The Numbers and Capacity Effect of a Profitable Industry Under Competitive Conditions

by

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The theoretical number of firms that will exist in a given industry under the assumption of perfect competition may or may not be determinate, depending upon what further assumptions are made regarding long-run average costs. 1/

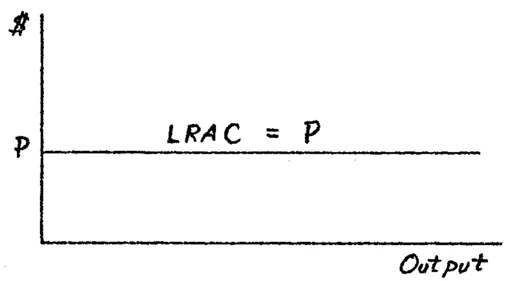

Figure 1.

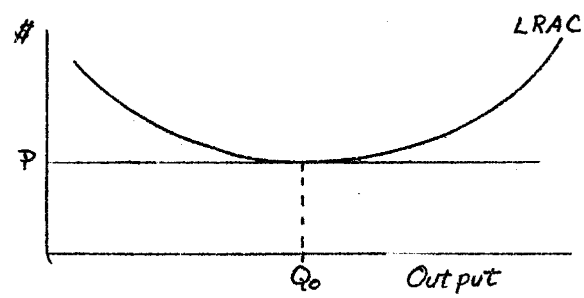

Figure 2.

For Figure 1, all that can be said is that there must be a very large number of producing firms to satisfy the perfect competition assumption. However, if in Figure 2, the industry minimizes long-run average costs, the optimum output is Q_o. Then if the output of the optimum size firm is known, say q_o, the optimum number of firms for the industry is

$$n_o = \frac{Q_o}{q_o}, \text{ so that } n = f(Q).$$

But suppose the assumption of perfect ease of entry is relaxed so that the number of firms will depend not only upon industry output but also upon the pool of capital available to the industry. Then for the analysis that follows, $n = f(K, Q)$.

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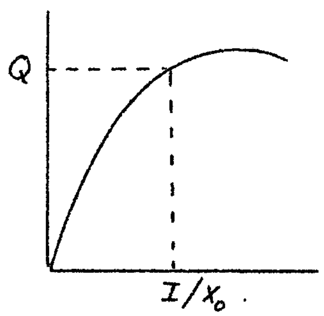
Let a definite period of time be specified during which capital is committed as a factor of production and let the actual amount invested, I, be less than or equal to the available pool of capital, K. If $I < K$, then K-I will be invested outside the industry at the market rate, r. I is related to Q through the industry's production function, (see Figure 3.)



Industry production function

X_o = all other factors held constant.

Figure 3.

If π is equal to the return to capital per unit of industry output, then the total return on K is equal to

$$R = (K-I)r + Q\pi \qquad \text{(eqn 1)}$$

or

$$K = (I + \frac{R}{r}) - (\frac{\pi}{r})\,Q \qquad \text{(eqn 2)}$$

and may be identified as a "return line".*

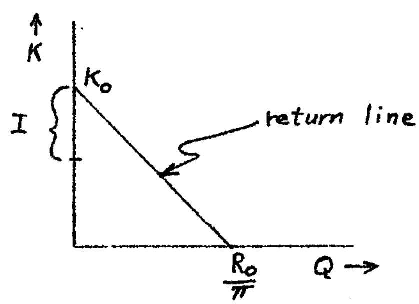


Figure 4.

If $Q = 0$, then $I = 0$ and $K = \frac{R}{r} = K_o$ for specified R_o and r_o. If $I = K$, then $Q = \frac{R_o}{\pi}$, or

$$\pi = \frac{R_o}{Q} \qquad \text{(eqn 3)}$$

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* π, in a sense may be thought of as profit, although strictly speaking, long run "profits" just cover factor costs.

Thus, if the industry restricts output, π rises or if output is increased, π decreases.

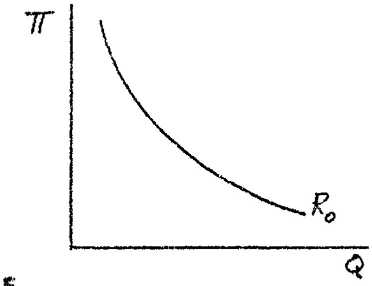

Figure 5.

Such an analysis lacks completeness and the actual phenomena can better be described in terms of a "numbers effect" and a "capacity effect" in much the same way as the effects of a price change for the consumer can be described in terms of an income effect and a substitution effect. 2/

From equation 2, the slope of the return line is $-\frac{\pi}{r}$. $\frac{\pi}{r} \geq 1$, with equality holding only if the premiums above the bare cost of money (e.g. risk) are equal. Any point on or below the return line is a potential combination with respect to output vs. invested capital. But for a given number of firms, n, there will be a preferred combination, P, provided that $\pi \geq r$ so that all points on or below the return line will be excluded, except P. If, in Figure 6, it is assumed that the industry would produce

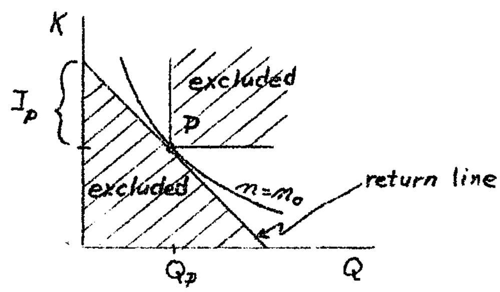

Figure 6.

Q_p with less investment preferred to more ($I \leq I_p$) together with the companion assumption that for the given level of investment I_p, more output is preferred to less ($Q \geq Q_p$), then an additional region above the line is excluded. 3/ Allowing the above, a smooth curve can be drawn through P, representing the actual combinations of capital and output for the given number of firms.* Thus, $n_o = f(K,Q)$ is an "isofirm line".

For n_o = given number of firms, the total differential is equal to zero:

$$dn_o = f_1 dK + f_2\, dQ = 0 \text{ or}$$

$$\frac{dQ}{dK} = \frac{f_1}{f_2}. \qquad \text{(eqn 4)}$$

Movement along the isofirm line represents output expansion or contraction, i.e., a capacity change.

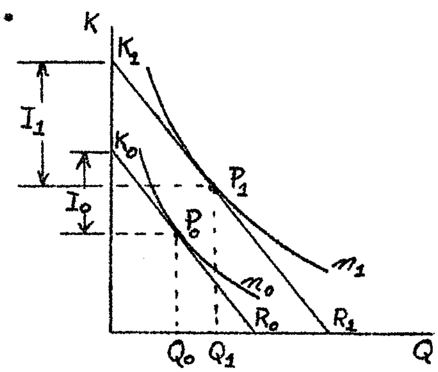

Figure 7.

From equation 1, $R = (K-I)r + Q\pi$. In Figure 7, π, r are constants. The other relationships to be noted are:

$$(K_1 - I_1) > (K_0 - I_0)$$

$$Q_1 > Q_0$$

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*It is convenient to assume that for each firm $q_1 = q_2 = \ldots\ldots = q_n$, at least to good approximation.

$R_1 = (K_1-I_1)r + Q_1 \pi$

$R_o = (K_o-I_o)r + Q_o \pi$

Observing the inequalities, it follows that $R_1 > R_o$, and since the capital requirement and output level at P_1 is greater than at P_o, conclusively then $n_1 > n_o$. As one proceeds outward from the origin, lines representing a constant number of firms are crossed having the ordering $n_o < n_1 < n_2$, etc.

From the standpoint of a return constraint, what is the maximum number of firms that would comprise the industry? The question has been answered graphically in Figure 6 in that once R_o and P have been specified, $n_o = f(K_p, Q_p)$. However, the problem is one of constrained maximization for which the Lagrangian multiplier technique is applicable.

Form the Lagrangian Function,

$V = f(K,Q) + \lambda[R_o - (K-I)r - Q\pi]$ and obtain the first partial derivatives, treating K, Q, and λ as the independent variables:

$$\frac{\delta V}{\delta K} = f_1 - \lambda r = 0 \qquad \text{(eqn 5)}$$

$$\frac{\delta V}{\delta Q} = f_2 - \lambda \pi = 0 \qquad \text{(eqn 6)}$$

$$\frac{\delta V}{\delta \lambda} = R_o - (K-I)r - Q\pi = 0 \qquad \text{(eqn 7)}$$

In addition, the relevant bordered Hessian determinant must be positive:

$$\Delta = \begin{vmatrix} f_{11} & f_{12} & -r \\ f_{21} & f_{22} & -\pi \\ -r & -\pi & 0 \end{vmatrix} > 0 = \qquad \text{(eqn 8)}$$

A maximum is obtained if equations 5, 6, 7 and 8 hold.

The marginal number of firms $= \frac{\delta f}{\delta R} = \lambda$ or from equations 5 and 6

$$\lambda = \frac{f_1}{r} = \frac{f_2}{\pi} \quad \text{or}$$

$$\frac{f_1}{f_2} = \frac{r}{\pi} = \text{inverse return ratio.}$$

From equation 4, $\frac{dQ}{dK} = -\frac{r}{\pi}$ (eqn 9)

Thus the maximum number of firms is determined by the points of tangency between

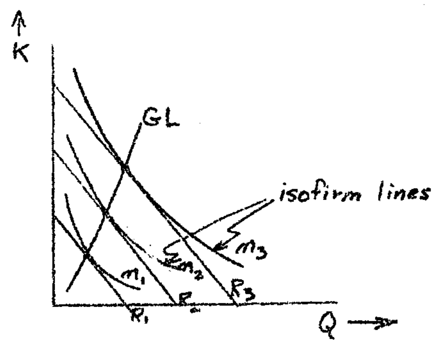

Figure 8.

the return lines and the isofirm lines. If the points are connected by a smooth curve, the "gradient line", GL, is obtained, i.e., the projection from the third dimension, m, of the line of steepest ascent. Any point on this line represents an efficiency standard for the industry.

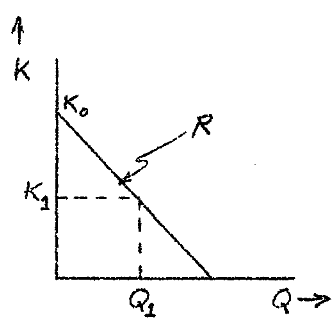

Figure 9.

$R = (K-I)\, r + Q\, \pi$ (eqn 1)

To accommodate the next phase, equation 1 can be simplified by the help of Figure 9. For example, an output Q_1 corresponding to an industry investment of $I = K_o - K_1$ leaves $K = K_1$ for outside investment. Thus, the return can be rewritten

$$R = Kr + Q\pi \qquad K \leq K_o \qquad \text{(eqn 10)}$$

Rewrite the Lagrangian function,

$$V = f(K,Q) + \lambda [R - Kr - Q\pi]$$

$$\frac{\delta V}{\delta K} = f_1 - \lambda r = 0 \qquad \text{(eqn 11)}$$

$$\frac{\delta V}{\delta Q} = f_2 - \lambda \pi = 0 \qquad \text{(eqn 12)}$$

$$\frac{\delta V}{\delta \lambda} = R - Kr - Q\pi = 0 \qquad \text{(eqn 13)}$$

Write the total differentials:

$$f_{11}dk + f_{12}dQ - rd\lambda = \lambda dr$$

$$f_{21}dK + f_{22}dQ - \pi d\lambda = \lambda d\pi$$

$$-rdK - \pi dQ = -dR + Kdr + Qd\pi \qquad \text{(eqn 14)}$$

In order to solve this system of three equations for the three unknowns, dK, dQ, and dλ, the terms on the right must be regarded as constants. Denote the system determinate by Δ and solve for dQ using the cofactor rule and expanding by column 2:

$$dQ = \frac{\begin{vmatrix} f_{11} & \lambda dr & -r \\ f_{21} & \lambda d\pi & -\pi \\ -r & -dR+Kdr+Qd\pi & 0 \end{vmatrix}}{\Delta}$$

$$dQ = \frac{\lambda \Delta_{12}\, dr + \lambda \Delta_{22}\, d\pi + \Delta_{32}}{\Delta} (-dR + Kdr + Qd\pi) \qquad \text{(eqn 15)}$$

Divide both sides by $d\pi$, assuming that r and R do not change, i.e., $dr = dR = 0$.

$$\frac{\delta Q}{\delta \pi} = \lambda \frac{\Delta_{22}}{\Delta} + Q \frac{\Delta_{32}}{\Delta} \qquad \text{(eqn 16)}$$

This partial derivative is the rate of change of output with respect to changes in "profit", all other things being equal. "Ceteris paribus", the rate of change with respect to total return is

$$\frac{\delta Q}{\delta R} = - \frac{\Delta_{32}}{\Delta} \qquad \text{(eqn 17)}$$

Finally, consider the effect of setting $f(K,Q) = m_o$ = constant so that the total differential yields

$$f_1 dK + f_2 \, dQ = 0 \qquad \text{(eqn 18)}$$

From equations 11 and 12 $f_2 = \frac{\pi}{r} f_1$ so that equation 18 becomes

$$f_1 \, dK + \frac{\pi}{r} f_1 \, dQ = 0 \quad \text{or}$$

$$dK + \frac{\pi}{r} \, dQ = 0$$

$$r \, dK + \pi dQ = 0 \qquad \text{(eqn 19)}$$

Substitute eqn. 19 into equation 14:

$$-dR + Kdr + Qd\pi = 0 \qquad \text{(eqn 20)}$$

Substitute equatian 20 together with r = constant ($dr = 0$) into equation 15:

$$\left(\frac{\delta Q}{\delta \pi}\right)_{m=m_o} = \lambda \frac{\Delta 22}{\Delta} \qquad \text{(eqn 21)}$$

Equation 17 and 21 in 16 yields

$$\frac{\delta Q}{\delta \pi} = \left(\frac{\delta Q}{\delta \pi}\right)_{m=m_o} - Q \left(\frac{\delta Q}{\delta R}\right)_{r,\pi} = \text{constant} \qquad \text{(eqn 22)}$$

The first term on the right hand side is the "capacity effect", or the rate at which output capacity changes when industry "profit" changes and the number of firms remains unchanged. The second term on the right is the "numbers effect" which states the industry's reaction with respect to change in output when the return varies, but with constant rates of return. The sum of the two terms gives the total effect on the output of the industry as profit changes. 4/

Suppose that a sizable return to capital, say π_2, exists:

(1) The existing firms will expand their output.

(2) π_2 is viewed as a substantial reward for the capital requirements of entry and consequently there is an increase in the total number of firms.

Both 1 and 2 are factors that tend to reduce industry "profit". The sign of the capacity effect is negative as was reasoned graphically in Figure 5, but can now be verified. By equation 21 the capacity effect is:

$$\left(\frac{\delta Q}{\delta \pi}\right)_{n=n_0} = \lambda \frac{\Delta_{22}}{\Delta}$$

By equation 8, $\Delta > 0$.

λ = marginal number of firms with respect to R and is positive.

$$\Delta_{22} = \begin{vmatrix} f_{11} & -r \\ -r & 0 \end{vmatrix} = -r^2$$

Thus $\left(\frac{\delta Q}{\delta \pi}\right)_{n=n_0} = - \frac{\lambda}{\Delta} r^2$ and is negative.

The numbers effect is $-Q\left(\frac{\delta Q}{\partial R}\right)_{r, \pi} = \text{constant}$

From equation 10, $R = Kr + Q\pi$:

$\frac{\delta R}{\partial Q} = \pi$. By previous assumption, $\pi \gtrless r > 0$.

Thus $\left(\frac{\delta Q}{\partial R}\right)_{r,\ \pi = \text{Constant}} > 0$ and the sign of the numbers effect is negative when $\pi > 0$. (The sign would be reversed if $\pi < 0$).

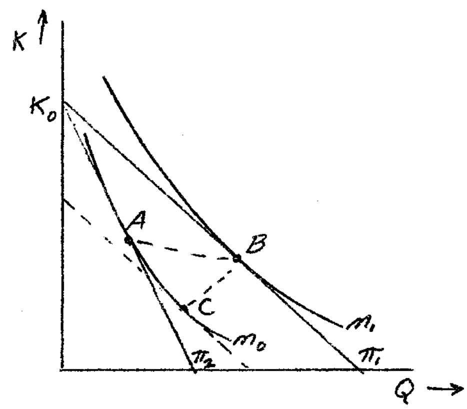

Figure 10.

The concepts are illustrated in Figure 10. The overall change from A to B may be broken down into two parts, A to C, and C to B. The original preferred combination is A at $\pi = \pi_2$. After the existing firms expand their output, the resulting combination is C, at a reduced level π_1. The movement along the isofirm line from A to C corresponds to the capacity effect. The prospects of π_2 induced an increase in the number of firms. Movement from C to B along the gradient line corresponds to the numbers effect. The movement from A to B is thus accounted for by the capacity effect and the numbers effect.

The same reasoning can be applied in reverse: To better the position of the industry, marginal firms are squeezed out, movement from B to C. The remaining firms restrict output, moving from C to A. The net result, movement from B to A, is an increase in π.*

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*Certainly other graphical representations are possible with different assumptions but demonstration of the numbers and capacity effect is all that is sought here.

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1. Chamberlin, E. H. : "Proportionality, Divisibility, and Economies of Scale", QJE, Vol. LXII, Feb. 1948, pp. 229-63.

2. Farris, P. L., editor: Market Structure Research, Iowa State University Press, p. 54.

3. The discussion of figure 6 is an application of consumer behavior theory taken from Welfare Economics class notes, J. R. Davidson, professor.

4. The development of equation 22 is an application of consumer behavior theory closely paralleling Henderson and Quandt, Microeconomic Theory, Chapter 2.