<u>Applications of Relaxed Constraint (RC) Models in Portfolio Optimization Subject to VaR, cVaR, and Related Risk Constraints</u>

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Presentation

Overview/Motivation of RC Models

Theory/Methodology

Application

Results

Overview of RC Concept

Consider the model:

$$
(1)\quad
\begin{aligned}
&\text{(a)}\ \underset{z}{\text{opt}}\ c \cdot z \\
&\text{(b)}\ A \cdot z \lesseqgtr b \\
&\text{(c)}\ Y \cdot z \lesseqgtr g \\
&\qquad z \geq 0
\end{aligned}
$$

where restrictions (1-b) must be satisfied but up to an endogenously identified proportion of constraints (1-c) can be violated or relaxed.

Overview of RC Concept

Variations of RC models can involve

(1) **Stochastic problems like portfolio optimization subject to VaR_q or probability of loss constraints**

or

(2) **Non-stochastic problems like quantile DEA where a model maximizes an efficiency metric while allowing a proportion of the DMU data to lie external to the DEA estimated technology hull.**

Overview of RC Concept

Insurance Companies or Banks

Regulatory Restrictions on VaR_q or $cVaR_q$ Exposure

Producer Level

Production or Operating Decisions Subject to Probabilistic Environmental Regulatory Restrictions (Europe)

Self Imposed Restrictions on Probability of Default or Failing to Meet Income Objectives

Binary (BRC) Example

VaR_q Constrained Portfolio Application

$$
\begin{aligned}
&\text{(a)}\ \max_{z,t,d}\ \mu_z \cdot z \\
&\text{(b)}\ Az \le b \\
(2)\quad &\text{(c)}\ Yz - t + Md \ge 0 \\
&\text{(d)}\ (\tfrac{1}{n}) \cdot d \le q \\
&\text{(e)}\ 1t = g_0 = \mathrm{VaR}_q(x) \\
&z \ge 0;\ t \text{ free};\ d_i = 0 \text{ or } 1
\end{aligned}
$$

where $y_{i,j}$ = per unit returns for asset j in state i = 1, …n, (1/n) is a vector, and $x_i = y_i \cdot z$ is the portfolio return in state i.

BRC Example

In System (2) we model joint possible returns as finitely discrete with Y an n x m matrix of n potential joint returns for m assets.

- Potential returns Y historical, simulated or generated with combined process.
- Marginal distributions (columns in Y) can be independently estimated or simulated and bound together with copulas.
- n must be "large" relative to m to avoid "data mining bias"
- n must be larger as q decreases if we desire to reasonably estimate multivariate tail risk.

Motivation of Continuous Relaxed Constraint (CRC) Concept

VaR_q constrained problems are usually "NP-hard" problems that are solved with MILP or BLP programming methods.

Mansini, R. W. Ogryczak, and M. Speranza. 2015

Often (as is demonstrated later) not practical to solve large MILP problems in reasonable amount of time.

Motivation of Continuous Relaxed Constraint (CRC) Concept

Note: system (1) is nested in system (2).

If $< \frac{1}{n}$, system (2) reverts to system (1)

CRC approach constructs an alternative system that also nests system (1) but is a continuous LP.

Utilizes partial moment stochastic inequality (Atwood-1985) that guarantees $\text{Prob}(x \le g) \le q$.

Theory-Methodology

Lower Partial Moment

$$(3)\quad \rho_{L}(\gamma,\mathbf{t}) = \int_{-\infty}^{\mathbf{t}} (\mathbf{t}-\mathbf{x})^{\gamma}\,\mathbf{f}(\mathbf{x})\,\mathbf{dx} \ \textbf{with}\ \gamma > \mathbf{0}$$

Upper Partial Moment

$$(4)\quad \rho_{U}(\gamma,\mathbf{t}) = \int_{\mathbf{t}}^{\infty} (\mathbf{x}-\mathbf{t})^{\gamma}\,\mathbf{f}(\mathbf{x})\,\mathbf{dx} \ \textbf{with}\ \gamma > \mathbf{0}$$

Theory-Methodology

Partial Moments have long been viewed as alternatives to variance as a measure of risk. (Bawa; Fishburn)

Consistent with varying degrees of stochastic dominance (Fishburn)

Semi-variance a special case (t = $\mu, \gamma = 2$)

Linear LPM ($\gamma = 1$) used in Tauer's Target MOTAD Model

Theory-Methodology

Partial Moments utilized in stochastic inequalities presented by Berck and Hihn(1982) and Atwood (1985).

Berck-Hihn used semivariance.

Atwood generalized for partial moments of any order and for all t levels.

Theory-Methodology

Atwood (1985) demonstrated that[1]:

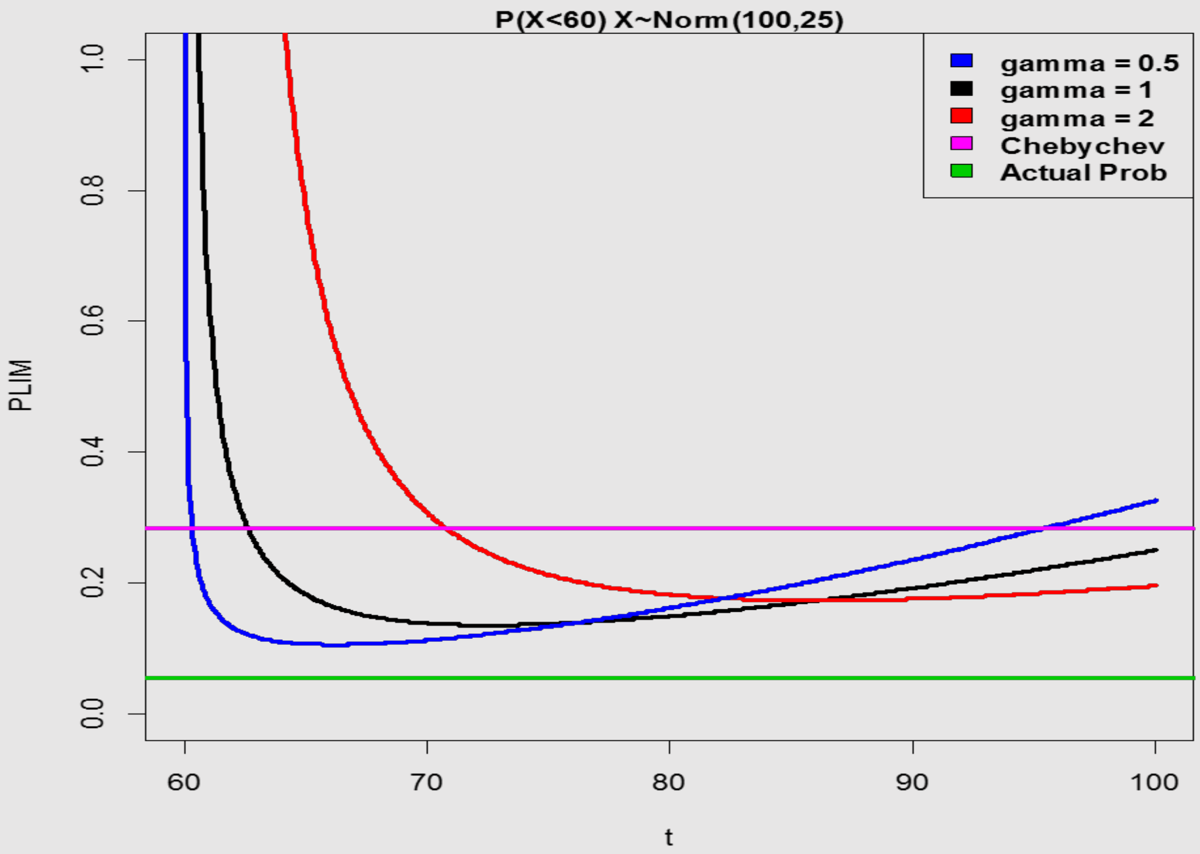
$$(5)\quad \mathbf{Prob(x \le g)} \le \frac{\rho_{\mathrm{L}}(\gamma, \mathbf{t})}{(\mathbf{t}-\mathbf{g})^{\gamma}} \text{ for all } \mathbf{t} > \mathbf{g} \text{ and } \gamma > 0$$

Similar manipulations with UPM give:

$$(6)\quad \mathbf{Prob(x \ge g)} \le \frac{\rho_{\mathrm{U}}(\gamma, \mathbf{t})}{(\mathbf{g}-\mathbf{t})^{\gamma}} \text{ for all } \mathbf{t} < \mathbf{g} \text{ and } \gamma > 0$$

Theory-Methodology

Lower Partial Moment (LPM) Probability Bounds



Theory-Methodology

Linear Lower Partial Moment Results:

(a) $\rho_L(t) = \int_{-\infty}^{t} (t - x) f(x)\,dx = F(t)[t - E\{x \mid x \leq t\}$

(b) $\text{Prob}(x \leq g) \leq \dfrac{\rho_L(t)}{(t - g) > 0}$

(7) Enforcing the constraint:

(c) $t - \frac{1}{q}\rho_L(t) \geq g \Rightarrow$

(d) $\text{Prob}(x \leq g) \leq \dfrac{\rho_L(t)}{(t - g) > 0} \leq q$

Theory-Methodology

Linear Upper Partial Moment Results

(a) $\rho_U(t) = \int_t^{\infty}(x - t)f(x)dx = (1 - F(t))\,[E\{x \mid x \geq t\} - t]$

(b) $\text{Prob}(x \geq g) \leq \dfrac{\rho_U(t)}{(g - t) > 0}$

(8) Enforcing the constraint:

(c) $t + {}^{1}\!/_{q}\,\rho_U(t) \leq g \Rightarrow$

(d) $\text{Prob}(x \geq g) \leq \dfrac{\rho_U(t)}{(g - t) > 0} \leq q$

Theory-Methodology

Atwood, Watts, Helmers, and Held (AWHH) (1988) presented the following continuous LP model:

$$
\text{(9)}\quad
\begin{aligned}
&\text{(a)}\ \max_{z,t,d,\rho_L}\ \mu_z \cdot z\\
&\text{(b)}\ Az \leq b\\
&\text{(c)}\ Yz - t + I\,d \geq 0\\
&\text{(d)}\ \left(\tfrac{1}{n}\right)\cdot d - \rho_L = 0\\
&\text{(e)}\ t - \tfrac{1}{q}\,\rho_L \geq g_0\\
&z \geq 0;\ t \text{ free};\ d_i;\ \rho_L \geq 0
\end{aligned}
$$

Theory-Methodology

AWHH (1988) and others noted:

The optimal solution aggregate returns vector $\widetilde{\boldsymbol{x}} = \boldsymbol{Y}\tilde{\mathbf{z}}$ from system (9) satisfied the probability constraint Prob$\left(\widetilde{\boldsymbol{x}} \leq \boldsymbol{g}_{\mathbf{0}}\right) \leq \boldsymbol{q}$

The solutions were often quite conservative in that Prob$\left(\widetilde{\boldsymbol{x}} \leq \boldsymbol{g}_{\mathbf{0}}\right)$ was often much less than $\boldsymbol{q}$.

Theory-Methodology

The reason for the conservative solutions in system (9) is easily demonstrated using results presented in the heavily cited paper by Rockafellar and Urysev (RU) (2000).

RU presented LP procedures for minimizing the conditional Value at Risk or $cVaR_q$ of an upside risk model.

<u>Theory-Methodology</u>

RU (2000) model equivalent to an UPM variant of AWHH (1988) model.

The g_0 in system (9) is actually the $cVaR_q(\tilde{x})$.

The optimal $\tilde{t}$ in system (9) is $VaR_q(\tilde{x}) \Rightarrow$

$$Prob(\tilde{x} \leq cVaR_q(\tilde{x})) < q \text{ and } Prob\left(\tilde{x} \leq VaR_q(\tilde{x})\right) = q$$

Theory-Methodology

The results that:

$$\boldsymbol{g_0 = cVaR_q(\tilde{x})}$$

and

$$\boldsymbol{\tilde{t} = VaR_q(\tilde{x})}$$

when system (9) is optimized can be easily shown using the definitions (7-a) or (8-a), the probability limits in (7-b) or (8-b), and assuming the constraints in (7-c) or (8-c) are binding.

Theory-Methodology

Results at this point:

We can use AWHH(1988) or RU(2000) models and LP to solve high dimension $cVaR_q$ problems.

- **Coherency of cVaR risk metrics attractive.**
- **However, the result solutions are commonly excessively conservative as approximations to a VaR_q constrained problem.**

Theory-Methodology

Question: Can we get less conservative VaR_q constrained solutions using continuous LP since we usually cannot use MILP procedures with high dimension problems.

Answer: Yes. If we use two insights. Which we use depends upon our objective.

Theory-Methodology

If we wish to maintain coherency in portfolio risk model

- **Exploit the $\tilde{t} = VaR_q$ and $g = cVaR_q$ result in the $cVaR_q$ restriction $\tilde{t} - \frac{1}{q}\,\rho_L(\tilde{t}) \geq \ g$**
- **Reduce g until $\tilde{t} = g_0$ i.e. the original VaR_q target.**
  - **Usually requires only two additional LP runs**
  - **The solution remains $cVaR_q$ constrained thus maintaining coherency by considering all outcomes below $\tilde{t}$**
  - **The solutions will still tend to be conservative relative to the optimal VaR_q constrained solution.**

Theory-Methodology

If we wish to find the "best" constraints to relax to more closely approximate the original VaR_q constrained problem's optimal solution

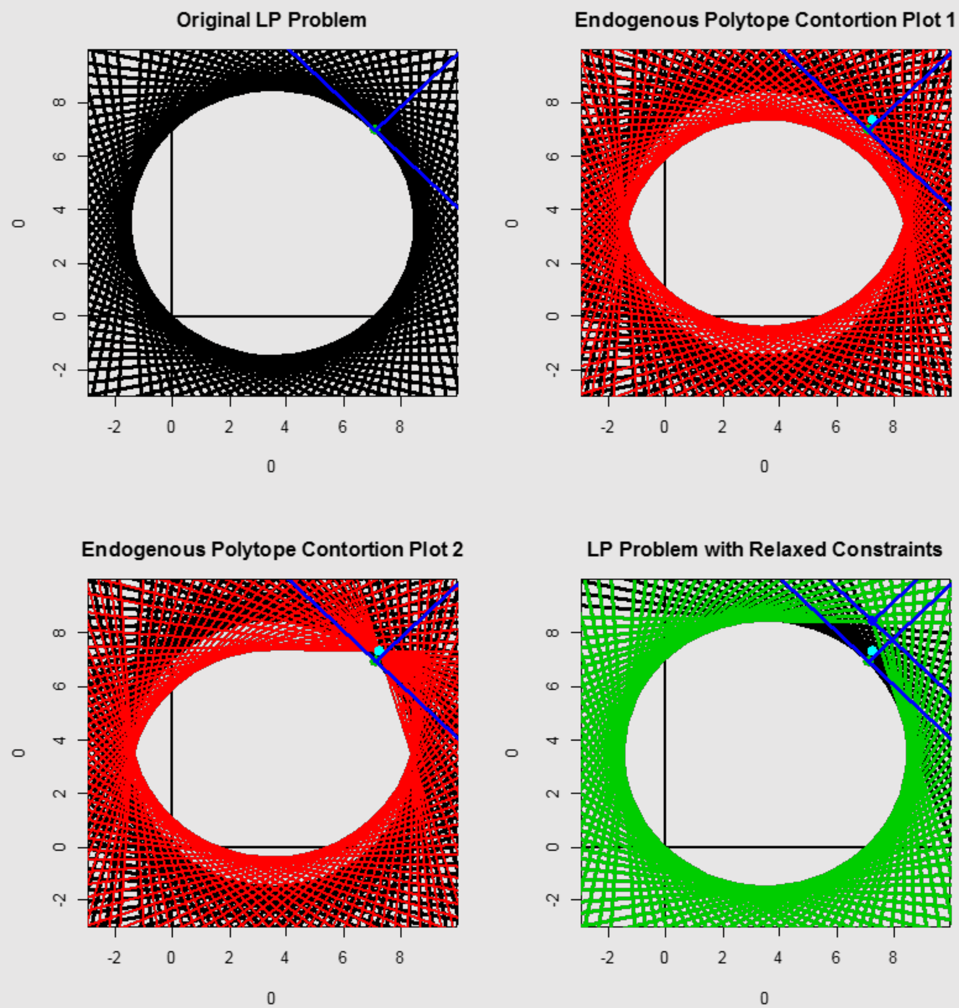
Let's examine what the CRC approach is doing to the nested LP problem.

Theory-Methodology

CRC Model

$$
(9)\quad
\begin{aligned}
&\text{(a)}\ \max_{z,t,d,\rho_L}\ \mu_z \cdot z \\
&\text{(b)}\ Az \le b \\
&\text{(c)}\ Yz - t + I\,d \ge 0 \\
&\text{(d)}\ \left(\tfrac{1}{n}\right) \cdot d - \rho_L = 0 \\
&\text{(e)}\ t - \tfrac{1}{q}\,\rho_L \ge g_0 \\
&z \ge 0;\ t \text{ free};\ d_i;\ \rho_L \ge 0
\end{aligned}
$$

CRC Theory-Methodology



CRC Theory-Methodology

- The positive d_i values in the partial moment model indicate constraints that the LP model has "stretched"

- Indicate potential constraints to be "relaxed" in original nested problem.

- When row restrictions on original problem are relaxed the resulting solutions may need to be iterated on several times to find nested solutions with q constraints actually being violated (i.e. qDEA)

EXAMPLE

(Example designed to illustrate concept)

- **Invest a lump sum amount in a four asset portfolio 25 years prior to retirement.**
- **Plan to purchase a 30-year fixed income annuity at retirement. Interest rate r = 0.04.**
- **Investment assets assumed to follow a correlated log-normal stochastic diffusion process.**
  - $\mu = (0.15, 0.12, 0.10, 0.075)$
  - $\sigma = (0.25, 0.20, 0.15, 0.05)$
  - $\rho_{i,j} = 0.5 \; for \; i,j \; = \; 1,2,3$
  - $\rho_{i,4} = 0 \; for \; i = 1,2,3$

EXAMPLE

The investor would like to estimate an optimal asset portfolio (with no rebalancing) and the minimal amount they would need to invest at time 0 to have no more than a 10% chance of their final retirement annuity falling below $45,000 per year.

Results

Results for Which Binary Solutions Could Be Found

rnames	nobs	g1	T1	BM1	pvalM1	g2	T2	BM2	pvalM2	BM22	pvalM22	BB	timeM	timeM2	timeGB	timeLNDO
Min	100	45000	47438	105936	0.030	34146	45000	89591	0.080	87279	0.080	71357	0.00	0.00	0.13	0.68
Q25	100	45000	49843	143100	0.040	38817	45000	126058	0.090	122913	0.090	117089	0.01	0.03	0.24	3.86
Median	100	45000	50974	147444	0.050	39726	45000	131221	0.090	127725	0.090	123021	0.01	0.03	0.30	4.82
Mean	100	45000	51280	147711	0.048	39556	45000	129849	0.092	127219	0.095	121441	0.01	0.03	0.34	4.73
Q75	100	45000	52168	154737	0.060	40627	45000	134994	0.100	133223	0.100	128212	0.02	0.03	0.38	5.76
Max	100	45000	59305	169018	0.070	42687	45000	150042	0.100	150042	0.100	143443	0.09	0.12	2.16	7.82
Min	250	45000	49216	140148	0.032	37368	45000	121067	0.092	115988	0.084	114145	0.01	0.03	0.84	8.78
Q25	250	45000	50651	148470	0.040	38773	45000	129134	0.096	127963	0.096	124112	0.02	0.05	3.77	14.42
Median	250	45000	51371	152173	0.044	39419	45000	133337	0.096	131253	0.096	127659	0.03	0.06	5.33	19.32
Mean	250	45000	51409	151850	0.045	39408	45000	132961	0.096	131627	0.097	127521	0.03	0.06	7.17	19.86
Q75	250	45000	52228	155286	0.048	39980	45000	136512	0.096	135284	0.100	131179	0.03	0.07	8.08	23.21
Max	250	45000	54190	165357	0.060	41145	45000	145915	0.100	145493	0.100	140196	0.10	0.13	47.88	46.25
Min	500	45000	49770	145661	0.034	37390	45000	125351	0.096	125083	0.090	116940	0.05	0.10	70.89	45.89
Q25	500	45000	50977	150323	0.040	38806	45000	131234	0.098	131006	0.098	127761	0.06	0.12	182.92	146.96
Median	500	45000	51690	153335	0.044	39176	45000	133574	0.098	133034	0.098	129505	0.07	0.14	260.09	211.61
Mean	500	45000	51618	153209	0.043	39243	45000	133588	0.098	133064	0.098	129809	0.07	0.16	391.80	310.18
Q75	500	45000	52183	155452	0.046	39724	45000	135595	0.098	135376	0.100	132702	0.08	0.22	455.70	377.47
Max	500	45000	54158	166736	0.054	40688	45000	144168	0.100	144080	0.100	138197	0.20	0.28	3599.99	1793.61

Results

Results for Which Binary Solutions Could Not Be Found

rnames	nobs	g1	T1	BM1	pvalM1	g2	T2	BM2	pvalM2	BM22	pvalM22	timeM	timeM2
Min	1000	45000	50209	146141	0.036	38085	45000	128427	0.098	128040	0.096	0.12	0.24
Q25	1000	45000	51154	151832	0.041	38966	45000	132351	0.099	131663	0.099	0.15	0.30
Median	1000	45000	51488	153465	0.043	39329	45000	133967	0.099	133687	0.099	0.17	0.31
Mean	1000	45000	51575	153347	0.043	39268	45000	133804	0.099	133351	0.099	0.17	0.32
Q75	1000	45000	51968	154951	0.045	39586	45000	135422	0.099	135166	0.100	0.17	0.33
Max	1000	45000	53171	159709	0.050	40332	45000	140682	0.100	140661	0.100	0.21	0.36
Min	5000	45000	51030	150878	0.039	38754	45000	131025	0.100	131005	0.099	5.19	9.39
Q25	5000	45000	51390	153025	0.042	39117	45000	133434	0.100	133328	0.100	6.04	10.34
Median	5000	45000	51604	153926	0.043	39241	45000	134326	0.100	134245	0.100	6.46	10.82
Mean	5000	45000	51592	153953	0.043	39251	45000	134285	0.100	134218	0.100	6.47	10.79
Q75	5000	45000	51768	154827	0.044	39404	45000	134997	0.100	134984	0.100	6.85	11.15
Max	5000	45000	52253	157231	0.045	39682	45000	136811	0.100	136797	0.100	8.57	12.81
Min	10000	45000	51118	151450	0.041	38897	45000	132261	0.100	132127	0.099	13.13	18.18
Q25	10000	45000	51440	153494	0.042	39186	45000	133948	0.100	133926	0.100	16.18	21.11
Median	10000	45000	51559	154108	0.042	39275	45000	134500	0.100	134414	0.100	17.26	22.00
Mean	10000	45000	51564	154027	0.042	39272	45000	134420	0.100	134372	0.100	17.37	22.06
Q75	10000	45000	51676	154613	0.043	39366	45000	134970	0.100	134954	0.100	18.32	22.90
Max	10000	45000	52060	156290	0.045	39614	45000	136395	0.100	136346	0.100	22.58	27.53

CONCLUSIONS

- Continuous CRC approach to relaxed constraint problems useful in portfolio optimization models.
- CRC has proven useful in CVaR problems
- Iterated CRC can give improved solutions to problems with VaR objectives or constraints.
- Two stage CRC approach has already proven use in quantile DEA applications.

NOTES:

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[1]**The LPM stochastic inequality is derived via:**

$$\rho_L(\gamma,t) = \int_{-\infty}^{t} (t-x)^{\gamma} f(x)\,dx = \int_{-\infty}^{g} (t-x)^{\gamma} f(x)\,dx + \int_{g}^{t} (t-x)^{\gamma} f(x)\,dx \Rightarrow$$

$$\rho_L(\gamma,t) \geq \int_{-\infty}^{g} (t-x)^{\gamma} f(x)\,dx \geq \int_{-\infty}^{g} (t-g)^{\gamma} f(x)\,dx = (t-g)^{\gamma} \int_{-\infty}^{g} f(x)\,dx \Rightarrow$$

$$\rho_L(\gamma,t) \geq (t-g)^{\gamma} F(g) \Rightarrow$$

$$\mathbf{Prob}(x \leq g) \leq \frac{\rho_{LPM}(\gamma,t)}{(t-g)^{\gamma}} \text{ for all } t > g \text{ and } \gamma > 0 \; .$$

NOTES

The preceding results use Fishburn's lower partial moment but can easily be modified to the use of upper partial moments (UPM) when computing limits on the probability of upside events. With the UPM and setting *g* > *t*, similar manipulations give:

$$\mathbf{Prob(x \geq g)} \leq \frac{\rho_{\mathbf{UPM}}(\gamma, \mathbf{t})}{(\mathbf{g}-\mathbf{t})^{\gamma}} \ \textbf{for all } \mathbf{g} > \mathbf{t} \textbf{ and } \gamma > 0 \ .$$

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