysis [(Just) and Table 1], it is likely that meaningful results can be obtained for some aggregate problems at the crop, county or state level. Ideally, time series data on microunits would provide the best environment for econometric risk response estimates.

The methodology discussed in this paper indicates that, though not without difficulty, structural econometric production systems prevalent in economic research can be amended to include risk aversion for simple utility models. Explicitly risk aversion may be estimated and its impact on input use.

Finally, econometric estimates of reduced form systems may benefit by imposing or testing structure which is implied by theory--such as the expected utility hypothesis. It is hoped that these procedures will find application in order to test explicitly for the effects of risk aversion on choice.

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TABLE 1

Factor Share Estimates for Risk Neutral and Risk Averse Cases - U.S. Aggregates

Factor	$\hat{a}_i$	$\hat{\tilde{a}}_i$	$\hat{b}_i$	$\hat{\tilde{b}}_i$
Labor (risk)	.543 (13.40)*	-3.05E -08 (-.113)	-.005 (-7.14)	8.88E -10 (.162)
Labor (riskless)**	.553 (20.32)	-.005 (-10.8)		
Operating inputs (risk)	.0204 (.427)	-3.86E -07 (1.11)	.005 (6.42)	-6.85E -09 (-1.06)
Operating inputs (riskless)	.063 (1.92)	.004 (7.92)		
Capital (risk)	-.116 (-2.238)	2.78E -07 (.736)	.006 (7.16)	-5.03E -09 (-.716)
Capital (riskless)	-.089 (-2.55)	.006 (9.74)		

* t-ratios are in parentheses.

** Riskless implies risk neutrality; risk implies risk aversion

FOOTNOTES

1/ This example can easily be extended to the multiproduct case as well, without substantively altering the methodology or conclusions.

2/ That is, the effect of risk aversion is to change the optimal q and hence the optimal input levels. Holding q constant in the cost function alleviates the risk effects. If the profit function were utilized, then revenue effects are present and inconsistent parameter estimates are obtained if risk neutrality is incorrectly assumed.

3/ A simple approach would be to include simple moving averages or insert last period's observation as $E(P_t)$ and the variance about this prediction for $V(P_t) = V(P)$.

4/ The production function is given by $q = e^{a_0} \prod_{i=1}^{N} x_i^{a_i} W$.

5/ This ad hoc approach may perhaps be rationalized by multiplying and dividing $\tilde{a}_i Z_{it}$ by W_t to obtain $(\tilde{a}_i/W_t) P_t q_t V(e_{0t})$. Treating $(\tilde{a}_i/W_t) V(e_{0t})$ as a parameter, it is clear that the parameter is random, but the distribution of estimates would be difficult to obtain. It is likely that the error term would be correlated with the exogenous variable.

6/ Such an argument would be similar in nature to Hoch's but would extend to the share equations as well. This seems to be a prevalent approach in the literature (see Burgess, Appelbaum).

7/ This does not imply that production uncertainty is assumed away, but as noted above, with some handwaving and given the assumption of constant variances of the random error in production and covariance with the price of output, the equations to be estimated are similar. It shall also be assumed that $\ln x_i$ is unrelated to the actual errors, e_0 and e_{it}.

8/ Ordinary least squares are efficient when no cross equation restrictions are present and all explanatory variables are identical. Predicting q from the exogenous variables did not improve the results.

9/ See Pope (1977) for a derivation of some of these results. Also, for the sake of brevity, no inequality restrictions will be presented here except the output supply and input demand results. Further, not all equality restrictions can be given here.

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