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"Simultaneous Optimization of n Input Levels in the Case of the Generalized Mitscherlich-Spillman Crop Response Function"

by

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We want to find an n-input combination (X_1, X_2, $X_3, \ldots, Xn$) for which the following function achieves a maximum:

$$S = P_o\left[M\prod_{i=1}^{n}(1 - R_i^{x_i})\right] - \sum_{i=1}^{n} P_i(X_i - X_i^o)$$

with the following constraints:

$P_i > 0$ ($i = 0, 1, \ldots, n$)

$M > 0$

$0 < R_i < 1 \quad (i = 1, \ldots, n)$

$X_i \geq X_i^o > 0$

where P_i($i = 1, 2, 3, \ldots, n$) are the input prices; P_o is the output price; the X_i are the amounts of nutrients in the soil after fertilizer application (i.e., they include both the quantities naturally occurring in the soil and the quantities added by farmers); the X_i^o ($i = 1, \ldots, n$) are the amounts of nutrients naturally occurring in the soil (so that $X_i - X_i^o$ would be the amounts that the farmer applies and pays for); S is profit; and $M\prod_{i=1}^{n}(1-R_i^{x_i})$ is the generalization to n inputs of the Mitscherlich-Spillman crop response function.

We proceed in the usual fashion by taking the partial derivatives of S w.r.t. the n independent variables and equate them to zero:

$$\frac{\partial S}{\partial X_i} = -P_o M(\ell n R_i)R_i^{x_i}\prod_{j\neq i}(1-R_j^{x_j}) - P_i = 0 \qquad (i = 1, 2, \ldots, n) \qquad (1)$$

Equation (1) could be expressed in terms of variables of the form $(1-R_i^{x_i})$:

$$P_o M(\ell n R_i)\prod_{j=1}^{n}(1-R_j^{x_j}) - P_o M(\ell n R_i)\prod_{j\neq i}(1-R_j^{x_j}) - P_i = 0 \qquad (2)$$

In other words, to the left side of the ith equation we have added and subtracted the term $P_o M(\ell n R_i)\prod_{j\neq i}(1-R_j^{x_j})$.

If we let:

$$C_i = P_o M(\ell n R_i) \tag{3}$$

$$W = \prod_{i=1}^{n} (1-R_i^{x_i}) \tag{4}$$

$$W_i = \prod_{j \neq i} (1-R_j^{x_j}) \tag{5}$$

then we can rewrite (2) as

$$P_i = C_i W - C_i W_i \quad (i = 1, \ldots, n) \tag{6}$$

with the restriction that

$$\prod_{i=1}^{n} W_i = W^{n-1} \tag{7}$$

i.e.,

$$\frac{W}{1-R_1^{x_1}} \cdot \frac{W}{1-R_2^{x_2}} \cdots \frac{W}{1-R_n^{x_n}} = \frac{W^n}{W} = W^{n-1} .$$

Hence, we obtain n+1 equations in n+1 unknowns: $W_1, W_2, \ldots, W_n, W$.

From (6) we get

$$W_i = W - \frac{P_i}{C_i} \qquad (i = 1, 2, \ldots, n) \tag{8}$$

Let $D_i = P_i/C_i$ and substitute into (7):

$$\prod_{i=1}^{n} (W - D_i) = W^{n-1} \tag{9}$$

i.e., the solution of the n equations (1) in n unknowns is equivalent to finding the abscissa of the intersection of the two functions $\prod_{i=1}^{n}(W-D_i)$ and W^{n-1} (equation (9)).

Multiplying out the left side of (9) we get:

$$W^n - (\Sigma D_i)W^{n-1} + (D_1D_2 + D_1D_3 + \cdots + D_{n-1}D_n)W^{n-2} + \cdots + (-1)^j (D_1D_2\cdots D_j + \cdots + D_{n-j+1}\cdots D_n)W^{n-j} + \cdots + (-1)^n \prod_{i=1}^{n} D_i = W^{n-1} .$$

If we let

$$Z_n = W^n - (\Sigma D_i+1)W^{n-1} + (D_1D_2 + \cdots + D_{n-1}D_n)W^{n-2} + \cdots + (-1)^j (D_1\cdots D_j + \cdots + D_{n-j+1}\cdots D_n)W^{n-j} + \cdots + (-1)^n \prod_{i=1}^{n} D_i$$

then we can say that we want to find values of W for which $Z_n = 0$; or which is the same, values such that $\prod_{i=1}^{n}(W-D_i)$ intersects with W^{n-1}.

From the definition of W as $\prod_{i=1}^{n} (1-R_i^{x_i})$ it is obvious that we are interested in the case when Z_n has positive roots (between zero and one).

The following theorem (known as Descartes' Rule of Signs) will be of help to us: "the number of positive real roots of an algebraic equation $Z(W) = 0$ with real coefficients, is either equal to the number of variations in sign of $Z(W)$ or less than that number by a positive even integer."

In our case

$$Z_n = b_n W^n + b_{n-1} W^{n-1} + \cdots + b_1 W + b_0$$

where

$$b_n = 1$$
$$b_{n-1} = -(\sum_{i=1}^{n} D_i + 1)$$
$$b_{n-2} = (D_1D_2 + D_1D_3 + \cdots + D_{n-1}D_n)$$
$$\cdots \qquad \cdots$$

$$b_{n-j} \;=\; (-1)^j\,(D_1D_2\cdots D_j+\cdots+D_{n-j+1}\cdots D_n)$$

$$\cdots \qquad \cdots$$

$$b_0 \;=\; (-1)^n \prod_{i=1}^{n} D_i \;.$$

Since the D_i's are negative (since the $\ln R_i$ are negative for $0 < R_i < 1$), and since in $b_{n-K} = (-1)^K\,(D_1D_2\cdots D_K+\cdots+D_{n-K+1}\cdots D_n)$, $K = 0, 2, 3, \ldots, n$, we have a sum of several terms, each term consisting of 2K negative factors (K D_i's and K (-1)'s), hence all the b_{n-K}'s will be positive with the possible exception of b_{n-1}.

If we assume $b_{n-1} < 0$, $Z_n(W)$ will have two changes of sign (i.e., two or zero positive real roots; the two could be a double root). On the other hand, if $b_{n-1} > 0$, we will have zero positive real roots. So we are interested only in the case when $b_{n-1} < 0$, i.e.,

$$1 > \sum_{i=1}^{n} (-D_i)\,. \tag{10}$$

It is a considerable advance to know that Z_n would at most have two positive real roots.

We can also prove that the positive roots of $Z_n = 0$ will all be less than 1. Indeed, we have that $W - D_i > W$ $(i = 1, \ldots, n)$ since $-D_i > 0$ $(i=1, \ldots, n)$; hence $\prod_{i=1}^{n-1} (W-D_i) > W^{n-1}$, and since for $W \geq 1$ we have that $(W-D_n) > 1$, it follows that $(W-D_n)\left[\prod_{i=1}^{n-1} (W-D_i)\right] > \prod_{i=1}^{n-1} (W-D_i) > W^{n-1}$, i.e., $\prod_{i=1}^{n} (W-D_i) > W^{n-1}$ for $W \geq 1$. Hence, intersection of $\prod_{i=1}^{n} (W-D_i)$ and W^{n-1} can occur only for $W < 1$.

At $W = 0$, we have $W^{n-1} = 0$, while $\prod_{i=1}^{n} (-D_i) > 0$.

Until further notice, we will assume in the remaining discussion that $Z_n = 0$ has two distinct positive real roots. The circumstances under which this assumption is justified will be considered at a later point in this paper. To find the largest positive root of $Z_n = 0$ numerical analysis methods can be applied.

As our discussion of the stability of the solution will show, of the two positive roots of $Z_n = 0$ we are interested only in α_G since it corresponds to the maximum whereas α_{G-1} corresponds to a saddle point of S.

Having found the $W^*(=\alpha_G)$ value that is the largest root of $Z_n = 0$, we solve for the set of X_i^*'s that maximizes S: for this purpose we use equations

$$W_i^* = W^* - P_i/C_i \qquad (i = 1,\dots,n)$$

to solve for the W_i^*'s.

Next, we solve for

$$1 - R_i^{X_i^*} = W^*/W_i^* \qquad (i = 1,\dots,n) \tag{11}$$

where from:

$$X_i^* = \frac{\ln\left[1 - (W^*/W_i^*)\right]}{\ln R_i} \qquad (i = 1,\dots,n).$$

In the following discussion we will consider the stability of the solution and its relation to the sign of $Z_n(W)$.

We have

$$\frac{\partial S}{\partial X_i} = P_0\, M(\ln R_i)\prod_{j=1}^{n}(1-R_j^{X_j}) - P_0 M(\ln R_i)\prod_{j\neq i}(1-R_j^{X_j}) - P_i \tag{12}$$

i.e., $$\frac{\partial S}{\partial X_i} = C_i\, W - C_i\, W_i - P_i \tag{13}$$

(from (3), (4), and (5)).

Obviously, (12) can be written as $\frac{\partial S}{\partial X_i} = MR_i - MC_i$, (i.e., as marginal profit = marginal revenue-marginal cost)

$$MR_i = C_i\, W - C_i\, W_i\,, \qquad MC_i = P_i$$

Wherefrom if $MR_i > MC_i$ we get from (13):

$$C_i W - C_i W_i - P_i > 0, \text{ or}$$

$$W - W_i < P_i / C_i \qquad (C_i < 0, \text{ since } \ln R_i < 0)$$

or

$$W - W_i < D_i \qquad (\text{definition of } D_i)$$

or

$$W - D_i < W_i \tag{14}$$

(14) and $MR_i > MC_i$ being equivalent.

On the other hand:

$$Z_n = \prod_{i=1}^{n} (W - D_i) - W^{n-1} \qquad (\text{definition of } Z_n)$$

and if $Z_n < 0$, then

$$\prod_{i=1}^{n} (W - D_i) < W^{n-1}, \text{ or}$$

$$\prod_{i=1}^{n} (W - D_i) < \prod_{i=1}^{n} W_i \qquad (\text{from (7)}) . \tag{15}$$

W is obtained by a monotonic transformation of the generalized Mitscherlich-Spillman production function $M \prod_{i=1}^{n} (1 - R_i^{X_i})$; hence a fixed value of W implies that we are on an isoquant. If we require that besides being on a particular isoquant that we also should be on the expansion path, then we should have the additional condition $MR_i = \lambda M C_i$ $(i = 1, \ldots, n)$, where λ is a Lagrange multiplier.

Hence on the expansion path we have three possibilities: a) $MR_i > MC_i$, all i; b) $MR_i = MC_i$, all i ; c) $MR_i < MC_i$, all i . Correspondingly, we can have three possible situations on the expansion path:

$W - D_i < W_i$; $W - D_i = W_i$; and $W - D_i > W_i$, for all i. Hence, we conclude that if for a particular $W = K$ we have $Z_n < 0$, that is (15) holds, then $MR_i > MC_i$ $(i = 1, \ldots, n)$ at the intersection of the isoquant MW = MK and the expansion path. That is, whenever $Z_n < 0$ it will pay to increase all inputs along the expansion path (the profit gradient $\underline{g}$ being positive at the intersection of expansion path and the isoquant for $W = K$).

Analogous reasoning will lead us to conclude that whenever $Z_n > 0$, then $\underline{g} < 0$ on the EP, and that $\underline{g} = 0$ on the EP if $Z_n = 0$. So that if $Z_n > 0$ we should reduce the use of all n inputs along the EP.

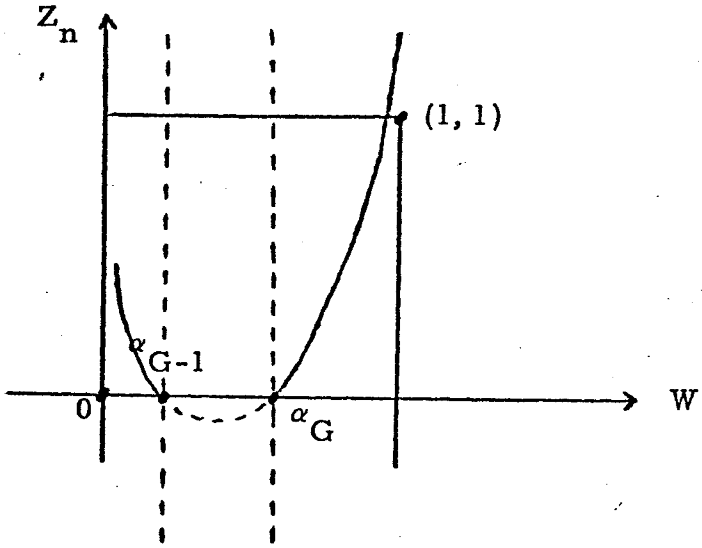

FIG. 1

The above discussion together with Fig. 1 makes it clear that while $W = \alpha_G$ corresponds to a stable equilibrium, $W = \alpha_{G-1}$ corresponds to a saddle point, hence we are interested in determining only α_G, the largest real root of $Z_n = 0$.

Observe also that in the case of a positive double root for $Z_n = 0$, according to the present analysis the solution will be unstable, since we will have an inflection point.

In the above discussion we have assumed the existence of the expansion path. We now proceed to show the existence of the EP without obtaining an explicit equation for it.

The following constrained minimization problem gives us points on the expansion path:

$$\text{minimize} \quad \sum_{i=1}^{n} P_i(x_i - X_i^0)$$

$$\text{subject to} \quad W^{**} = \prod_{i=1}^{n} (1 - R_1^{X_i}),$$

i.e., minimize costs subject to the fact that yield equals MW^{**}.
This is equivalent to minimizing the Lagrangian function:

$$L = \sum_{i=1}^{n} P_i(X_i - X_i^0) + \lambda\left[W^{**} - \prod_{i=1}^{n} (1 - R_i^{X_i})\right],$$

where λ is the Lagrange multiplier. The necessary conditions for the minimum of L are:

$$\frac{\partial L}{\partial X_i} = P_i + \lambda \prod_{j \neq i} (1 - R_j^{X_j})(-\ell n R_i) R_i^{X_i} = 0 \qquad (16)$$

$$(i = 1, \ldots, n)$$

and

$$W^{**} - \prod_{i=1}^{n} (1 - R_i^{X_i}) = 0.$$

If we define $W_i^{**} = \dfrac{W^{**}}{(1-R_i^{X_i})}$, then (16) can be written as

$$\frac{\partial L}{\partial X_i} = P_i^{*} - \lambda\left[W^{**} - W_i^{**}\right] \ell n\, R_i = 0 \qquad (i = 1, \ldots, n)$$

wherefrom

$$\frac{W_K^{**} - W^{**}}{W_i^{**} - W^{**}} = \frac{P_K^*}{P_i^*}, \text{ where } P_K^* = P_K / |\ell n R_K|,\ P_i^* = P_i / |\ell n R_i|$$

Let $Y_i = W_i^{**} - W^{**}$, so that $W_i^{**} = Y_i + W^{**}$, then

$$\frac{Y_K}{P_K^*} = \frac{Y_i}{P_i^*}, \quad \text{so that}$$

$$Y_i = P_i^*(Y_K/P_K^*) \qquad (i = 1, \ldots, n)$$

and then from the definitions of W_i^{**} we get

$$(W^{**})^{n-1} = \prod_{i=1}^{n} W_i^{**}, \text{ or}$$

$$(W^{**})^{n-1} = \prod_{i=1}^{n} (Y_i + W^{**})$$

$$(W^{**})^{n-1} = \prod_{i=1}^{n} \left[P_i^*(Y_K/P_K^*) + W^{**}\right]$$

$$(P_K^*)^n (W^{**})^{n-1} = \prod_{i=1}^{n} (P_i^* Y_K + P_K^* W^{**})$$

$$(P_K^*)^n (W^{**})^{n-1} = \left(\prod_{i=1}^{n} P_i^*\right) Y_K^n + \left[(P_K^* W^{**}) \sum_{i_1 > i_2 \cdots > i_{n-1}} P_{i_1}^* P_{i_2}^* \cdots P_{i_{n-1}}^*\right] Y_K^{n-1}$$

$$\cdots + \left[(P_K^* W^{**})^{n-m} \left(\sum_{i_1 > i_2 \cdots > i_m} P_{i_1}^* P_{i_2}^* \cdots P_{i_m}^*\right)\right] Y_K^m + \cdots$$

$$\cdots + (P_K^* W^{**})^n ,$$

or

$$\left(\prod_{i=1}^{n} P_i^*\right)Y_K^n + \cdots + \left[(P_K^* W^{**})^{n-m}\left(\sum_{i_1>i_2\cdots>i_m} P_{i_1}^* P_{i_2}^* \cdots P_{i_m}^*\right)\right]Y_K^m + \cdots$$

$$\cdots + P_K^{*n}\left[(W^{**})^n - (W^{**})^{n-1}\right] = 0$$

obtaining an nth degree polynomial in Y_K with all coefficients positive except for $P_K^{*n}\left[(W^{**})^n - (W^{**})^{n-1}\right]$ which is negative, since $0 < W^{**} < 1$ and $P_i^* > 0$ $(i = 1, \ldots, n)$.

So that by the Descartes' Rule of Signs (17) will have one root, proving the existence of EP (we can let $K = 1, \ldots, n$; and knowing Y_K and W^{**} we can obtain W_K^{**}, $R_K^{X_K}$, and finally X_K).

We study next the necessary second order conditions for a maximizing S (in combination with the first order conditions the second order conditions are sufficient for a maximum).

The necessary second order conditions for a maximum of S require that the following expressions be positive:

$$(-1)^H \begin{vmatrix} S_{11} & S_{12} \cdots & S_{1H} \\ S_{21} & S_{22} \cdots & S_{2H} \\ --- & --- & --- \\ S_{H1} & S_{H2} & S_{HH} \end{vmatrix} > 0 \qquad (H = 1, \ldots, n) \tag{18}$$

where

$$S = P_0 M \prod_{i=1}^{n} (1 - R_i^{X_i}) - \sum_{i=1}^{n} P_i (X_i - X_i^0), \text{ and}$$

$$S_{ij} = \frac{\partial^2 S}{\partial X_i \, \partial X_j}$$

Let us denote, for the sake of convenience:

$$K = P_0 M$$
$$L_i = R_i^{X_i}$$
$$T_i = 1 - L_i$$
$$-a_i = \ln R_i, \qquad a_i > 0$$

then (18) can be rewritten

$$(-1)^H \begin{vmatrix} -a_1^2 K L_1 \prod_{i\neq 1} T_i, & a_1 a_2 K L_1 L_2 \prod_{i\neq 1,2} T_i, & \cdots & a_1 a_H K L_1 L_H \prod_{i\neq 1,H} T_i \\ a_1 a_2 K L_1 L_2 \prod_{i\neq 1,2} T_i, & -a_2^2 K L_2 \prod_{i\neq 2} T_i, & \cdots & a_2 a_H K L_1 L_H \prod_{i\neq 1,H} T_i \\ \cdots & \cdots & & \cdots \\ a_1 a_H K L_1 L_H \prod_{i\neq 1,H} T_i, & a_2 a_H K L_2 L_H \prod_{i\neq 2,H} T_i, & - & a_1^2 K L_H \prod_{i\neq H} T_i \end{vmatrix} > 0$$

(H=1, . . . , n)

Multiplying ith row by T_i and jth column by T_j:

$$\frac{(-1)^H}{(\prod T_i)^2} \begin{vmatrix} -a_1^2 K L_1 T_1 \prod T_i, & a_1 a_2 K L_1 L_2 \prod T_i, & \cdots & a_1 a_H K L_1 L_H \prod T_i \\ a_1 a_2 K L_1 L_2 \prod T_i, & -a_2^2 K L_2 T_2 \prod T_i, & \cdots & a_1 a_H K L_2 L_H \prod T_i \\ \cdots & \cdots & & \cdots \\ a_1 a_H K L_1 L_H \prod T_i, & a_2 a_H K L_2 L_H \prod T_i, & \cdots & a_H^2 K L_H T_H \prod T_i \end{vmatrix} > 0$$

(H = 1, . . . , n)

Dividing ith row by $(a_i L_i)$ and jth column by $a_j K(\prod T_i)$:

$$\frac{(-1)^H}{(\prod T_i)^2} (\prod a_i)^2 (K \prod T_i)^H (\prod L_i) \begin{vmatrix} -T_i & L_2 & \cdots & L_H \\ L_1 & -T_2 & \cdots & L_H \\ \cdots & \cdots & & \cdots \\ L_1 & L_2 & \cdots & -T_H \end{vmatrix} > 0$$

(H = 1, . . . , n)

Adding all other columns to the first column:

$$(\prod a_i)^2(-K)^H(\prod T_i)^{H-2}(\prod L_i)\begin{vmatrix} -1+\sum L_i, & L_2, & \cdots & L_H \\ -1+\sum L_i, & -1+L_2, & \cdots & L_H \\ -1+\sum L_i, & L_2, & \cdots & -1+L_H \end{vmatrix} > 0$$

$$(H = 1, \ldots, n)$$

Subtracting the first row from all other rows:

$$(\prod a_i)^2(-K)^H(\prod T_i)^{H-2}(\prod L_i)\begin{vmatrix} -1+\sum L_i, & L_2 \cdots & L_H \\ 0 & -1 \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & -1 \end{vmatrix} > 0$$

$$(H = 1, \ldots, n)$$

But a determinant of a trangular matrix equals the product of its diagonal elements:

$$(\prod a_i)^2 (-K)^H (\prod T_i)^{H-2} (\prod L_i)(1-\sum L_i)(-1)^H > 0$$

$$(H = 1, \ldots, n)$$

or

$$(\prod a_i)^2 \cdot (K)^H (\prod T_i)^{H-2} (\prod L_i)(1-\sum L_i)(-1)^{2H} > 0$$

$$(H = 1, \ldots, n)$$

or

$$1 - \sum_{i=1}^{H} L_i > 0 \qquad (H = 1, \ldots, n)$$

or

$$1 > \sum_{i=1}^{H} R_i^{X_i} \qquad (H = 1, \ldots, n) \tag{19}$$

Obviously, condition (19) is (since $R_i^{X_i} \geq 0$, all i) equivalent to

$$1 > \sum_{i=1}^{n} R_i^{X_i} \ . \qquad (20)$$

Observe that if the first order conditions for a maximum are satisfied, then from (1) we obtain

$$R_i^{X_i} = -P_i/\left[P_0 M(\ell n R_i) \prod_{j \neq i} (1 - R_j^{X_j}), \text{ all } i.\right.$$

i.e.,

$$R_i^{X_i} = -D_i / \prod_{j \neq i} (1 - R_j^{X_j}), \text{ all } i \qquad (21)$$

Wherefrom

$$R_i^{X_i} > -D_i, \text{ all } i$$

So that $1 > \sum_{i=1}^{n} R_i^{X_i'} \Rightarrow 1 > \sum_{i=1}^{n} (-D_i) \Rightarrow b_{n-1} < 0 \Rightarrow Z_n$ has two changes of sign. However, (10) and (21) do not together imply the stronger condition (20). Let $R_i^{X_i} = L_i$, then $1 > \sum_{i=1}^{n} R_i^{X_i}$ becomes $1 > \sum_{i=1}^{n} L_i$, which is equivalent to $-1 < \sum_{i=1}^{n} -L_i$, and in turn to $n-1 < \sum_{i=1}^{n} (1-L_i)$, or

$$n - 1 < \sum_{i=1}^{n} T_i \qquad (T_i = 1 - L_i, \text{ all } i) \ . \qquad (22)$$

Then if we write the generalized Mitscherlich-Spillman production function as

$$Y = M \prod_{i=1}^{n} T_i$$

we will have the elasticity of production with respect to the vector of the Ti's given by

$$\epsilon(\underline{T}) = \sum_{i=1}^{n} \left(\frac{\partial Y}{\partial T_i}\right) \frac{T_i}{Y} \ , \text{ i.e.,}$$

$$\epsilon(\underline{T}) = \sum_{i=1}^{n} Y/Y = n$$

wherefrom condition (20), equivalent to (22), can be stated as

$$\varepsilon(\underline{T}) - 1 < \sum_{i=1}^{n} T_i .$$

We proceed to show an alternative derivation of the second order conditions.

Define

$$F(W) = \int Z_n(W)\,dW ,$$

so that $Z_n(W) = F'(W)$.

From Fig. 1 we see that $F'(\alpha_G) = 0$, and $F''(\alpha_G) = Z'_n(\alpha_G) > 0$; hence at α_G we minimize $F(W)$.

The sufficient condition for $F(W)$ being minimum at α_G is

$$\frac{\partial^2 F(\alpha_G)}{\partial W^2} > 0, \text{ or}$$

$$Z'_n > 0, \text{ or}$$

$$\frac{\partial}{\partial W}\left[\prod_{i=1}^{n}(W-D_i) - W^{n-1}\right]_{W=\alpha_G} > 0, \text{ or}$$

$$\left\{\left[\sum_{i=1}^{n} \prod_{j\neq i}(W-D_j)\right] - (n-1)W^{n-2}\right\}_{W=\alpha_G} > 0 ,$$

but at $W = \alpha_G$ we have $W-D_i = W_i$ $(i = 1, \ldots, n)$, from first order conditions;

$$\left[\sum_{i=1}^{n} \prod_{j\neq i} W_j\right] - (n-1)W^{n-2} > 0$$

$$\left[\sum_{i=1}^{n} \prod_{j\neq i} \frac{W}{1-R_j}X_j\right]$$

$$\left[\sum_{i=1}^{n} W^{n-1} \prod_{j\neq i} \frac{1}{1-R_j}X_j\right] - (n-1)W^{n-2} > 0$$

$$\left[\sum_{i=1}^{n} W^{n-1} \cdot \frac{1}{W_i}\right] - (n-1)W^{n-2} > 0$$

$$\left[\sum_{i=1}^{n} W \cdot \frac{1}{W_i}\right] - (n-1) > 0$$

$$\sum_{i=1}^{n} (1-R_i^{X_i}) - (n-1) > 0$$

$$n - \sum_{i=1}^{n} R_i^{X_i} - (n-1) > 0$$

$$- \sum_{i=1}^{n} R_i^{X_i} + 1 > 0$$

$$1 > \sum_{i=1}^{n} R_i^{X_i} . \quad \text{Q.E.D.}$$

Next, we proceed to show that the elasticity of substitution is less than one.

By definition

$$\sigma_{ij} = - \frac{d\left[\ln(X_i/X_j)\right]}{d\left[\ln(MP_i/MP_j)\right]}$$

where

$$MP_i = f_i = \frac{\partial f}{\partial X_i} = \frac{\partial}{\partial X_i}[MW]$$

$$= \frac{\partial}{\partial X_i}\left[M \prod_{K=1}^{n} (1 - R_K^{X_K})\right]$$

$$= M(-\ln R_i)R_i^{X_i} \prod_{K \neq i} (1 - R_K^{X_K})$$

$$= M(\ln R_i)\left[W - \prod_{K \neq i} (1 - R_K^{X_K})\right]$$

$$= M(\ln R_i)\, W\left(1 - \frac{1}{1-R_i^{X_i}}\right)$$

$$= M(\ln R_i)\, W \left(\frac{-R_i^{X_i}}{1-R_i^{X_i}} \right)$$

$$= M\, a_i\, W\, (L_i/T_i)$$

(Since $a_i = -\ln R_i$; $L_i = R_i^{X_i}$; $T_i = 1 - L_i$),

wherefrom:

$$\sigma_{ij} = - \frac{d\left[\ln (X_i/X_j)\right]}{d\left[\ln\left(\frac{a_i L_i T_j}{a_j L_j T_i}\right)\right]} \quad , \quad \text{or}$$

$$\sigma_{ij} = - \frac{\left(\frac{X_j}{X_i}\right) \frac{X_j\, d\, X_i - X_i\, d\, X_j}{X_j^{\,2}}}{\frac{a_j L_j T_i}{a_i L_i T_j} \cdot \frac{a_j L_j T_i\, d(a_i L_i T_j) - a_i L_i T_j\, d(a_j L_j T_i)}{a_j^2\, L_j^2\, T_i^2}}$$

But since for $i \neq j$

$$\frac{\partial}{\partial X_i} L_i = - a_i L_i , \quad \frac{\partial}{\partial X_j} L_i = 0$$

$$\frac{\partial}{\partial X_i} T_i = a_i L_i , \quad \frac{\partial}{\partial X_j} T_i = 0$$

It follows that

$$a_j\, L_j\, T_i\, d(a_i L_i T_j) - a_i\, L_i\, T_j\, d(a_j L_j T_i)$$

$$= a_j L_j T_i\, (-a_i^2\, L_i\, T_j\, d\, X_i + a_i\, a_j\, L_i\, L_j\, d\, X_j) -$$

$$- a_i L_i T_j (-a_j^2\, L_j\, T_i\, d\, X_j + a_i\, a_j\, L_i\, L_j\, d\, X_i)$$

$$= - a_i^2\, a_j\, L_i\, L_j\, T_j\, (T_i + L_i) d\, X_i \; +$$

$$+ a_i\, a_j^2\, L_i\, L_j\, T_j\, (T_j + L_j) d\, X_j$$

$$= -a_i^2 a_j L_i L_j T_j \, d X_i + a_i a_j^2 L_i L_j T_i \, d X_j$$

since $T_i + L_i = T_j + L_j = 1$.

Wherefrom,

$$\sigma_{ij} = - \frac{\left(\frac{X_j}{X_i}\right) \cdot \frac{X_j \, dX_i - X_i \, dX_j}{X_j^2}}{\frac{a_j L_j T_i}{a_i L_i T_j} \cdot \frac{(-a_i^2 a_j L_i L_j T_j dX_i + a_i a_j^2 L_i L_j T_i dX_j)}{a_j^2 L_j^2 T_i^2}}$$

$$\sigma_{ij} = \frac{-\frac{dX_i}{X_i} + \frac{dX_j}{X_j}}{-\frac{a_i dX_i}{T_i} + \frac{a_j dX_j}{T_j}}$$

$$\sigma_{ij} = \frac{-\frac{1}{X_i} + \left(\frac{1}{X_j}\right) \frac{dX_j}{dX_i}}{\frac{-a_i}{T_i} + \frac{a_j}{T_j} \cdot \frac{dX_j}{dX_i}} \quad .$$

But along an isoquant we have

$$MP_i \, dX_i + MP_j \, dX_j = 0 \, , \text{ i.e.,}$$

$$M a_i \, W\left(\frac{L_i}{T_i}\right) d X_i + M a_j \, W\left(\frac{L_j}{T_j}\right) dX_j = 0$$

Wherefrom:

$$\frac{dX_j}{dX_i} = - \frac{a_i \, L_i \, T_j}{a_j \, L_j \, T_i} \, ,$$

so that:

$$\sigma_{ij} = \frac{-\frac{1}{X_i} - \left(\frac{1}{X_j}\right) \frac{a_i L_i T_j}{a_j L_j T_i}}{-\frac{a_i}{T_j} - \frac{a_j}{T} \cdot \frac{a_i L_i T_j}{a_j L_j T_i}}$$

divide numerator and denominator by $- a_i L_i / T_i$:

$$\sigma_{ij} = \frac{\dfrac{T_i}{a_i X_i L_i} + \dfrac{T_j}{a_j X_j L_i}}{\dfrac{1}{L_i} + \dfrac{1}{L_j}} ,$$

$$\sigma_{ij} = \frac{\dfrac{1 - L_i}{a_i X_i L_i} + \dfrac{1 - L_j}{a_j X_j L_j}}{\dfrac{1}{L_i} + \dfrac{1}{L_j}} ,$$

$$\sigma_{ij} = \frac{\dfrac{1 - e^{-a_i X_i}}{(a_i X_i) e^{-a_i X_i}} + \dfrac{1 - e^{-a_j X_j}}{(a_j X_j) e^{-a_j X_j}}}{e^{a_i X_i} + e^{a_j X_j}} ,$$

let $Y_K = a_K X_K$, then

$$\sigma_{ij} = \frac{\dfrac{1 - e^{-Y_i}}{Y_i\, e^{-Y_i}} + \dfrac{1 - e^{-Y_d}}{Y_j\, e^{-Y_j}}}{e^{Y_i} + e^{Y_j}} ,$$

$$\sigma_{ij} = \frac{\dfrac{e^{Y_i} - 1}{Y_i} + \dfrac{e^{Y_j} - 1}{Y_j}}{e^{Y_i} + e^{Y_j}} .$$

However, $\dfrac{e^u - 1}{u} < e^u$, for $u > 0$,

since $$\frac{e^u - 1}{u} = \frac{\left(\sum_{K=0}^{\infty} \dfrac{u^K}{K!}\right) - 1}{u} = \frac{\sum_{K=1}^{\infty} \dfrac{u^K}{K!}}{u} =$$

$$= \sum_{K=1}^{\infty} \frac{u^{K-1}}{K!} = \sum_{K=0}^{\infty} \frac{u^K}{(K+1)!} < \sum_{K=0}^{\infty} \frac{u^K}{K!} = e^u .$$

In turn, $\frac{e^u - 1}{u} < e^u$ for $u > 0$, implies that $0 < \sigma_{ij} < 1$ for $X_i, X_j > 0$.

We suggest the Newton's method for finding roots of equations, to find the largest positive root of $Z_n = 0$. In this method, W_n, the nth approximation to a root of Z(W) is given by:

$$W_n = W_{n-1} - \frac{Z(W_{n-1})}{Z'(W_{n-1})}, \qquad n = 2, 3, \ldots$$

It can be proved (see Appendix B) that if a function $Z_n(W)$ and its second derivative have the same sign on [a, b] and ξ is the only root of $Z_n(W) = 0$ on [a, b], then the Newton method converges to ξ if our first approximation to ξ is in [a, b]. We are going to show that if $Z_n = 0$ has two roots on (0, 1) then $Z_n(W)$ and $Z_n''(W)$ are both positive for $W > \alpha_G$ where α_G is the largest real root of $Z_n(W) = 0$. Indeed, the nth degree polynomial

$$Z_n = b_n W^n + b_{n-1} W^{n-1} + \cdots + b_1 W + b_0$$

can be expressed

$$Z_n = u_1(W) \cdots u_K(W)\,(W - \alpha_1) \cdots (W - \alpha_G) \tag{23}$$

where:

$n = 2K + G$

$u_i(W) > 0 \qquad (i, \ldots, K)$

K = number of conjugate pairs of complex roots of $Z_n = 0$

G = number of real roots of $Z_n = 0$

α_j = jth real root of $Z_n = 0$ listed in ascending order of magnitude.

From (23) it is obvious that $Z_n > 0$ for $W > \alpha_G$.

A theorem of Rolle states: "Between two consecutive (real) roots α_j and α_{j+1} of a polynomial Z(W) there is at least one and at any rate an odd number of roots of its derivative $Z'(W)$."

If $n \geq 3$, then if Z_n has two changes of sign, so will Z_n'; so that $Z_n' = 0$ has either zero or two positive real roots. By Rolle's theorem if $Z_n = 0$ has two positive real roots α_G and α_{G-1} then $Z_n' = 0$ has one root on (α_{G-1}, α_G), and we conclude that $Z_n' = 0$ must have two real positive roots (β_G and β_{G-1}). The other positive root of $Z_n' = 0$ is between on $(0, \alpha_{G-1})$. Indeed, if $n \geq 3$, then $Z_n' > 0$ at $W = 0$. Since $Z_n = 0$ at α_{G-1} we conclude that $Z_n' = 0$ somewhere on $(0, \alpha_{G-1})$.

Analogous reasoning will show that if Z_n has two changes of sign and $n \geq 4$, then Z_n'' will have two real roots γ_{G-1} and γ_G, with $0 < \gamma_{G-1} < \beta_{G-1} < \gamma_G < \beta_G$. So that $Z_n''(W) > 0$ for $W > \alpha_G$, $(\alpha_G > \beta_G > \gamma_G)$. Hence, since $\alpha_G < 1$ and since $(Z_n/Z_n'') > 0$ for $W > \gamma_G$, we conclude that $W = 1$ is a convenient approximation to α_G when using the Newton method given that Z_n has two real roots.

The Newton method can also tell us if $Z_n = 0$ has real positive roots: if when applying the Newton method we start with $W_0 = 1$ and after a few iterations: either, a) the jth approximation W_j is negative, or, b) $W_{j+1} > W_j$, or, c) $Z_n' = 0$ at $W = W_j$ while $Z_n(W_j) \neq 0$; then we can conclude that $Z_n = 0$ has no real roots.

APPENDIX A

The Special Case of Three Inputs

We can be more definite about the existence of real positive roots of $Z_n = 0$ when $n = 3$ ($n = 1, 2$ being trivial cases).

The discriminant of a cubic equation b, $W^3 + b_2 W^2 + b_1 W^1 + b_0 = 0$ is given by

$$\Delta \equiv 18\, b_2\, b_1\, b_0 - 4\, b_2^3\, b_0 + b_2^2\, b_1^2 - 4\, b_1^3 - 27\, b_0^2.$$

And the following important relations hold true:

if $\Delta < 0$, one root is real and two are complex;

if $\Delta = 0$, all roots are real and two are equal;

if $\Delta > 0$, the three roots are real and unequal.

Hence, after having determined that $b_2 = \left(\sum_{i=1}^{3} D_i + 1\right)$ is negative (so that we have either zero or two real positive roots for $Z_3 = 0$, we check for the sign of the discrimininant, and if $\Delta > 0$ we are assured that our problem has a solution, since two of the three real roots will be positive and unequal.

APPENDIX B

Convergence of the Newton Method

In the Newton method the (n+1)th approximation to a root is given by

$$X_{n+1} = \ominus(X_n), \text{ where } \ominus(X) = X - \frac{f(X)}{f'(X)} .$$

Theorem: If f(X) is continuous on [a, b], $f'(X) \neq 0$ on [a, b], $f(a) < 0$, $f(b) > 0$, $f(X) f''(X) > 0$ for $a \le X \le b$, and ξ is the only root of $f(X) = 0$ on [a, b], then the Newton method converges to ξ (assuming X_0 is in $(\xi, b]$.)

Proof: First note that $f(X) f''(X) \Rightarrow \ominus'(X) =$

$$= 1 - \frac{[f'(X)]^2 - f(X) f''(X)}{[f'(X)]^2} = \frac{f(X) f''(X)}{[f'(X)]^2} > 0 ,$$

i.e.,

$$\ominus'(X) > 0 \qquad \text{on} \quad [a, b] \tag{B-1}$$

Next, note that

$$\ominus(X) < X \qquad \text{for} \qquad \xi < X \ge b , \tag{B-2}$$

for if there is some X_1 such that $\xi > X \ge b$ and $X_1 \le \ominus(X_1)$, then

$$\ominus(X_1) = X_1 - \frac{f(X_1)}{f'(X_1)} \quad \Rightarrow \quad - \frac{f(X_1)}{f'(X_1)} \ge 0$$

(since $X_1 \le \ominus(X_1)$), wherefrom we have $f(X_1) \le 0$, since $f'(X_1 > 0$ (because $f(a) < 0$, $f(b) > 0$, and $f'(X) \neq 0$). But $f(X_1) \le 0$ is a contradiction since we have assumed $f(a) < 0$, $f(b) > 0$, and only one root of $f(X) = 0$ on [a, b].

We proceed to show that if $X_0 > \xi$, then $\xi < X_1 < X_0$.

Indeed,

$$X_0 > \xi \;\Rightarrow\; \Theta(X_0) > \Theta(\xi) \qquad \text{(by (B-1))}$$

$$\Rightarrow\; X_1 > \xi \;\text{ since } \Theta(X_0) = X_1, \text{ and } \Theta(\xi) = \xi\,,$$

$$\text{since } f(\xi) = 0\,.$$

Also,

$$X_0 > \xi \;\Rightarrow\; X_0 > \Theta(X_0) \quad \text{by} \quad \text{(B-2)}$$

$$\Rightarrow\; X_0 > X_1\,.$$

By induction, we have $X_0 > X_1 > \cdots > X_n > \xi$, if $X_0 > \xi$. Hence we have a bounded monotone sequence. Therefore a limit exists, i.e., $\lim\limits_{n\to\infty} X_n = \ell$.

We proceed to show that $\ell = \xi$:

Since $X_{n+1} = X_n - \dfrac{f(X_n)}{f'(X_n)}$, hence

$$\lim_{n\to\infty} X_{n+1} = \lim_{n\to\infty} X_n - \frac{f(\lim\limits_{n\to\infty} X_n)}{f'(\lim\limits_{n\to\infty} X_n)}$$

$$\ell = \ell - \frac{f(\ell)}{f'(\ell)}, \quad \text{where from}$$

$f(\ell) = 0$, and since ξ is the only root on $[a, b]$, we have $\ell = \xi$.

Hence the sequence $X_0, X_1, \ldots, X_n$ converges to ξ.

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