Resource Allocation and Asset Pricing

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The modern theories of the productive firm and the financial firm are, for the most part, distinct. Apart from some pioneering attempts to model firms facing price uncertainty for physical commodities in the presence of financial markets (Danthine, 1978; Holthausen, 1979; Anderson and Danthine, 1981) and some recent research on production-based asset pricing and financial innovations (Cochrane, 1991; Rahi, 1995), these two literatures have developed separately. A common view seems to be that recently articulated by LeRoy and Werner (2001, p.13) that "...production theory...does not lie within the scope of finance as usually defined, and not much is gained by combining exposition of the theory of asset pricing with that of resource allocation".

Cochrane (2001) discusses the rationales for treating production and finance separately, observing that while finance theory typically takes production decisions as given, and examines asset price determination, permanent income macroeconomics does the opposite. For an S-state world, the fixed output and fixed asset price models may be interpreted, respectively, as fixed-proportions and linear approximations to a general technology, coinciding at a given equilibrium point. These approximations work well in characterizing an equilibrium, but less well in a comparative-static setting. Cochrane (2001, p. 43) cautions that

> We routinely think of betas and factor risk prices ... as determining expected returns. But the whole consumption process, discount factor and factor risk premia change when the production technology changes. Similarly, we are on thin ice if we say anything about the effects of policy interventions, new markets and so on.

Given that many productive firms routinely operate in financial markets, the idea that production and financial decisions should be treated separately appears problematic. For such firms, the primitive ability to separate production decisions from financial decisions must rest either on some form of separability in the firm's objective function, which allows consideration of production decisions separately from financial decisions, or on some form of orthogonality between productive and financial operations. Intuitively, this requires either the implausible assumption that firms have fundamentally different preferences over income earned from these two sources or the equally unappealing assumption that the assets traded in financial markets are unrelated to the risks faced by firms engaged in real production.

The same point may be expressed in terms of arbitrage arguments. The view that financial and production decisions may be treated separately requires the assumption either that firms ignore arbitrage opportunities between financial and production operations, or that no such arbitrage opportunities exist. The first assumption is inconsistent with standard assumptions about the objectives of the firm and its shareholders, while the second is inconsistent with the basic presumption that financial markets are created to manage the risks associated with production and consumption.

A strong argument for not combining the theories of resource allocation and asset pricing would be increased theoretical simplicity and tractability gained from considering them separately. However, when one compares the literatures on arbitrage pricing and producer decisionmaking, one must be struck by their similarities. In the axiomatic approach to production, one typically proceeds from the specification of a technology, defined by convex sets, to a minimal cost characterization of the technology. There is an obvious analogy with the construction of derivative assets as replicating or super-replicating elements derived from a set of feasible portfolios, which is typically a convex cone. Milne (1976; 1988; 1995), in particular, utilizes this observation to induce both individual preference structures over asset holdings and producer technologies over asset returns and input use.

The true difficulty in combining these theories, therefore, is not that they are fundamentally different. In fact, it seems obvious that they are virtually identical. Rather the problem appears to be that until recently, the theory of production under uncertainty was specified in terms of stochastic production functions in a manner that was distinct from either modern finance theory or modern production theory. Chambers and Quiggin (2000) have shown how to transport modern non-stochastic, axiomatic production theory virtually *en masse* to firms facing stochastic technologies using the Arrow-Debreu state-space specification of a technology.

A crucial feature of the state-contingent production technologies considered by Chambers and Quiggin (2000) is that they allow for substitution between outputs in different states of nature. Substitutability between outputs in different state of nature, in turn, allows for subsitutability between incomes generated by those outputs, and is essential to the manner in which firms respond to difference in state-claim prices. Stochastic production functions rule this substitutability out by assumption. Hence, they cannot yield useful information on the relative behavior of state-claim prices for a given time period, and more importantly they severely circumscribe the range of arbitrage opportunities open to the firm.

The most immediate implications of this substitutability, or lack thereof, relate to the theory of arbitrage pricing for productive firms. Ross's (1978) pathbreaking demonstration of the existence of positive state claim prices relies on the solution to a simple linear program, which minimizes the cost of purchasing an investment portfolio that ensures the firm at least a zero return in each state of nature. In a frictionless economy, the result is a linear valuation operator which places a positive value on each contingent claim. This basic observation has been extended in a multiplicity of directions to account for various forms of financial market frictions including, among many others, pre-existing tax structures, the existence of bid-ask price spreads, and general convex friction functions as well as empirical analyses of these effects (Prisman, 1986; Ross, 1987; Dermody and Prisman, 1988; Dermody and Rockafellar, 1991; Clark, 1993; Jouini and Kallal, 1995 (a,b); Luttmer, 1996; Jouini, Kallal, and Napp, 2001).[1]

The arbitrage problem solved by Ross (1978) is mathematically equivalent to the cost minimization problem for a firm facing a linear multi-output technology. Because the 'technology' used to produce payouts in the second period is linear, it exhibits constant returns to scale generally and linearity over the cone defined by the asset structure. Thus, the law of one price and the existence of positive state-claim prices can be recognized in modern production-theoretic terms as mathematical reflections of the basic properties of this 'technology'. The natural conclusion seems to be that the arbitrage problem and the firm's resource allocation problem are essentially equivalent. Hence, it should be relatively easy and hopefully fruitful to re-unite the analysis of the productive and financial decisions of the firm under a common paradigm of minimizing the cost of assembling a derivative financial asset by combining financial and productive operations. This paper attempts to initiate that

[1]The literature in this area is voluminous, and an adequate citation of all relevant contributions would entail a reference list exceeding the length of the current paper.

reunification.

Our paper is perhaps most closely related in spirit and method of analysis on the finance side to Prisman's (1986) analysis of arbitrage pricing in the face of suitably convex frictions and on the production side to the contribution of Chambers and Quiggin (2000). The latter, in particular, adapts a long line of contributions in axiomatic production analysis, including Shephard (1970), McFadden (1978), Diewert (1982), and Färe (1988), to the Arrow-Debreu state-space representation of stochastic technologies. The arguments in this paper are thus largely reworkings, or perhaps better, convex combinations of arguments familiar from these literatures. Most, therefore, when placed in their proper context should be very familiar from basic microeconomic theory.

For example, when one compares our problem with that of Prisman (1986) and Ross (1987), the main difference is that we have replaced convex frictions on the financial asset structure with a convex cost structure for a stochastic production technology. This change, while in a sense relatively minor, leads us to focus on a completely different set of issues than in either of these earlier works, or in the numerous studies that have followed them. In particular, we concentrate on the interplay between the financial asset structure and the technology in creating the derivative asset. On the production side, our analysis can be recognized as a reinterpretation of a special case of the theory of cost minimization for output-nonjoint technologies (Chambers, 1988).

In what follows, we first introduce some notation and concepts used throughout the paper. Then we turn to the specification of a stochastic technology in Arrow-Debreu state-space terms and a frictionless financial market structure. This structure is briefly compared with the technology that is characterized by a stochastic production function. The financial market and the firm's state-space technology are then used to deduce what we term a derivative-cost function, which represents the minimal cost to the firm of assembling a financial asset or 'derivative' through real and financial operations. The derivative-cost function is shown to be convex in the derivative asset, and its basic properties are then derived and formally proved.

We then turn to an analysis of virtual state-claim prices, virtual risk-free rates, and virtual asset prices for the firm in terms of the subdifferentials of the derivative-cost function and its associated directional derivatives. It is shown, among other results, that cost minimizing production decisions can always be interpreted in terms of virtual profit maximization, and that as a consequence virtual state-claim prices (and thus virtual asset prices) can often be deduced directly from the physical production technology. This reinforces Cochrane's (1991) observation that there exists a production-based approach to asset pricing, which is completely analogous to the more familiar consumption-based approach. The main difference is that our pricing rules are in terms of a cost function for a more general technical specification than that permitted by Cochrane (1991), and, therefore, should be applicable to a greater range of actual problems.

The tools developed for the virtual pricing of state-claims are then used to examine two problems drawn, respectively, from the literatures on asset pricing and the literature on producer decisionmaking under uncertainty. The first is the existence of no-arbitrage pricing of state claims. We show that these no-arbitrage prices of state-claims and assets are our virtual pricing rules. After that we take up the issue of when the decisionmaker's production choices are independent of his or her attitudes towards risk. This has been referred to as separation. We use our results to provide necessary and sufficient conditions for such a separation between the production choices and decisionmaker's risk attitudes for general asset structures and technologies.

The penultimate section of the paper considers three brief applications of our approach, and the final section concludes.

1. Notation

Denote the unit vector by $\mathsf{1} \in \Re^S_+$. Define e_i as the i-th row of the $S \times S$ identity matrix

$$\mathsf{e}_i = (0, ..., 1, 0, ..., 0).$$

For $\mathsf{m}, \mathsf{m}' \in \Re^S$, the notation $\mathsf{m} \cdot \mathsf{m}'$ denotes the componentwise product of the two vectors, i.e., $\mathsf{m} \cdot \mathsf{m}' = (m_1 m_1', ..., m_s m_s')$, while the notation $\frac{\mathsf{m}}{\mathsf{m}'}$ denotes the componentwise ratios of the two vectors, i.e., $\frac{\mathsf{m}}{\mathsf{m}'} = \left(\frac{m_1}{m_1'}, ..., \frac{m_S}{m_S'}\right)$ The notation $\mathsf{m}\mathsf{m}'$ for two conformable vectors denotes the usual inner product. For a linear subspace, H, denote its orthogonal complement by $H^{\perp}$. For the $S \times J$ matrix, B, denote its right inverse if it exists by B^R and the left inverse by B^L. Denote the relative interior of a convex set $A \subseteq \Re^S$ by riA, and the convex hull of a set A by coA.

For a convex function[2] $f : \Re^S \to \Re$, its *subdifferential* at m is the closed, convex set:

$$\partial f(\mathsf{m}) = \left\{\mathsf{k} \in \Re^S : f(\mathsf{m}) + \mathsf{k}(\mathsf{m}' - \mathsf{m}) \leq f(\mathsf{m}') \text{ for all } \mathsf{m}'\right\}. \tag{1}$$

The elements of $\partial f(\mathsf{m})$ are referred to as *subgradients*. The *one-sided directional (Gateaux differential) derivative* of f in the direction of n is defined by

$$f'(\mathsf{m};\mathsf{n}) = \lim_{\lambda \to 0^+} \left\{\frac{f(\mathsf{m}+\lambda\mathsf{n}) - f(\mathsf{m})}{\lambda}\right\}.$$

For f convex, $f'(\mathsf{m};\mathsf{n})$ is positively linearly homogeneous and convex in n. Moreover,

$$f'(\mathsf{m};\mathsf{n}) \geq -f'(\mathsf{m};-\mathsf{n}).$$

When $f'(\mathsf{m};\mathsf{n}) = -f'(\mathsf{m};-\mathsf{n})$, we say that f is *smooth in the direction of* n *at* m. When f is smooth in all directions at m, it is differentiable. Moreover, if f is differentiable at m, $\partial f(\mathsf{m})$ is a singleton and corresponds to the usual gradient, which we denote by $\nabla f(\mathsf{m}) = [f_1(\mathsf{m}), ..., f_s(\mathsf{m})]$ where subscripts denote partial derivatives. If $\partial f(\mathsf{m})$ is a singleton, f is differentiable at m. By basic results, for f convex and finite at m:

$$f'(\mathsf{m};\mathsf{n}) \geq \{\mathsf{kn} : \mathsf{k} \in \partial f(\mathsf{m})\}. \tag{2}$$

The convex conjugate of f is denoted

$$f^*(\mathsf{k}) = \sup_{\mathsf{m}} \{\mathsf{km} - f(\mathsf{m})\}.$$

If f is proper and closed,[3] then f^* is proper and closed and

$$f(\mathsf{m}) = \sup_{\mathsf{v}} \{\mathsf{km} - f^*(\mathsf{k})\}, \tag{3}$$

and on the relative interior of their domains

$$\mathsf{k} \in \partial f(\mathsf{m}) \Leftrightarrow \mathsf{m} \in \partial f^*(\mathsf{k}). \tag{4}$$

[2] Apart from our notion of smoothness in a particular direction, these results on convex functions are all drawn directly from Rockafellar (1970).

[3] f is proper if $f(\mathsf{x}) < \infty$ for at least one x, and $f(\mathsf{x}) > -\infty$ for all x. A proper convex function is closed if is lower-semicontinuous.

2. State-Contingent Technologies, Asset Structures, and the Derivative Cost Function

We model a price-taking firm facing a stochastic environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain. Uncertainty is resolved by 'Nature' making a choice from $\Omega = \{1, 2, ..., S\}$. Each element of Ω is referred to as a state of nature. The firm's stochastic production technology is represented by a single-product, state-contingent input correspondence.[4] To make this explicit, let $\mathsf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathsf{z} \in \Re_+^S$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0. If state $s \in \Omega$ is realized (picked by 'Nature'), and the producer has chosen the *ex ante* input-output combination (x, z), then the realized or *ex post* output in period 1 is z_s.

The firm's technology is characterized by a continuous input correspondence, $X : \Re_+^S \to \Re_+^N$, which maps state-contingent output vectors into input sets that are capable of producing that state-contingent output vector.[5] It is defined

$$X(\mathsf{z}) = \{\mathsf{x} \in \Re_+^N : \mathsf{x} \text{ can produce } \mathsf{z}\}.$$

Intuitively, $X(\mathsf{z})$ is associated with everything on or above the isoquant for the state-contingent output vector z. At points it will be convenient to consider an alternative representation, which we refer to as the state-contingent output set,

$$Z(\mathsf{x}) = \left\{\mathsf{z} \in \Re_+^S : \mathsf{x} \in X(\mathsf{z})\right\}.$$

Intuitively, $Z(\mathsf{x})$ can be thought of as all state-contingent outputs on or below a state-contingent product transformation curve. We impose the following axioms on $X(\mathsf{z})$:

X.1 $X(\mathsf{0}_{MxS}) = \Re_+^N$ (no fixed costs), and $\mathsf{0} \notin X(\mathsf{z})$ for $\mathsf{z} \geq \mathsf{0}$ and $\mathsf{z} \neq \mathsf{0}$ (no free lunch).

X.2 $\mathsf{z}' \leq \mathsf{z} \Rightarrow X(\mathsf{z}) \subseteq X(\mathsf{z}')$.

[4] For a generalization to the multiple-output case, see Chambers and Quiggin (2000, Chapter 4). Our results extend straightforwardly to that case.

[5] In general, different firms need not have the same technology, so that, if there are N firms, the technology for firm n will be represented by a correspondence X^{n}. Since we mainly discuss the decisions of a single firm, we will normally suppress the index n.

X.3 $\mathsf{x}' \geq \mathsf{x} \in X(\mathsf{z}) \Rightarrow \mathsf{x}' \in X(\mathsf{z})$.

X.4 $\lambda X(\mathsf{z}) + (1-\lambda)X(\mathsf{z}') \subseteq X(\lambda \mathsf{z} + (1-\lambda)\mathsf{z}') \qquad 0 \leq \lambda \leq 1.$

X.5 X is continuous.

The first part of X.1 says that doing nothing is always feasible, while the second part of X.1 says that realizing a positive output in any state of nature requires the commitment of some inputs. X.2 says that if an input combination can produce a particular mix of state-contingent outputs then it can always be used to produce a smaller mix of state-contingent outputs. X.3 implies that inputs have non-negative marginal productivity. X.4 ensures that the cost function developed below is convex in state-contingent outputs.

Period 0 prices of inputs are denoted by $\mathsf{w} \in \Re^N_{++}$ and are non-stochastic. Output price is stochastic, and we denote by $\mathsf{p} \in \Re^S_{++}$ the vector of state-contingent output prices corresponding to the vector of state-contingent outputs. Producers take these state-contingent output prices and the prices of all inputs as given. The state-contingent revenue vector, denoted $\mathsf{p} \cdot \mathsf{z} \in \Re^S_{+}$, has typical elements of the form $p_s z_s$.

Financial markets are frictionless, and the *ex ante* financial security payoffs are given by the $S \times J$ non-negative matrix A. The assumption that financial markets are frictionless can be easily modified by subsuming the financial frictions within the production technology with little change in the analysis that follows. The vector of state-contingent payoffs on the jth financial asset is denoted $\mathsf{A}_j \in \Re^S_{+}$. Denote the span of the matrix A by M . The prices of the financial securities are given by $\mathsf{v} \in \Re^J_{+}$. The firm's portfolio vector is denoted $\mathsf{h} \in \Re^J$.

Dual to the input correspondence is the cost function, $c : \Re^N_{++} \times \Re^S_{+} \to \Re_{+}$, defined as

$$c(\mathsf{w}, \mathsf{z}) = \min_{\mathsf{x}} \{\mathsf{wx} : \mathsf{x} \in X(\mathsf{z})\} \quad \mathsf{w} \in \Re^N_{++}$$

if there exists an $\mathsf{x} \in X(\mathsf{z})$ and ∞ otherwise. Mathematically, $c(\mathsf{w}, \mathsf{z})$ is equivalent to the multi-product cost function familiar from non-stochastic production theory (Färe 1988). Let

$$\mathsf{x}(\mathsf{w}, \mathsf{z}) \in \arg\min_{\mathsf{x}} \{\mathsf{wx} : \mathsf{x} \in X(\mathsf{z})\}$$

If the input correspondence satisfies properties X, $c(\mathsf{w}, \mathsf{z})$ satisfies (Chambers and Quiggin, 2000):

C.1. $c(\mathsf{w},\mathsf{z})$ is positively linearly homogeneous, non-decreasing, concave, and continuous on $\Re^N_{++}$;

C.2. If $\mathsf{x}(\mathsf{w},\mathsf{z})$ is unique, c is differentiable in w and $\mathsf{x}(\mathsf{w},\mathsf{z}) = \nabla_{\mathsf{w}} c(\mathsf{w},\mathsf{z})$. If $\nabla_{\mathsf{w}} c(\mathsf{w},\mathsf{z})$ exists, then $\mathsf{x}(\mathsf{w},\mathsf{z})$ is unique, and $\mathsf{x}(\mathsf{w},\mathsf{z}) = \nabla_{\mathsf{w}} c(\mathsf{w},\mathsf{z})$ (Shephard's Lemma).

C.3. $c(\mathsf{w},\mathsf{z}) \geq 0$, $c(\mathsf{w}, 0_S) = 0$, and $c(\mathsf{w},\mathsf{z}) > 0$ for $\mathsf{z} \geq \mathsf{0}, \mathsf{z} \neq \mathsf{0}$;

C.4. $\mathsf{z}^o \geq \mathsf{z} \Rightarrow c(\mathsf{w},\mathsf{z}^o) \geq c(\mathsf{w},\mathsf{z})$.

C.5 $c\left(\mathsf{w},\mathsf{z}\right)$ is convex on $\Re^S_+$ and continuous on the interior of the region where it is finite.

For $\mathsf{q} \in \Re^S_+$, the convex conjugate of c,

$$c^*\left(\mathsf{w},\mathsf{q}\right) = \sup_{\mathsf{z}} \left\{\mathsf{qz} - c\left(\mathsf{w},\mathsf{z}\right)\right\},$$

can be interpreted as a profit-function for the 'price' vector q. Let $\mathsf{z}' \in \arg\sup \left\{\mathsf{qz} - c\left(\mathsf{w},\mathsf{z}\right)\right\}$, then

$$\mathsf{qz}' - c\left(\mathsf{w},\mathsf{z}'\right) \geq \mathsf{qz} - c\left(\mathsf{w},\mathsf{z}\right),$$

for all z, and hence $\mathsf{q} \in \partial c\left(\mathsf{w},\mathsf{z}'\right)$. By (4), we then obtain $\mathsf{z}' \in \partial c^*\left(\mathsf{w},\mathsf{q}\right)$, which restates Hotelling's lemma in terms of subdifferentials.

2.1. The stochastic production function

It is worthwhile to briefly compare the Arrow-Debreu state-space representation with the technology most commonly considered in the literature on production under uncertainty as well as in many financial applications (e.g., Cochrane, 1991). A stochastic production function is typically specified

$$z_s = f\left(x, \varepsilon_s\right)$$

where x is a scalar input chosen by the producer and $\boldsymbol{\varepsilon}$ is a random input taking the value ε_s in state s.

Chambers and Quiggin (1998, 2000) show that the cost function for this technology[6] is

[6] Chambers and Quiggin (2000) consider the more general multiple input case and show that

$$c\left(\mathsf{w},\mathsf{z}\right) \geq \max\left\{c_1\left(\mathsf{w}, z_1\right), \ldots c_S\left(\mathsf{w}, z_S\right)\right\}$$

where $c_i\left(\mathsf{w}, z_i\right)$ is the cost function corresponding to $f\left(\mathsf{x}, \varepsilon_i\right)$.

given by

$$\begin{aligned} c(\mathsf{w}, \mathsf{z}) &= \inf \{wx : f(x, \varepsilon_s) \geq z_s \forall s\} \\ &= \sup_s \left\{ w f^{-1}(z_s; \varepsilon_s) \right\} \end{aligned}$$

Its output set,

$$Z(x) = \{\mathsf{z} : f(x, \varepsilon_s) \geq z_s, s \in \Omega\}.$$

Figure 1, drawn from Cochrane (2001), illustrates by comparing a general state-contingent output set and that associated with a stochastic production function that agrees with the general set at a given equilibrium point. Notice, in particular, that $Z(x)$ maintains that state-contingent outputs are not substitutable for one another. Chambers and Quiggin (2000) offer a three-dimensional illustration of the same-technology, which they refer to as output-cubical. In particular, if the technology does not allow X.2, and

$$Z(x) = \{\mathsf{z} : f(x, \varepsilon_s) = z_s, s \in \Omega\},$$

the associated state-contingent output set is a single point. The range of state-contingent outputs, which can be feasibly produced, is then even more severely circumscribed and corresponds to a manifold emanating from the origin.[7] This specification, thus, severely limits the ability of the firm to use its physical technology in conjunction with operations in financial markets to construct derivative assets. And, in some instances, for example, those of additive or multiplicative uncertainty, this specification can render the physical technology redundant. More generally, as we illustrate below, this restriction circumscribes the ability of the physical technology to provide virtual prices for arbitrary derivative assets in a way that does not occur in the more general specification.

2.2. The derivative-cost function

Define the *derivative-cost function* $C : \Re^S_+ \to \Re$, by

$$C(\mathsf{y}) = \min_{\mathsf{h},\mathsf{z}} \{c(\mathsf{w}, \mathsf{z}) + \mathsf{vh} : \mathsf{Ah} + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y}\} \tag{5}$$

[7] This remains true for the multiple input case.

if there exists $\mathsf{Ah} + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y}$ and ∞ otherwise. $C(\mathsf{y})$, thus, represents the minimal cost (maximal buying price) to the firm of constructing the derivative financial asset, y, either through operations in financial markets or through its production operations.

The firm's derivative-cost minimization problem can be visualized with the aid of a Lerner-Pearce diagram, familiar from the theory of factor-price equalization, as adapted to state-contingent output space. In the case where $S = 2$ and $J = 1$, and there is no output price uncertainty, we can illustrate as in Figure 2a. Here financial markets are incomplete, and the span of the financial market is given by the ray through the origin labelled M. Shorting the asset corresponds to operating in the non-positive quadrant, while long positions are represented by the positive quadrant. Isocost curves for the state-contingent technology are drawn as exhibiting an increasing marginal rate of state-contingent transformation, reflecting the convexity of c. The problem is to choose an asset position and a point on an isocost curve to reach point y at minimal cost. Points B and C in the figure represent such a potential solution. As illustrated, the firm's equilibrium internal rate of transformation between state claims is largely dictated by the physical cost structure.

When $J = 2$, and there is no redundant asset, the analogy with factor-price equalization is complete. In that case, the decisionmaker's cost of raising (r_1, r_2) in financial markets is given by

$$r_1 \left(\frac{v_1 A_{22} - v_2 A_{12}}{A_{22} - A_{21} A_{12}} \right) + r_2 \left(\frac{v_2 - v_1 A_{21}}{A_{22} - A_{21} A_{12}} \right),$$

which has linear isocost cost lines as illustrated in Figure 2b. Cost minimization, therefore, takes place at points where the slope of this isocost line is tangent to an isocost contour for $c(\mathsf{w}, \mathsf{z})$ and $\mathsf{r} + \mathsf{z} = \mathsf{y}$. The equilibrium slopes of the isocost curve and line correspond to the firm's equilibrium internal supporting state-claim prices. It is easy to see that if the isocost curve associated with $c(\mathsf{w}, \mathsf{z})$ is kinked at the equilibrium, as it would be with a stochastic production function specification, then even though the state-claim prices are uniquely determined in financial markets, the only useful information that can be inferred from the physical cost structure is the magnitude of z and the level of cost. In this case, which has been the basis of most theoretical and empirical analyses of financial and physical production interaction, the physical technology does not generally permit inferences about state-claim prices, other than to require that they lie somewhere in $co\{\mathsf{e}_1, .., \mathsf{e}_S\}$.

There are alternative, but equivalent, ways to define C, which allow for different intuitive interpretations and, which may prove analytically useful in differing contexts. Studies in the financial literature (e.g., Garmann and Ohlson, 1981; Prisman, 1986; and Ross, 1987) have investigated the properties of 'minimal investment functions' defined by the linear program

$$c^i(\mathsf{y}) = \min\{\mathsf{vh} : \mathsf{Ah} \geq \mathsf{y}\},$$

if $\{\mathsf{h} : \mathsf{Ah} \geq \mathsf{y}\}$ is nonempty and ∞ otherwise.

These minimal investment functions are equivalent to multiple-output cost functions for a linear production technology, well understood from the theory of production correspondences and duality.[8] In addition to satisfying homogeneity and concavity properties in v, they are positively linearly homogeneous and convex in y. And for $\mathsf{y} \in M$, they are linear.[9] Thus, they naturally form the basis for the linear pricing of assets defined on M. Prisman (1986), Ross (1987), Dermody and Prisman (1988), Dermody and Rockafellar (1991) and others have developed generalizations of c^i to cover financial frictions in the forms of transactions costs, taxes, bid-ask spreads etc. Clark (1993) has shown how to extend pricing derived from such linear asset valuations beyond the asset span.

One can thus redefine the derivative-cost function as

$$C(\mathsf{y}) = \min_{\mathsf{r},\mathsf{z}}\left\{c(\mathsf{w},\mathsf{z}) + c^i(\mathsf{r}) : \mathsf{r} + \mathsf{p}\cdot\mathsf{z} \geq \mathsf{y}\right\}.$$

Viewed in this fashion, C corresponds to a multiple-output cost function for an output-nonjoint technology, where production operations in the construction of $\mathsf{p}\cdot\mathsf{z}$ and r are independent, and interdependence between the two technologies only results from the firm's arbitrage activities in constructing y at minimal cost (Chambers, 1988). By restricting attention to a single state and setting $p_s = 1$, this cost minimization problem can be visualized

[8] The main difference is that the 'inputs', h, can be either positive or negative because of the ability to go short in frictionless markets.

[9] Linearity can be established by noting that so long as $\mathsf{y} \in M$, $c^{\mathsf{i}}(\mathsf{y}) = r^{\mathsf{i}}(\mathsf{y})$, where

$$r^{\mathsf{i}}(\mathsf{y}) = \max\{\mathsf{vh} : \mathsf{Ah} \leq \mathsf{y}\}.$$

$r^{\mathsf{i}}(\mathsf{y})$ is the firm's minimal selling price of the asset y and is equivalent mathematically to multiple-output revenue function for a linear technology. Hence, it is positively linearly homogeneous and concave in y by standard arguments. This establishes that on M, $c^{\mathsf{i}}(\mathsf{y})$ is both convex and concave. Hence, it must be linear.

in terms of the 'beaker diagram' illustrated in Figure 3, where the horizontal dimension of the beaker is y_s, the marginal production cost is graphed against the left axis, and the marginal cost of the financial holding is graphed agains the right axis. In equilibrium, the marginal costs must be equalized between financial and physical operations.

This observation illustrates a general point. The developments in this paper, the developments in Prisman (1986), Ross (1987), Dermody and Rockafellar (1991) and many others are all special cases of a more general income production problem where vectors of inputs/assets (which allow short selling) are used/purchased to assemble derivative assets using a generalized state-contingent production technology that encompasses the output-nonjoint form and the others as special cases. One can generalize axioms X to permit the possibility of short selling and then, by replacing z there by y, axiomatically derive a cost function for that technology along the general lines demonstrated by Chambers and Quiggin (2000).

The advantages of proceeding in this latter fashion are clear, increased generality and elegance. However, the cost is equally clear. As one moves to a more canonical state-contingent income production structure, one loses insight into the actual arbitrage activities undertaken by the firm, whether they are due to responses to the structure of financial assets, the presence of convex tax structures, transactions costs, or physical production opportunities.

3. Properties of the Derivative-Cost Function

At points in the following discussion, it will be useful to assume that no $\mathsf{y} \in \Re^S_+$ can have a cost that is arbitrarily negative. Formally, this assumption's role is to ensure that $C(\mathsf{y})$ is a proper convex function, which allows us to invoke standard results on subdifferentials and conjugate duality for convex functions (Rockafellar, 1970).

Assumption 1 $C(\mathsf{y}) > -\infty, \quad \mathsf{y} \in \Re^S_+$.

We list some basic properties of C. (Proofs not included in the text are in an appendix.)[10]

[10] C is a cost function, and thus it possesses standard properties in terms of the input prices (v, h). These can be gleaned from any good microeconomics text, and thus in the interest of notational economy and

Theorem 1. *C satisfies:*

1. $C(\mathsf{y})$ is a nondecreasing, convex function that is continuous on the interior of the region where it is finite. If $\mathsf{z}' \geq \mathsf{z}, \mathsf{z}' \neq \mathsf{z} \Rightarrow c(\mathsf{w}, \mathsf{z}') > c(\mathsf{w}, \mathsf{z})$, then $\mathsf{y}' \geq \mathsf{y}, \mathsf{y}' \neq \mathsf{y} \Rightarrow C(\mathsf{y}') > C(\mathsf{y})$.

2. If C is subdifferentiable at $\mathsf{y} \in ri\Re^S_+$, $\partial C(\mathsf{y}) \subset \Re^S_+$. If C is strictly monotonic and C is subdifferentiable at $\mathsf{y} \in \Re^S_{++}$, $\partial C(\mathsf{y}) \subset \Re^S_{++}$.

3. Under Assumption 1, if $C(\mathsf{y})$ is finite for any $\mathsf{y} \in ri\Re^S_+$, then C is subdifferentiable at y, and

$$C'(\mathsf{y}; \mathsf{z}) = \sup\{\mathsf{qz} : \mathsf{q} \in \partial C(\mathsf{y})\}$$

is finite for every z.

4. $C(\mathsf{0}) \leq 0$.

5. $C(\mathsf{y} + \boldsymbol{\delta}\mathsf{A}_j) = C(\mathsf{y}) + \delta v_j \quad \mathsf{y} + \boldsymbol{\delta}\mathsf{A}_j \in \Re^S_+$.

Theorem 1.1 ensures that we can invoke standard methods from convex analysis in analyzing C. In a sense, therefore, it motivates everything else that follows. Theorem 1.2 gives consequences of monotonicity for subdifferentials of C. Property 3 shows that C will be subdifferentiable at all points in the relative interior of the positive orthant. Thus, for any strictly positive derivative asset that can be constructed at finite cost, C is always subdifferentiable implying that there exist well defined supporting hyperplanes for the graph of C. As usual, these supporting hyperplanes have the natural interpretation as being the firm's virtual prices (supporting state-claim prices) for the state contingent incomes (state claims). Property 3 also relates the subdifferentials to the directional derivative of the cost function. Later developments will show that this representation, in conjunction with property 5, is key in determining virtual state-claim prices for assets which are not marketed. The fourth part of the theorem simply exploits the absence of fixed costs for the production technology to show that the firm would never pay a strictly positive price for a non-stochastic zero payoff in each state of Nature. It also demonstrates that the firm can never make a negative profit in its construction of the $\mathsf{0}$ asset, presuming of course that the latter is priced at zero. As we

parsimony, we do not discuss them here. But at points in the development, we use these properties when needed.

see below, a strict inequality here is equivalent to the presence of an arbitrage opportunity, and thus ruling out such a strict inequality forms a fundamental part of the argument in establishing the presence of no-arbitrage prices.

Theorem 1.5 shows that translating the derivative asset by some multiple of an existing financial asset just changes derivative-cost by the same multiple of the price of the asset. This a pre-requisite for the derivative asset to have been assembled efficiently. Suppose, for example, that $C\left(\mathsf{y}+\boldsymbol{\delta}\mathsf{A}_j\right) > C(\mathsf{y}) + \delta v_j$. The firm could assemble y at a cost of $C(\mathsf{y})$, and then purchase of δ units of A_j, giving the firm claim to $\mathsf{y}+\boldsymbol{\delta}\mathsf{A}_j$, but at a cost less than the minimal cost of constructing $\mathsf{y}+\boldsymbol{\delta}\mathsf{A}_j$. This contradicts the definition of the minimal cost function. On the other hand suppose that $C\left(\mathsf{y}+\boldsymbol{\delta}\mathsf{A}_j\right) < C(\mathsf{y}) + \delta v_j$. Then, by a parallel argument, it follows that the firm can assemble $\mathsf{y}+\boldsymbol{\delta}\mathsf{A}_j$ at a cost of $C\left(\mathsf{y}+\boldsymbol{\delta}\mathsf{A}_j\right)$, sell $\boldsymbol{\delta}\mathsf{A}_j$ and realize a return of y and a profit on the operation, which is again a contradiction.

We have the following immediate consequence of Theorem 1:

Corollary 2. $C'\left(\mathsf{y};\mathsf{A}_j\right) = v_j$, *and* $C(\mathsf{y})$ *is smooth in the direction of* A.

3.1. Virtual State-Claim Prices

Most of the paper is devoted to an analysis of what happens in the generalization of Figure 2a and not of Figure 2b. Our interest is largely focussed on what supporting state-claim prices look like when markets are not complete. Our next theorem provides a conjugate representation of C that relates the firm's virtual state-claim prices to the asset structure and to $c(\mathsf{w},\mathsf{z})$. To motivate this result, notice that the convex conjugate of C,

$$C^*(\mathsf{q}) = \sup_{\mathsf{y}} \{\mathsf{q}\mathsf{y} - C(\mathsf{y})\},$$

can be interpreted as the firm's virtual profit function for the state-claim prices q. Intuitively, it seems clear that this virtual profit function is unboundedly large if there exist any arbitrage opportunities in financial markets at state-claim prices q, that is $\mathsf{q}\mathsf{A} \neq \mathsf{v}$. When there are no such opportunities, then the firm's virtual profit is given by the maximal virtual profit realized from the production of z. More formally, for $\mathsf{q} \in \Re_+^S$

$$\begin{aligned}
C^*(\mathsf{q}) &= \sup_{\mathsf{y}} \{\mathsf{qy} - C(\mathsf{y})\} \\
&= \sup_{\mathsf{y}} \left\{ \mathsf{qy} - \min_{\mathsf{h,z}} \{c(\mathsf{w,z}) + \mathsf{vh} : \mathsf{Ah} + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y}\} \right\} \\
&= \sup_{\mathsf{y,h,z}} \{\mathsf{qy} - \mathsf{c}(\mathsf{w,z}) - \mathsf{vh} : \mathsf{Ah} + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y}\} \\
&= \sup_{\mathsf{h,z}} \{\mathsf{q}(\mathsf{Ah} + \mathsf{p} \cdot \mathsf{z}) - \mathsf{c}(\mathsf{w,z}) - \mathsf{vh}\} \\
&= \begin{cases} \infty & \mathsf{qA} \neq \mathsf{v} \\ c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p}) & \mathsf{qA} = \mathsf{v} \end{cases}.
\end{aligned}$$

For any $\mathsf{q}^* \in \arg\sup \{\mathsf{qy} - C^*(\mathsf{q})\}$,

$$\mathsf{q}^*\mathsf{y} - C^*(\mathsf{q}^*) \geq \mathsf{qy} - C^*(\mathsf{q}),$$

so that $\mathsf{y} \in \partial C^*(\mathsf{q}^*)$ and by (4), $\mathsf{q}^* \in \partial C(\mathsf{y})$.

We conclude by conjugacy.

Theorem 3. *Under Assumption 1 for* $\mathsf{y} \in ri\Re_+^S$, $C(\mathsf{y}) = \sup_{\mathsf{q}} \{\mathsf{qy} - c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p}) : \mathsf{qA} = \mathsf{v}\}$.

Before we say what this theorem implies more formally, let's first see what it is saying intuitively. Consider the associated Lagrangean expression in the case of a smooth technology

$$L = \mathsf{qy} - c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p}) - (\mathsf{qA} - \mathsf{v})\boldsymbol{\mu},$$

where $\boldsymbol{\mu} \in \Re_+^J$ is a vector of Lagrangean multipliers. First-order conditions here include

$$\mathsf{y} - \boldsymbol{\nabla} c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p}) \cdot \mathsf{p} - \mathsf{A}\boldsymbol{\mu} = 0,$$

where $\boldsymbol{\nabla} c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p})$ is the gradient of $c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p})$ in $\mathsf{q} \cdot \mathsf{p}$. By basic duality theory, $\boldsymbol{\nabla} c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p})$ is the vector of virtual profit maximizing supplies of state-contingent outputs. By standard optimization theory, $\boldsymbol{\mu}$ corresponds to the vector of marginal costs of $C(\mathsf{y})$ with respect to v. But notice that from our original formulation of the derivative-cost problem and Shephard's lemma, these marginal costs of $C(\mathsf{y})$ with respect to v just correspond to the optimal positions, h, taken in the financial market. Thus, rewriting the derivative-cost problem in this fashion is intuitively equivalent to recasting the problem into one of locating the virtual state-claim prices and financial market positions for which the firm's supply of the derivative asset just equals its demand, as given by y.

If $\mathsf{q} \in \partial C(\mathsf{y})$, then two conditions must hold. First, these virtual state-claim prices must allow no virtual profit from financial operations. Because the asset structure effectively exhibits constant returns to scale between the firm's position in financial markets and payouts, virtual profit is either zero or infinitely large. If such a profit existed, the firm could always create the asset associated with the position, sell it, and then use the proceeds to finance the purchase of y. Because this could be repeated an infinite number of times, the cost of assembling y would thus be driven to $-\infty$. Second, the firm's cost minimizing choice of z in creating the derivative asset, y , must also maximize virtual profit in terms of the virtual state-claim prices. This observation echoes Working's (1953) long-ago assertion that the primary reason for a productive firm (in his case millers) engaging in financial hedging activity (in his case wheat futures) was not risk avoidance, but profit enhancement. By this last observation, if $\mathsf{q} \in \partial C(\mathsf{y})$, then $\mathsf{q} \cdot \mathsf{p} \in \partial c(\mathsf{w}, \mathsf{z})$ at any z which solves the derivative-cost problem.

If $c(\mathsf{w}, \mathsf{z})$ is strictly convex, then maximal virtual profit is finite. Consider, however, the case where the state-contingent production technology exhibits constant returns to scale so that $c(\mathsf{w}, \mu\mathsf{z}) = \mu c(\mathsf{w}, \mathsf{z})$, $\mu > 0$. Then either $c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p})$ is zero or it is unboundedly large. Suppose, for example, that $(\mathsf{q} \cdot \mathsf{p})\mathsf{z} - c(\mathsf{w}, \mathsf{z}) > 0$. Then for these virtual state-claim prices, the firm can always ensure itself of creating the asset y at a virtual profit of infinity, which is equivalent to having produced it at a cost of minus infinity. The virtual state-claim prices, therefore, must adjust to eliminate this virtual profit. More formally, we conclude:

Corollary 4. *Under Assumption 1 for* $\mathsf{y} \in ri\Re^S_+$, *if* $c(\mathsf{w}, \mu\mathsf{z}) = \mu c(\mathsf{w}, \mathsf{z})$, $\mu > 0$, *then* $C(\mathsf{y}) = \sup_{\mathsf{q}} \{\mathsf{q}\mathsf{y} : c^*(\mathsf{w}, \mathsf{q} \cdot \mathsf{p}) = 0, \mathsf{q}\mathsf{A} = \mathsf{v}\}$.

3.2. Virtual Risk-Free Rates and Virtual Asset Prices

Suppose that A contains a riskless asset, offered for a price of 1 with a riskless rate of r. By Corollary 2

$$C'(\mathsf{y}; (1+r)\mathsf{1}) = 1 = \sup\{(1+r)\mathsf{q}\mathsf{1} : \mathsf{q} \in \partial C(\mathsf{y})\}.$$

The marginal cost of a sure increase in y of $(1+r)$ dollars is one dollar. Because the firm can always realize such a sure increase in y by purchasing one unit of the safe asset, this condition is necessary for the firm to have exhausted all its arbitrage opportunities in constructing the derivative asset. Theorem 3 implies in this instance that for all $\mathsf{q} \in \partial C(\mathsf{y})$, $(1+r)\,\mathsf{q}$ can be interpreted as *risk-neutral probabilites.*

On the other hand, even when the riskless asset is not actively traded, the firm's marginal cost of a sure increase in y of $t > 0$ dollars is $C'(\mathsf{y};t\mathsf{1}) = tC'(\mathsf{y};\mathsf{1})$, where we have used the linear homogeneity of the directional derivative. This leads us to define the firm's virtual risk-free rate

$$r^{+}(\mathsf{y}) = \frac{1}{C'(\mathsf{y};\mathsf{1})} - 1,$$

as the risk-free rate which yields the firm no opportunity for profitable arbitrage between production operations and purchasing one unit of the riskless asset if it were traded. Symmetrically, when the riskless asset is not traded then the risk-free rate which yields the firm no opportunity for arbitrage if one unit of the riskless asset were constructed and sold would be

$$\begin{aligned} r^{-}(\mathsf{y}) &= \frac{-1}{C'(\mathsf{y};-\mathsf{1})} - 1 \\ &= \frac{1}{\inf\{\mathsf{q}\mathsf{1} : \mathsf{q} \in \partial C(\mathsf{y})\}} - 1 \end{aligned}$$

By the properties of directional derivatives $r^{+}(\mathsf{y}) \geq r^{-}(\mathsf{y})$. If the riskless asset is traded with a rate of r, then by Corollary 2 $r^{+}(\mathsf{y}) = r = r^{-}(\mathsf{y})$.

More generally, directional derivatives offer means for computing virtual asset prices, whether the assets are traded or not. By Corollary 2 when A_j is traded

$$\begin{aligned} C'\left(\mathsf{y};\mathsf{A}_j\right) &= \sup\left\{\mathsf{q}\mathsf{A}_j : \mathsf{q} \in \partial C(\mathsf{y})\right\} \\ &= v_j. \end{aligned}$$

If A_j were not actively traded, then $C'\left(\mathsf{y};\mathsf{A}_j\right)$ corresponds to the maximal price the firm should be willing to pay to acquire one unit of it. Thus, it represents an appropriate virtual price of the firm for this asset.

These virtual asset prices inherit several very attractive properties of directional derivatives. First, they are linearly homogeneous indicating that simply renumbering the units in which returns for assets are denominated renumbers virtual prices in a matching fashion. Second, they are convex, which, when coupled with linear homogeneity, implies that they are subadditive, i.e.,

$$C'(\mathsf{y};\mathsf{y}^o+\mathsf{y}^*)\leq C'(\mathsf{y};\mathsf{y}^o)+C'(\mathsf{y}';\mathsf{y}^*).$$

As noted above in the case of the riskless asset, when an asset is not actively traded in the market, there is no reason to expect that the virtual price that the firm would be willing to pay to add one unit of it to its portfolio is equivalent to the minimal price that the firm would be willing to accept in divesting itself of one unit of the asset. This minimally acceptable virtual price for y^o is

$$\inf\{\mathsf{q}\mathsf{y}^o\colon \mathsf{q}\in\partial C(\mathsf{y})\},$$

which under the conditions of Theorem 1 corresponds to $-C'(\mathsf{y};-\mathsf{y}^o)$. To distinguish $-C'(\mathsf{y};-\mathsf{y}^o)$ from $C'(\mathsf{y};\mathsf{y}^o)$, we refer to the former as the virtual selling price. By the basic properties of directional derivatives for convex functions, $-C'(\mathsf{y};-\mathsf{y}^o)$ is linearly homogeneous and concave, and therefore superadditive, in y^o with $-C'(\mathsf{y};-\mathsf{y}^o)\leq C'(\mathsf{y};\mathsf{y}^o)$. When these virtual prices coincide, as they would in the presence of a marketed asset, $C'(\mathsf{y};\mathsf{y}^o)$ is smooth in the direction of y^o.

3.3. Production and the Efficient Set

Theorem 3 has implications for $\partial c(\mathsf{w},\mathsf{z})$, and the related cost minimizing production equilibrium. We start by supposing that the riskless asset is actively traded, then for $\mathsf{q}\in\partial C(\mathsf{y})$

$$\mathsf{q}\cdot\mathsf{p}\in\partial c(\mathsf{w},\mathsf{z}), \tag{6}$$

at the cost minimizing z, and (2) gives for this z

$$c'\left(\mathsf{w},\mathsf{z};(1+r)\frac{\mathsf{1}}{\mathsf{p}}\right)\geq \mathsf{q}\mathsf{1}=1. \tag{7}$$

The latter condition shows that the marginal cost of a sure increase in y of $(1+r)$ dollar via a change in the state-contingent production vector can never be less than a dollar at the optimal choice of z. If this condition did not hold, then the firm could always sell one unit of its current holding of the riskless asset and replace it with a 'riskless asset' of its own construction, thus lowering its total cost.

Theorem 3 and expressions (6) and (7) admit another interpretation in terms of production practices for expected-profit maximizing individuals. Rewriting expression (6), the cost minimizing z must satisfy

$$(1+r)\,\mathsf{q}\cdot\frac{\mathsf{p}}{1+r}\in\partial c\,(\mathsf{w},\mathsf{z})\,,$$

so that the cost minimizing z must be consistent with the state-contingent output choice of an expected profit maximizer (in current dollars), who faced the probability vector $(1+r)\,\mathsf{q}$, which corresponds to the 'risk-neutral probabilities' defined earlier. Expression (7), in these terms, then manifests the equilibrium condition that such an individual would never miss an opportunity to raise virtual profit with certainty. Following this interpretation, Chambers and Quiggin (2000) have defined the set of $\mathsf{z}'s$, which satisfy (7), as the *efficient set*, and noted that its members consist of precisely those state-contingent output vectors which could be optimal for an expected profit maximizing risk-neutral individual.[11]

More generally, even if the riskless asset is not actively traded, by Theorem 3, it remains true that

$$\begin{gathered}\left(1+r^{+}\,(\mathsf{y})\right)\mathsf{q}\cdot\frac{\mathsf{p}}{1+r^{+}\,(\mathsf{y})}\in\partial c\,(\,\mathsf{w},\mathsf{z})\\ c'\left(\mathsf{w},\mathsf{z};\frac{1}{\mathsf{p}}\right)\geq\frac{1}{1+r^{+}\,(\mathsf{y})}.\end{gathered}\tag{8}$$

Hence,

Corollary 5. *Under Assumption 1, if $C\,(\mathsf{y})$ is finite for $\mathsf{y}\in\mathsf{ri}\Re^{S}_{+}$, then the cost minimizing choice of z must be in the efficient set for $r^{+}\,(\mathsf{y})$.*

For certain classes of technologies, these observations can yield even further information about the range of potentially cost minimizing state-contingent outputs. For example, a cost function exhibits translation homotheticity in the direction of $\mathsf{g}\in\Re^{S}_{+}$ in state-contingent

[11] Chambers and Quiggin (2000) considered individuals whose preferences were defined over net returns, and thus they implicitly considered the case where $r=0$.

outputs (Chambers and Färe, 1998) if

$$c(\mathsf{w}, \mathsf{z}) = \hat{c}(\mathsf{w}, T(\mathsf{z}, \mathsf{w})),$$

where $T : \Re^S_+ \times \Re^N_+ \rightarrow \Re_+$ is nondecreasing and continuous in z, and

$$\begin{aligned} T(\mathsf{z}, \lambda \mathsf{w}) &= T(\mathsf{z}, \mathsf{w}) \quad \lambda > 0 \\ T(\mathsf{z} + \delta \mathsf{g}, \mathsf{w}) &= T(\mathsf{z}, \mathsf{w}) + \delta \quad \delta \in \Re. \end{aligned}$$

Visually, a translation homothetic cost structures possesses isocost curves that have equal slope as one proceeds in the direction g. By definition,

$$T'(\mathsf{z}, \mathsf{w}; \mathsf{g}) = 1.$$

Supposing for simplicity that $\hat{c}$ is smooth and convex in T, we then have for this class of technologies that

$$c'(\mathsf{w}, \mathsf{z}; \mathsf{g}) = \hat{c}_T(\mathsf{w}, T).$$

Applying these facts to what we have already learned, it follows immediately that if there exists a traded riskless asset with a risk-free rate of r, and c is translation homothetic in the direction $\frac{1}{\mathsf{p}}$,[12] then the boundary of the efficient set is given by

$$c'\left(\mathsf{w}, \mathsf{z}; \frac{1}{\mathsf{p}}\right) = \hat{c}_T(\mathsf{w}, T) = \frac{1}{1+r},$$

and convexity of $\hat{c}$ then implies that an unique T corresponds to the boundary of the efficient set. We thus conclude:

Corollary 6. *If $c(\mathsf{w}, \mathsf{z})$ is translation homothetic in the direction of $\frac{1}{\mathsf{p}}$,, all elements of the efficient set are equally costly.*

Translation homotheticity, in effect, implies that there exists an aggregate output, T. This aggregate output has the property that movements in the direction of g just increase it one unit for each unit of g that is added. That is, the marginal cost of moving the associated

[12] Because such technologies possess isocost curves whose slope does not change as one raises income by the same sure amount in all states of nature, Chambers and Quiggin (2000), in analogy with the literature on utility structures, refer to such technologies as being constant absolutely risky.

output bundle in the direction of g equals one. Therefore for such structures, finding the efficient set just reduces to finding the aggregate output where the marginal cost of increasing T is just equal to the reciprocal of the return on the risk-free asset.

Another implication of Theorem 3 is the natural extension of the discussion surrounding Figure 2b to the case of incomplete financial markets. If the technology is suitably smooth, its virtual state-claims prices can be deduced from $\partial c(\mathsf{w}, \mathsf{z})$. Because the firm's technology is, in principle, both observable and measurable via econometric analysis, this observation offers a method for computing firms' virtual state-claim prices, and hence virtual asset prices, directly from estimated versions of the stochastic production technology.

So, for example, if the riskless asset is not actively traded, expression (8) can be used to place a lower bound on the virtual risk-free rate. As we have shown above, these calculations become particularly simple when the cost technology is translation homothetic in the direction of $\frac{1}{\mathsf{p}}$, suggesting that translation homotheticity might offer a particularly parsimonious parametric specification for measuring virtual risk-free rates. Similarly, the firm's virtual price for the asset y^* has upper bound

$$c'\left(\mathsf{w}, \mathsf{z}; \frac{\mathsf{y}^*}{\mathsf{p}}\right).$$

If, for example, the objective is to create a virtual price for a specific non-marketed asset, then by arguments parallel to those above, translation homotheticity in the direction of $\frac{\mathsf{y}^*}{\mathsf{p}}$ may offer a particularly tractable specification. In the case where the state-contingent technology is suitably smooth, as it will be in most econometric applications, these bounds turn into equalities. Naturally, as above, we can construct lower bounds for the virtual selling prices as $-c'\left(\mathsf{w}, \mathsf{z}; -\frac{\mathsf{y}^*}{\mathsf{p}}\right)$.

However, this approach is generally less useful, when one restricts attention to the stochastic production function specification of a state-contingent technology, as the following example illustrates.

Example 7. *Consider*

$$c(\mathsf{w}, \mathsf{z}) = \hat{c}\left(\mathsf{w}, \max\left\{\frac{z_1}{\varepsilon_1}, ..., \frac{z_S}{\varepsilon_S}\right\}\right),$$

which corresponds to the a stochastic production function with a multiplicative error structure (Chambers and Quiggin, 2002). Presuming that $\hat{c}$ is smooth in its arguments and that *the decisionmaker preferences are strictly monotonic in both cost and state-contingent returns (see our discussion of no-arbitrage below), the decisionmaker always chooses to locate at* z *such that*

$$z_s = \frac{\varepsilon_s}{\varepsilon_1} z_1,$$

for all s*, where*

$$\partial c(\mathsf{w}, \mathsf{z}) = c_2\left(\mathsf{w}, \frac{z_1}{\varepsilon_1}\right) co\{\mathsf{e}_1, .., \mathsf{e}_S\},$$

is noninformative about the relative state-claim prices.

Upper bounds on the virtual price of an asset, y*, however, can be obtained from*

$$c'(\mathsf{w}, \mathsf{z}; \mathsf{y}) = c_2\left(\mathsf{w}, \frac{z_1}{\varepsilon_1}\right) \max\left\{\frac{y_1}{p_1}, ..., \frac{y_S}{p_S}\right\},$$

and lower bounds from

$$-c'(\mathsf{w}, \mathsf{z}; -\mathsf{y}) = c_2\left(\mathsf{w}, \frac{z_1}{\varepsilon_1}\right) \min\left\{\frac{y_1}{p_1}, ..., \frac{y_S}{p_S}\right\}.$$

Even in the presence of a marketed asset, these bounds will not generally coincide.

More generally, in the stochastic production function specification, if x *is held fixed, we have that the bounding hyperplanes of the set*

$$\{\mathsf{z} : z_s \leq f(\mathsf{x}, \varepsilon_s), \quad s \in \Omega\}$$

correspond to $co\{\mathsf{e}_1, .., \mathsf{e}_S\}$*, so that information about relative state-claim prices can be recaptured from the primal technology only by placing additional structure upon the technology. At the poles, assuming that the dimension of the input space is the same as the dimension of the state space allows one to infer the supporting state-claim prices, while as illustrated earlier in the section on the stochastic production function, nothing can be inferred in the case of a single input or an aggregate input. Because most empirical applications will not be consistent with the state space being spanned by the input space, the stochastic production function generally is an inconvenient specification for capturing information about state-claim prices.*

A number of related observations follow immediately when there is further information available upon the structure of asset returns. For example, suppose that financial markets are complete in the sense that rank $\mathsf{A} = S$. Then the right inverse of A exists and is given by

$$\mathsf{A}^R = \mathsf{A}'\left(\mathsf{A}\,\mathsf{A}'\right)^{-1},$$

and by Theorem 3, $\mathsf{q} \in \partial C(\mathsf{y}) \Rightarrow \mathsf{q} = \mathsf{v}\mathsf{A}^R$, and z optimal for (5) satisfies

$$\mathsf{z} \in \arg\sup\left\{\left(\mathsf{v}\mathsf{A}^R\right)(\mathsf{p}\cdot\mathsf{z}) - c(\mathsf{w},\mathsf{z})\right\},$$

which simply reconfirms that in the presence of complete financial markets, and thus complete state-claims markets, production equilibrium maximizes the surplus given by these state-claim prices.

However, there will exist cases of interest where markets are not necessarily complete, but for which further information is available about the relationship between asset payoffs and their prices. We state two immediate consequences of Theorem 3

Corollary 8. *Under Assumption 1, suppose that there exists a vector of objective probabilities,* $\boldsymbol{\pi}$*, such that* $\boldsymbol{\pi}\mathsf{A} = \mathsf{v}(1+r)$. *Then either* $\boldsymbol{\pi}/(1+r) \in \partial C(\mathsf{y})$ *or* $\mathsf{q} \in \partial C(\mathsf{y}) \Rightarrow \boldsymbol{\pi}/(1+r) - \mathsf{q} \in M^{\perp}$. *For any for any* $\boldsymbol{\pi} \in \Re^S$ *such that* $\boldsymbol{\pi}\mathsf{A} > (<)\, \mathsf{v}(1+r)$, $\mathsf{q} \in \partial C(\mathsf{y}) \Rightarrow \left(\frac{\pi}{1+r} - \mathsf{q}\right)\mathsf{A} > (<)\, \mathsf{0}$.

This corollary addresses the case where financial markets are 'fair' in the sense of offering a uniform expected return on all assets for some objective set of probabilities. In this case, we can strengthen our earlier results on expected profit maximizing choices to conclude that either the firm makes it choices of cost minimizing z as though it were a current period expected profit maximizer for these objective probabilities, or it makes its cost minimizing choice of z as though it were a current period expected profit maximizer for the state-claim prices obtained by projecting $\frac{\pi}{(1+r)}$ onto M.

More generally, because $\mathsf{q}\cdot\mathsf{p} \in \partial c(\mathsf{w},\mathsf{z})$ for the cost minimizing z, $\left(\frac{\pi}{1+r} - \mathsf{q}\right)$ measures the firm's departure from risk-neutral production practices (for the probabilities $\boldsymbol{\pi}$) in its choice of z to create the derivative asset y. In the presence of a fair market these divergences must be orthogonal to the asset span. When the market is not fair in this sense, the direction of its unfairness then determines how divergences from risk-neutral production behavior will match with the structure of the asset market.

4. Arbitrage and Separation

The primary use of minimum investment functionals, c^i, has been in the theory of linear valuation operators for financial assets under different market structures. In the simplest case, where there are no market frictions, $c^i(\mathsf{y})$ is linear on the span of the market (Prisman, 1986; Ross, 1987; Clark, 1993) and positively linearly homogenous and sub-additive (convex, sub-linear) elsewhere. This linearity, in turn, guarantees the existence of nonnegative, state-claim prices, corresponding to the subdifferential of $c^i(\mathsf{y})$ at the origin. When convex market frictions are introduced, $c^i(\mathsf{y})$ becomes nonlinear but remains convex. However, to ensure the absence of arbitrage, there must exist a positively linearly homogeneous and sub-linear valuation providing a lower bound for $c^i(\mathsf{y})$. Thus, the absence of arbitrage in the presence of frictions also requires the existence of non-negative state-claim prices (Prisman, 1986; Ross, 1987; Dermody and Prisman, 1988; Dermody and Rockafellar, 1991; Clark, 1993; Jouini and Kallal, 1995 (a,b); Jouini, Kallal, and Napp, 2001).

Much attention has also been focused on isolating circumstances in which production decisions are independent of attitudes toward risk (Townsend, 1978; Danthine, 1978; Holthausen, 1979; Anderson and Danthine, 1981; Chambers and Quiggin, 1997). When this happens, it is said that there exists 'separation' between the decisionmakers production decisions and his risk attitudes, and thus his hedging activities. It is a textbook result (Hirshleifer and Riley, 1992) that this occurs in the presence of complete markets (as well as a trivial consequence of Corollary 8 above).

In this section, we show how the derivative-cost function can be used to generalize existing results on no-arbitrage pricing to the case of productive firms and to derive necessary and sufficient conditions for separation to occur.

4.1. Arbitrage

A *strong production-financial arbitrage* exists at (h,z) if there is $(\mathsf{h}',\mathsf{z}')$ such that

$$\mathsf{vh}' + c(\mathsf{w},\mathsf{z}') < \mathsf{vh} + c(\mathsf{w},\mathsf{z}),\; \mathsf{Ah}' + \mathsf{p}\cdot\mathsf{z}' \geq \mathsf{Ah} + \mathsf{p}\cdot\mathsf{z}.,$$

or in words a strictly cheaper portfolio and production plan yielding at least the same income in each state of Nature as (z,h). If the firm's objective function is nondecreasing in state claims, the existence of a strong production-financial arbitrage at (z, h) is inconsistent with (z, h) occuring in equilibrium. This local notion of an arbitrage encompasses, as a special case, the more traditional notion of a strong financial arbitrage, which is the existence of an h^* such that

$$\mathsf{vh}^* < 0, \mathsf{Ah}^* \geq 0,$$

by setting $\mathsf{z}' = \mathsf{z}$ and taking $\mathsf{h}^* = \mathsf{h}' - \mathsf{h}$. It also incorporates, as a special case, the notion of a strong production arbitrage at z, which is the existence of a strictly cheaper state-contingent output vector, which yields no lower return in any state of Nature,

$$c(\mathsf{w}, \mathsf{z}') < c(\mathsf{w}, \mathsf{z}), \mathsf{p} \cdot \mathsf{z}' \geq \mathsf{p} \cdot \mathsf{z}.$$

Hence, if no strong-production financial arbitrage exists, then no strong financial arbitrage exists and no strong production arbitrage exists. Moreover, if (z, h) is cost minimizing for $\mathsf{y} = \mathsf{Ah} + \mathsf{p} \cdot \mathsf{z}$, the absence of a strong production-financial arbitrage requires that there exist no strong arbitrage at y in the form of a strictly cheaper y' with returns as large in each state of Nature,

$$C(\mathsf{y}') < C(\mathsf{y}), \mathsf{y}' \geq \mathsf{y}.$$

Theorem 1 ensures that a strong arbitrage of this form can never exist at any y. Firms constructing their derivative assets at minimal cost, that is rational firms, automatically eliminate such strong production-financial arbitrages. Hence, applying the no-arbitrage condition to such points brings no further information than that already embedded in $C(\mathsf{y})$.

The absence of a strong production-financial arbitrage at (v, h) guarantees the existence of nonnegative state-claim prices.

Theorem 9. *If there is no strong production-financial arbitrage at* (z, h), *there exists a* $\mathsf{q} \in \Re^S_+$ *such that*

$$\begin{aligned} \mathsf{qA} &= \mathsf{v}, \\ (\mathsf{q} \cdot \mathsf{p})\,\mathsf{z} - c(\mathsf{w}, \mathsf{z}) &\geq (\mathsf{q} \cdot \mathsf{p})\,\mathsf{z}' - c(\mathsf{w}, \mathsf{z}'), \quad \mathsf{p} \cdot \mathsf{z}' \geq \mathsf{p} \cdot \mathsf{z}. \end{aligned}$$

These no-arbitrage prices ensure that virtual profit in financial markets is zero, and that virtual marginal profit associated with moving from z to z' is nonpositive. Hence, for any rational y constructed at minimal cost, the Theorem implies that state-claim prices, q, exist and satisfy

$$\mathsf{q} \cdot \mathsf{y} - C(\mathsf{y}) \geq \mathsf{q} \cdot \mathsf{y}' - C(\mathsf{y}'), \quad \mathsf{y}' \geq \mathsf{y}.$$

Because all the members of $\partial C(\mathsf{y})$ satisfy this inequality for all y, they are state-claim prices consistent with the absence of a strong production-financial arbitrage. Thus, the no-arbitrage risk free rate

$$r^+(\mathsf{y}) = \frac{1}{C'(\mathsf{y};\mathsf{1})} - 1,$$

and the no-arbitrage price of the asset y^o, $C'(\mathsf{y};\mathsf{y}^o)$, can be taken as the virtual risk-free rate and virtual asset price at y. These no-arbitrage asset prices are positively linearly homogeneous and convex (sub-additive) by the properties of directional derivatives. The no-arbitrage selling price of the asset is now $-C'(\mathsf{y};-\mathsf{y}^o)$. When y^o is in the span of the market, by previous results $C'(\mathsf{y};\mathsf{y}^o) = -C'(\mathsf{y};-\mathsf{y}^o)$ implying that the no-arbitrage pricing operator is smooth in the direction of the asset structure.

One result of particular interest emerges from considering the case $(\mathsf{h},\mathsf{z}) = \mathsf{0}$. An immediate consequence of the absence of an arbitrage there is that there exists no $\mathsf{y} \geq \mathsf{0}$ such that $C(\mathsf{y}) < 0$. This observation shows that Assumption 1 must be satisfied, and in conjunction with Theorem 1 proves that the decisionmaker cannot realize a strictly positive profit from constructing the zero asset and selling it for a price of zero.

Theorem 10. *The absence of a strong production-financial arbitrage at* $(\mathsf{h},\mathsf{z}) = \mathsf{0}$ *implies* $C(\mathsf{0}) = 0$.

A *production-financial arbitrage* exists for the firm at (h,z) if there exists an $(\mathsf{h}',\mathsf{z}')$ such that

$$\mathsf{v}\mathsf{h}' + c(\mathsf{w},\mathsf{z}') \leq \mathsf{v}\mathsf{h} + c(\mathsf{w},\mathsf{z}), \mathsf{A}\mathsf{h}' + \mathsf{p}\cdot\mathsf{z}' \geq \mathsf{A}\mathsf{h} + \mathsf{p}\cdot\mathsf{z}$$

with at least one strict inequality. Such an arbitrage is inconsistent with equilibrium if the decisionmaker preferences are strictly increasing in y.

This notion of an arbitrage can be used to generate no-arbitrage state-claim prices, and it is a relatively easy extension of Theorem 9 to demonstrate that these prices must be strictly positive. These state-claim prices then yield no-arbitrage virtual asset prices and risk-free rates just as above. We leave the details of this to the reader, and turn our attention instead to the implications of the absence of a production-financial arbitrage for $C(\mathsf{y})$. If (h, z) is cost minimizing, then the absence of a production financial arbitrage requires that there is no $\mathsf{y}' \geq \mathsf{y}$, $\mathsf{y}' \neq \mathsf{y}$, such that $C(\mathsf{y}') \leq C(\mathsf{y})$. This implies:

Theorem 11. *If (z, h) is cost minimizing for y, and there is no production-financial arbitrage at (z, h), then $C(\mathsf{y})$ is strictly increasing in all its arguments at y.*

Example 12. *This example illustrates the difference between a strong production-financial arbitrage and a production-financial arbitrage, and the different implications they have for asset pricing and monotonicity of $C(\mathsf{y})$. Consider the cost structure*

$$c(\mathsf{w}, \mathsf{z}) = \hat{c}(\mathsf{w}) \max\left\{\frac{z_1}{\varepsilon_1}, \frac{z_2}{\varepsilon_2}\right\},$$

and suppose that output price is nonstochastic and normalized for convenience to one. As Figure 4 illustrates, this cost structure is nonsubstitutable in state-contingent outputs (it corresponds to a linearly homogeneous state-contingent production function with multiplicative effort). For any z such that

$$z_2 > \frac{\varepsilon_2}{\varepsilon_1} z_1,$$

$\mathsf{z}' = z_2\left(\frac{\varepsilon_1}{\varepsilon_2}, 1\right)$ represents a production arbitrage but not a strong production arbitrage. Because $c(\mathsf{w}, \mathsf{z}') = c(\mathsf{w}, \mathsf{z})$. $C(\mathsf{y})$ is only weakly monotonic, and the supporting state-claim prices are given by $\partial c(\mathsf{w}, \mathsf{z}) = \frac{\hat{c}(\mathsf{w})}{\varepsilon_2}\mathsf{e}_2$, which is nonnegative but not strictly positive. At z', there exists no production arbitrage and the supporting state-claim prices are given by $\partial c(\mathsf{w}, \mathsf{z}') = \hat{c}(\mathsf{w})\, co\left(\frac{\mathsf{e}_1}{\varepsilon_1}, ..., \frac{\mathsf{e}_S}{\varepsilon_S}\right)$, which contains strictly positive elements.

It is well-known, that the existence of a linear valuation implies that the valuation can be expressed as an expectation for a particular martingale (Harrison and Kreps, 1979; Cox, Ingersoll, and Ross, 1985). Clark (1993) has demonstrated the formal linkages between linear valuations derived from the absence of arbitrage and the subjective probability theory of de Finetti (1937). A similar connection exists here:

Corollary 13. *There are no production-financial arbitrage opportunities at y if and only if there exists an expectations operator E such that*

$$C(\mathsf{y}) \geq \left(1 + r^{+}(\mathsf{y})\right) \mathsf{E}\mathsf{y}, \mathsf{y} \in \Re^S_+$$

with

$$\left(1+r^{+}\left(\mathsf{y}\right)\right)E\mathsf{A}_{j}=v_{j},\quad j=1,...,J,.$$

4.2. Separation

The question of when the firm's output will be independent of the preferences of its owners (assuming only that these preferences are monotonic) has arisen in a variety of contexts. For example, as mentioned above, this happens when there exist complete markets. It is legitimate to ask, therefore, whether other circumstances may exist in which, from the firm's perspective as a producer, markets are effectively complete in the sense that all firms facing the same technology and the same financial market structure will make the same production decisions. Because the output decisions of the firm generally depend on the decisionmaker's risk attitudes, this would be an important step forward in removing some of the ambiguity surrounding such decisions. This section provides necessary and sufficient conditions for separation for general technologies and frictionless financial markets.

We start by noting that any z that is produced must belong to

$$\left\{\mathsf{z}':c^{i}\left(\mathsf{p}\cdot\mathsf{z}\right)\geq c\left(\mathsf{w},\mathsf{z}\right)\right\},$$

because for any z not in this set, the decisionmaker is always better off assembling $\mathsf{p}\cdot\mathsf{z}$ in financial markets. This observation leads us to the crudest kind of separation result. Namely, if this set is empty, then the decisionmaker's production decisions are always independent of his risk attitudes, because all rational decisionmakers would operate at $\mathsf{z}=\mathsf{0}$, regardless of the magnitude of y. This is the case, where the technology is entirely redundant in the presence of asset markets.

We now seek instances other than complete production redundancy where production decisions are independent of the decisionmaker's risk attitudes. In considering cost minimizing production choices, Theorem 3 allows us to restrict attention to the set

$$Z^{*}=\cup_{\mathsf{q}}\left\{\mathsf{z}:\mathsf{z}\in\partial c^{*}\left(\mathsf{w},\mathsf{q}\cdot\mathsf{p}\right)\right\}.$$

Z^{*} corresponds to the efficient set for all possible $r^{+}\left(\mathsf{y}\right)$.

We say that *separation exists over a set* Y, if for all $\mathsf{y} \in Y$ the decisionmaker's cost minimizing production choices satisfy

$$\mathsf{z} \in \partial c^{*}\left(\mathsf{w}, \mathsf{q} \cdot \mathsf{p}\right), \tag{9}$$

for the same q. In other words, for all $\mathsf{y} \in Y$, $\mathsf{q} \in \partial C\left(\mathsf{y}\right)$. Consider the set obtained by translating the span of the market by the *ex ante* values of the cost minimizing production choices:

$$Y^{*} = M + \mathsf{p} \cdot Z^{*}.$$

By the basic properties of c^{i}, $c^{i}\left(\mathsf{r}\right) = \mathsf{q}^{M}\mathsf{r}$, $\mathsf{r} \in M$. Thus, for $\mathsf{y} \in Y^{*}$:

$$\begin{aligned} C\left(\mathsf{y}\right) &= \min_{\mathsf{z} \in Z^{*}, \mathsf{r}} \left\{ c\left(\mathsf{w}, \mathsf{z}\right) + c^{i}\left(\mathsf{r}\right) : \mathsf{p} \cdot \mathsf{z} + \mathsf{r} \geq \mathsf{y} \right\} \\ &= \min_{\mathsf{z} \in Z^{*}} \left\{ c\left(\mathsf{w}, \mathsf{z}\right) + c^{i}\left(\mathsf{y} - \mathsf{p} \cdot \mathsf{z}\right) \right\} \\ &= \min_{\mathsf{z} \in Z^{*}} \left\{ \mathsf{q}^{M}\mathsf{y} - \mathsf{q}^{M}\left(\mathsf{p} \cdot \mathsf{z}\right) + c\left(\mathsf{w}, \mathsf{z}\right) \right\} \\ &= \mathsf{q}^{M}\mathsf{y} - c^{*}\left(\mathsf{w}, \mathsf{q}^{M} \cdot \mathsf{p}\right), \end{aligned}$$

implying separation over $M + \mathsf{p} \cdot Z^{*}$.

Suppose that separation exists over Y, and let the common q be q^{M}. Then by Theorem 3, for all $\mathsf{y} \in Y$

$$C\left(\mathsf{y}\right) = \mathsf{q}^{M}\mathsf{y} - c^{*}\left(\mathsf{w}, \mathsf{q}^{M} \cdot \mathsf{p}\right)$$

with $\mathsf{q}^{M}\mathsf{A} = \mathsf{v}$. Together these expressions imply that the optimal choice of $\mathsf{r} = \mathsf{A}\mathsf{h}$ and z satisfy

$$c^{i}\left(\mathsf{r}\right) = \mathsf{q}^{M}\mathsf{r}, \tag{10}$$

and (9). Expression (10) implies that c^{i} is linear in r, and hence by the basic properties of c^{i}, $\mathsf{r} \in M$. Hence, $\mathsf{y} \in M + \mathsf{p} \cdot Z^{*}$.

Theorem 14. *Separation occurs over* Y *if and only if* $Y \subset M + \mathsf{p} \cdot Z^{*}$.

Several things should be noted here. If $\mathsf{p} \cdot Z^{*} \subset M$, then separation always occurs, and the decisionmaker's production decision is completely independent of his or attitudes towards risk. We might call this strong separation, and state this obvious fact as a Corollary

Corollary 15. *If* $\mathsf{p}\cdot Z^* \subset M$*, separation occurs for all* y*, and*

$$Z^* = \arg\sup\left\{\mathsf{q}^M(\mathsf{p}\cdot\mathsf{z}) - c(\mathsf{w},\mathsf{z})\right\},$$

where $c^i(\mathsf{r}) = \mathsf{q}^M\mathsf{r}$, $\mathsf{r} \in M$.

Second, even in the presence of separation between production decisions and preferences, full insurance is not typically available and the income accruing to the firm's owners need not lie in the financial span M. Rather, the owners of the firm are subject to 'background' risk that cannot be offset either by production choices or by financial transactions. We close this section with an example which illustrates an application of the basic theorem in the case of strong separation.

Example 16. *Consider again costs in the form*

$$c(\mathsf{w},\mathsf{z}) = \hat{c}\left(\mathsf{w}, \max\left\{\frac{z_1}{\varepsilon_1},...,\frac{z_S}{\varepsilon_S}\right\}\right).$$

Suppose that $\left(\frac{p_1}{\varepsilon_1},\frac{p_2\varepsilon_2}{\varepsilon_1},...,\frac{p_S\varepsilon_S}{\varepsilon_1}\right) \in M$*, then*

$$\begin{aligned} C(\mathsf{y}) &= \min_{\mathsf{z},\mathsf{r}}\left\{\hat{c}\left(\mathsf{w},\max\left\{\frac{z_1}{\varepsilon_1},...,\frac{z_S}{\varepsilon_S}\right\}\right) + c^i(\mathsf{r}) : \mathsf{r}+\mathsf{p}\cdot\mathsf{z} \geq \mathsf{y}\right\} \\ &= \min_{z_1,\mathsf{r}}\left\{\hat{c}\left(\mathsf{w},\frac{z_1}{\varepsilon_1}\right) + c^i(\mathsf{r}) : \mathsf{r}+z_1\mathsf{p}\cdot\left(\frac{1}{\varepsilon_1},\frac{\varepsilon_2}{\varepsilon_1},...,\frac{\varepsilon_S}{\varepsilon_1}\right) \geq \mathsf{y}\right\} \\ &= \min_{z_1,\mathsf{r}}\left\{\hat{c}\left(\mathsf{w},\frac{z_1}{\varepsilon_1}\right) + c^i\left(\mathsf{y}-z_1\mathsf{p}\cdot\left(\frac{1}{\varepsilon_1},\frac{\varepsilon_2}{\varepsilon_1},...,\frac{\varepsilon_S}{\varepsilon_1}\right)\right)\right\} \\ &= \mathsf{q}^M\mathsf{y} - \max\left\{z_1\mathsf{q}^M\mathsf{p}\cdot\left(\frac{p_1}{\varepsilon_1},\frac{p_2\varepsilon_2}{\varepsilon_1},...,\frac{p_S\varepsilon_S}{\varepsilon_1}\right) - \hat{c}\left(\mathsf{w},\frac{z_1}{\varepsilon_1}\right)\right\}. \end{aligned}$$

5. Other Applications

The results above can be applied to a wide range of problems involving firms that undertake nontrivial production and also buy or sell financial assets. To illustrate, in this section, we present brief applications of our methods that study: the relationship between equilibrium and no-arbitrage pricing, the special case of price but not production uncertainty, and comparative static issues.

5.1. Equilibrium and No-Arbitrage Prices

The no-arbitrage conditions ensurethe existence of positive state-claim prices, which allow decisionmakers to price out any potential assets. These asset prices generally do not coincide with 'equilibrium prices'. In this section, we briefly review the connection between the no-arbitrage prices and the 'equilibrium prices' as they are usually understood. To that end, we need a fuller characterization of the economy in which we are operating, and to keep things simple we make several assumptions. First, we assume that all firms have access to the same technology. Second, we assume that all firms agree on p, and that these prices are taken as given by all. Third, we assume that the economy wide endowment of the factors of production is x^*. And fourth, we assume that there are a fixed number of firms, N, with period 0 wealth of ω^n, and preferences $W^n\left(c^0, \mathsf{y}\right)$ which is strictly increasing in all its arguments.

To examine the equilibrium, it is convenient to enrich our notation by reintroducing the suppressed parameters v and w into the cost structure, while recalling that Shephard's Lemma applies to $C\left(\mathsf{y}, \mathsf{w}, \mathsf{v}\right)$ just as it does for $c\left(\mathsf{w}, \mathsf{z}\right)$. Because we presume that preferences are strictly increasing in all arguments, in equilibrium all y must be consistent with the absence of a production-financial arbitrage, and hence with locally strictly monotonic costs. Therefore, with little true loss of generality we presume that C is smooth.

Our equilibrium concept is any v, w, and $\left(\mathsf{y}^1, ..., \mathsf{y}^N\right)$ such that

$$\begin{aligned}
\mathsf{y}^n &\in \arg\max\left\{W^n\left(\omega^n - C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right), \mathsf{y}^n\right)\right\} \\
\sum_n \nabla_{\mathsf{v}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right) &= \mathsf{0}, \\
\sum_n \nabla_{\mathsf{w}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right) &= \mathsf{x}^*.
\end{aligned}$$

The first n conditions are self explanatory, the second two sets of conditions are market-clearing conditions in the asset market and the market for inputs expressed in terms of Shephard's lemma. Assuming the first-order conditions are necessary and sufficient for an optimum allows us to rewrite the above as the system

$$\begin{aligned}\nabla_{\mathsf{y}} W^n - W_1^n \nabla_{\mathsf{y}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right) &= \mathsf{0}, n = 1, ..., N \\ \sum_n \nabla_{\mathsf{v}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right) &= \mathsf{0}, \\ \sum_n \nabla_{\mathsf{w}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right) &= \mathsf{x}^*.\end{aligned}$$

Upon inspection, $\nabla_{\mathsf{y}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right)$ must be a no-arbitrage price vector for decisionmaker n, and they must coincide, in equilibrium, to the risk-neutral probabilities defined by the decisionmakers preferences, $\nabla_{\mathsf{y}} W^n / W_1^n$. Because decisionmakers can have different attitudes towards risk, there is no *a priori* reason, even in this very simplified economy, for all decisionmakers to have the same no-arbitrage prices, even though all face the same technical and financial possibilities. In fact, if all decisionmakers had the same no-arbitrage prices, then markets must be effectively complete because the no-arbitrage prices could then taken as Arrow state-claim prices. The structure of the no-arbitrage prices, even at the firm level, are dependent on the input endowment within the economy as well as the technology and the asset structure. So the determination of no-arbitrage prices in a world where firms engage in both productive and financial operations is considerably more complex than in the more stylized setting familiar from finance theory, where the asset prices are determined solely by the structure of financial market. In particular, the no-arbitrage prices associated with absence of arbitrage in financial markets alone, $\partial c^i\left(\mathsf{0}\right)$, only correspond to $\nabla_{\mathsf{y}} C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right)$ if the decisionmaker's net position in financial markets is zero, which suggests that financial activities are not being used to balance the production risk faced by the decisionmakers in equilibrium. These observations reinforce and formalize, in our framework, our introductory quotation from Cochrane (2001).

The interaction between preferences, technology, effective market completeness, and no-arbitrage prices can be examined further with a simple example. Suppose that,

$$W^n\left(\omega^n - C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right), \mathsf{y}^n\right) = \sum \pi_s y_s - C\left(\mathsf{y}^n, \mathsf{w}, \mathsf{v}\right) + \omega^n.$$

Producers are risk-neutral for a set of either objective probabilities or a set of commonly shared subjective probabilities. Equilibrium is then characterized by

$$\begin{aligned}\boldsymbol{\pi}-\nabla_{\mathsf{y}}C(\mathsf{y},\mathsf{w},\mathsf{v}) &= \mathsf{0}, n=1,...,N\\ \nabla_{\mathsf{v}}C(\mathsf{y},\mathsf{w},\mathsf{v}) &= \mathsf{0},\\ N\nabla_{\mathsf{w}}C(\mathsf{y},\mathsf{w},\mathsf{v}) &= \mathsf{x}^{*}.\end{aligned}$$

Markets are effectively complete, and $\nabla_{\mathsf{y}}C(\mathsf{y},\mathsf{w},\mathsf{v})$ is both a vector of Arrow state-claim prices and a vector of no-arbitrage prices. Because they are risk-neutral and identical, decisionmakers take no position in the asset market in equilibrium. If they had a strong incentive to take a non-zero position, it must offer unboundedly large profit, and this is inconsistent with equilibrium. If, however, producers faced different technologies, they would generally take offsetting positions in asset markets, offering zero profit, but which exploited the relative cost advantages across technologies in raising overall surplus.

We can use this simplified equilibrium setting to illustrate how C can be used to examine the potential for innovation of new financial assets via the introduction of physical production technologies. We consider the simplest possible case, where the existing equilibrium can be entirely described by equilibrium in asset markets, and no physical production takes place. Then if an equilibrium does exist it must be true (in an obvious notation) that

$$\begin{aligned}\boldsymbol{\pi} &\in \partial c^{i}(\mathsf{y},\mathsf{v})\\ \nabla_{\mathsf{v}}c^{i}(\mathsf{y},\mathsf{v}) &= \mathsf{0}.\end{aligned}$$

Suppose y is in the span of the market, in which case, the above reduces to

$$\begin{aligned}\boldsymbol{\pi} &= \partial c^{i}(\mathsf{0},\mathsf{v})\\ \nabla_{\mathsf{v}}c^{i}(\mathsf{0},\mathsf{v}) &= \mathsf{0},\end{aligned}$$

because c^{i} is linear over the span of the market. Hence, equilibrium can emerge in the span of the market if and only if the decisionmakers subjective probabilities are rational in the sense that they mirror the no-arbitrage prices generated by the asset structure, $\partial c^{i}(\mathsf{0},\mathsf{v})$. There remains the possibility that an equilibrium might emerge outside of the span of the market. However, this is also impossible because if the decisionmaker found it profitable in an expected-value sense to operate at y, he could realize infinitely large expected profits by repeatedly replicating that market position. Formally, this is a consequence of the fact that $c^i(\mathsf{y},\mathsf{v})$ is always positively linearly homogeneous.

For this setting, the only sensible equilibrium is the 'rational' one where decisionmakers basically adopt the objective probabilities dictated by the asset structure. In essence, this is the trivial equilibrium, where there are no firms, and no one takes a position in the market. Now suppose that a new technology becomes available. Then innovation will take place in equilibrium if and only if there exists a $\mathsf{y} \neq \mathsf{0}$, v, and w such that

$$\begin{aligned} \boldsymbol{\pi} &\in \partial C(\mathsf{y},\mathsf{v},\mathsf{w}) \\ \nabla_{\mathsf{v}} C(\mathsf{y},\mathsf{v},\mathsf{w}) &= \mathsf{0} \\ \nabla_{\mathsf{w}} C(\mathsf{y},\mathsf{v},\mathsf{w}) &\leq \mathsf{x}^*. \end{aligned}$$

5.2. Nonstochastic Production

Our treatment thus far has restricted attention to the case where production is stochastic. However, an extremely large literature has developed around the case where the producer faces price uncertainty but not production uncertainty. We now briefly consider such situations in this framework. Let $c^c(\mathsf{w},z)$ denote a cost function for a scalar, nonstochastic output z. This cost function satisfies the same basic properties in w as $c(\mathsf{w},\mathsf{z})$.[13] In particular, we assume that it is nondecreasing and convex in the nonstochastic scalar z. The convex conjugate of $c^c(\mathsf{w},z)$, $c^{c*} : \Re^n_{++} \times \Re_+ \to \Re$, is defined by

$$c^{c*}(\mathsf{w},\tilde{q}) = \sup\{\tilde{q}z - c^c(\mathsf{w},z)\}$$

The firm's problem is

$$C(\mathsf{y}) = \min_{z,\mathsf{h}}\{c^c(\mathsf{w},z) + \mathsf{vh} : z\mathsf{p} + \mathsf{Ah} \geq \mathsf{y}\}, \tag{11}$$

if there exists (z,h) such that $z\mathsf{p} + \mathsf{Ah} \geq \mathsf{y}$ and ∞ otherwise. We have the slight modification of Theorem 3:

[13] In fact, it may be regarded as the special case of the stochastic production function technology where all ε are equal.

Theorem 17. *For problem (11) under Assumption 1,*

$$C(\mathsf{y}) = \sup_{\mathsf{q}} \left\{ \mathsf{qy} - c^{c*}\left(\mathsf{w}, \sum q_s p_s\right) : \mathsf{qA} = \mathsf{v} \right\}.$$

A particularly striking example arises if price as well as output is nonstochastic, so $\mathsf{p} = p\mathsf{1}$. We then obtain

$$c^{c*}\left(\mathsf{w}, p\sum q_s\right).$$

If a riskless asset exists, for $\mathsf{q} \in \partial C(\mathsf{y})$, $\sum q_s = \frac{1}{1+r}$. If a riskless asset is not traded, $\sum q_s$ stills yields the firm's virtual valuation of the non-stochastic vector $\mathsf{1}$, and thus one can interpret

$$\frac{1}{\sum q_s} - 1,$$

as measuring the firm's virtual risk-free rate as we have noted above. Theorem 17 establishes that the firm makes its non-stochastic production decisions so that it maximizes its virtual discounted profit. If a riskless asset exists and is traded, the firm then always maximizes the period 0 profit in choosing its level of output.

This result, of course, reflects the parallel result developed in (7). There we saw that the firm should always choose its state-contingent production vector to exhaust all opportunities for increasing state-contingent return for sure in each state of nature. Here, because there is neither price or production risk, the parallel result is that the firm should exhaust all opportunity for discounted profit. If it did not, it could always profitably sell off some of its endowment of the riskless asset and replace it with production from the firm.

Because $p\sum q_s \in \partial c^c(\mathsf{w}, z)$ at the cost minimizing z, this observation and the discussion in the previous section on virtual prices implies that the that the firm's virtual risk-free rate can be inferred directly from its marginal cost of output. However, because production and price are non-stochastic, the firm cannot use its technology to 'construct' derivative assets, and therefore analogous results for virtual asset pricing are not available. More generally, for this technology, even in the case where c^c is smoothly differentiable in z, only bounds on virtual prices for arbitrary assets can be obtained from the production technology, though of course the virtual asset prices themselves remain available from $C(\mathsf{y})$.

Suppose, for example, that the technology is smooth and that the firm is considering moving its derivative asset y in the direction of A_j. It has at least two choices, one is to purchase a unit of A_j in the market for v_j. Alternatively, it can increase its production by the amount $\max\left\{\frac{A_{1j}}{p_1},...,\frac{A_{Sj}}{p_S}\right\}$ at marginal cost of

$$max\left\{\frac{A_{1j}}{p_1},...,\frac{A_{Sj}}{p_S}\right\}c_z^c(\mathsf{w},z).$$

However, if this latter term is strictly less than v_j the firm could not have constructed y at minimal cost, and thus

$$max\left\{\frac{A_{1j}}{p_1},...,\frac{A_{Sj}}{p_S}\right\}c_z^c(\mathsf{w},z)\geq v_j.$$

More generally, we conclude that

$$max\left\{\frac{y_1^*}{p_1},...,\frac{y_S^*}{p_S}\right\}c_z^c(\mathsf{w},z),$$

must always represent an upper bound on the firm's virtual price for y^*. Alternatively, one can view this imprecision in asset pricing as resulting from the fact that the non-stochastic case is the special case of the stochastic production function, where $\varepsilon_s = \varepsilon$ for all s.

We now turn to separation results in the presence of nonstochastic production. A fundamental result from the expected utility literature on futures and forward markets (Danthine, 1978; Holthausen, 1979; and Anderson and Danthine, 1981) is that a firm facing a single forward market for the commodity produced offering a sure price of v^* for the amount hedged, and stochastic spot prices, p, should have its production decisions independent of its risk attitudes. This result may be recognized as an immediate corollary of Theorem 14.

If the amount hedged is denoted h, then the firm creates the income stream $\mathsf{y} = \mathsf{p}(z-h)$ at a cost to itself of $c^c(\mathsf{w},z) + v^*h$. Making the simple substitution of $a = z - h$, this implies that the firm creates an asset with return $\mathsf{y} = \mathsf{p}a$ at a cost of $c^c(\mathsf{w},z) + v^*(a-z)$. Regardless of the level of a chosen, the firm always perceives its marginal cost of varying z as $\partial c^c(\mathsf{w},z) - v^*$, and it thus minimizes this cost by choosing the same z regardless of the choice of y.

In this setting since $\mathsf{y} = \mathsf{p}a$, and the span of the asset market and p coincide leading to the conclusion that the price of the asset and the marginal cost of the non-stochastic output must also coincide. Alternatively, the firm is combining two assets, one has a return of p, and the other has a return of $-\mathsf{p}$. Their marginal prices must then match. Our next result specializes Theorem 14 to the case of a general asset market structure where p lies in the span of the asset market.

Theorem 18. *For problem (11), if* $\mathsf{p} \in M$, *there exists* $\mathsf{a} \in \Re^J$ *such that* $\mathsf{Aa} = \mathsf{p}$ *and*

$$C(\mathsf{y}) = c^i(\mathsf{y}) - c^{c*}(\mathsf{w}, \mathsf{va}).$$

By Theorem 18, therefore, the firm constructs the derivative asset as best as it can via purely financial operations. It then uses its physical production activity simply to defray the costs of doing so by taking an offsetting position in the real market which exhausts the possibilities for profitable behavior in that market and for arbitrage between the two markets. Production decisions are, thus, determined by equating marginal cost of output to va. Perhaps the most intuitive way to interpret this result is in the context of a production operation, which itself is relatively riskless, but for which the firm faces price uncertainty, but also has good opportunities to hedge that price risk in financial markets. These stylized facts, for example, might closely approximate the milling industry considered in Working's (1953) classic study of price hedging in the presence of well-functioning futures markets. Theorem 18 would then imply that the firm's ultimate production decisions would not depend upon its attitudes towards risk, but would instead be driven by it attempting to maximize the sure profit associated with va, and then using operations in financial markets to cope with the price risk that it faces

5.3. Some Observations on Comparative Statics

As we noted much earlier in a footnote, in addition to y, C is functionally dependent on $(\mathsf{w}, \mathsf{v}, \mathsf{p}, \mathsf{A})$. Moreover, C is also obviously dependent upon the state of the technology in the current period when production decisions are taken. Now let us return to a quotation from Cochrane (2001) that we offered in the introduction to this paper.

> We routinely think of betas and factor risk prices ... as determining expected returns. But the whole consumption process, discount factor and factor risk premia change when the production technology changes. Similarly, we are on thin ice if we say anything about the effects of policy interventions, new markets and so on.

From the preceding developments, it is now obvious that these parameters of C, which we have largely suppressed so far, have an important role in addressing the problem being raised by Cochrane (2001). Moreover, it is also apparent that C inherits the property characteristic of all dual representations of technology of affording a means of efficient and simple comparative-static analysis. In this section, we illustrate how the representation that we have developed can be used to make concrete statements about the effects of such developments on virtual asset prices and virtual risk-free rates.

For the sake of economy of notation, we assume that we are interested in a single parameter of C, which we denote as t, and refer to as an index of the technology in current period. Thus, we can think of what follows as the simple comparative statics of technical change for asset pricing. We refer to an increase in t as technical change. However, we emphasize that the principles used, and the arguments are perfectly general, and we will close the section by noting how similar arguments can be adapted for the introduction of new assets.

Let's first address the issue of whether virtual state-claim prices can be independent of t, and thus invariant to technical change in the physical production technology. Denoting the subdifferential of $C(\mathsf{y};t)$ in y by $\partial C(\mathsf{y};t)$, we have

$$\partial C(\mathsf{y};t) = \{\mathsf{q} : +\mathsf{q}(\mathsf{y}'-\mathsf{y}) \geq C(\mathsf{y}';t) - C(\mathsf{y};t), \quad \text{all } \mathsf{y}'\},$$

from which we conclude that the virtual state-claim prices for the firm can be independent of t if and only if

$$C(\mathsf{y};t) = \hat{C}(\mathsf{y}) + m(t).$$

This observation and the definition of C yield:

Theorem 19. *If the state-contingent production technology satisfies,*

$$c(\mathsf{w},\mathsf{z},t) = \hat{c}(\mathsf{w},\mathsf{z}) + m(\mathsf{w},t)$$

with $m(\mathsf{w},t)$ positively linearly homogeneous and concave in w, then

$$\partial C(\mathsf{y};t) = \partial \hat{C}(\mathsf{y})$$

for all t, and the no-arbitrage asset prices and risk-free rates are independent of the state of the technology.

The technology in Theorem 19 corresponds to one consisting of the sum of two state-contingent input sets, one which can be thought of as a fixed-cost component that depends upon the t. The other component, which depends upon z but not t, can be interpreted as variable cost. Notice that this technology is only sensible if we relax X.1 to permit the existence of fixed costs associated with the state of the technology.

Now suppose that the technology index t describes movements in the technology applicable to the economy is a whole, and that these movements lead to an uniform increase in the payoffs of all assets as well to a proportional reduction in the firm's production costs of the form

$$\begin{aligned} \mathsf{A}(t) &= \frac{\mathsf{A}}{m(t)}, \\ c(\mathsf{w},\mathsf{z},t) &= m(t)\, c(\mathsf{w},\mathsf{z}), \end{aligned}$$

with $m(t) > 0$ monotonically decreasing in t. We have

$$\begin{aligned} C(\mathsf{y};t) &= \min_{\mathsf{h},\mathsf{z}} \left\{ m(t)\, c(\mathsf{w},\mathsf{z}) + \mathsf{vh} : \frac{\mathsf{A}}{m(t)}\mathsf{h} + \mathsf{p}\cdot\mathsf{z} \geq \mathsf{y} \right\} \\ &= \min_{\mathsf{h},\mathsf{z}} \left\{ m(t)\, c(\mathsf{w},\mathsf{z}) + \mathsf{vh} : \mathsf{A}\frac{\mathsf{h}}{m(t)} + \mathsf{p}\cdot\mathsf{z} \geq \mathsf{y} \right\} \\ &= \min_{\mathsf{h},\mathsf{z}} \left\{ m(t)\, c(\mathsf{w},\mathsf{z}) + m(t)\, \mathsf{v}\frac{\mathsf{h}}{m(t)} : \mathsf{A}\frac{\mathsf{h}}{m(t)} + \mathsf{p}\cdot\mathsf{z} \geq \mathsf{y} \right\} \\ &= m(t) \min_{\mathsf{h}/m,\mathsf{z}} \left\{ c(\mathsf{w},\mathsf{z}) + \mathsf{v}\frac{\mathsf{h}}{m} : \mathsf{A}\frac{\mathsf{h}}{m} + \mathsf{p}\cdot\mathsf{z} \geq \mathsf{y} \right\} \\ &= m(t)\, C(\mathsf{y}). \end{aligned}$$

The converse follows by duality and so,

Theorem 20. $C(\mathsf{y};t) = m(t)\, C(\mathsf{y})$ *if and only if, one can write*

$$\begin{aligned} \mathsf{A}(t) &= \frac{\mathsf{A}}{m(t)}, \\ c(\mathsf{w},\mathsf{z},t) &= m(t)\, c(\mathsf{w},\mathsf{z}). \end{aligned}$$

For this specification, virtual prices are independent of the state of the technology. It follows immediately that relative no-arbitrage prices will also be independent of t. More generally, by basic separability results, relative virtual prices are independent of t if and only if $C(\mathsf{y};t) = \hat{C}(\bar{c}(\mathsf{y}), t)$. This latter restriction requires that the joint technology, defined by the asset structure and $c(\mathsf{w}, \mathsf{z})$, exhibit output-Hicks-neutral technical change (Chambers and Färe, 1994).

Now consider the effect of technical change on the no-arbitrage pricing rules that we have derived above. For simplicity, we restrict attention to the case $C(\mathsf{0};t)$. The no-arbitrage prices are the virtual state-claim prices associated with

$$\partial C(\mathsf{0};t) = \{\mathsf{q} : \mathsf{qy} \geq C(\mathsf{y};t), \quad \text{all } \mathsf{y}\},$$

from which we conclude:

Theorem 21. *If $t' > t$ implies $c(\mathsf{w}, \mathsf{z}, t') \leq c(\mathsf{w}, \mathsf{z}, t)$, $C'(\mathsf{0}; t; \mathsf{y}^o)$, rises as a result of technical change, and the no-arbitrage risk-free rate, $r^+(\mathsf{0})$, falls.*

These theorems are trivial consequences of our specification of the derivative-cost function, but despite their triviality they answer potentially important questions in the literature on asset pricing. And when this recognition is combined with our earlier observations on the ability to infer asset pricing directly from $c(\mathsf{w}, \mathsf{z}, t)$, these results can be made empirically meaningful using traditional econometric methods of analysis.

It is equally trivial to deduce similar results for the entire range of the remaining parameters $(\mathsf{w}, \mathsf{v}, \mathsf{p}, \mathsf{A})$ of C. For example, it is easy to demonstrate that C is nonincreasing in both w and v. Hence, it follows immediately by parallel arguments that the no-arbitrage asset prices are decreasing in both w and v, and that the no-arbitrage risk-free rate is increasing. Notice, moreover, that the introduction of a new financial asset, as opposed to the introduction of a physical technology as discussed above, can be interpreted as a comparative-static change of the form of moving from an $S \times J$ dimensional matrix A containing as one of its columns the null vector to one containing no null vectors. For simplicity, let that correspond to the Jth column. Then innovating the new financial asset A_j increases all no-arbitrage prices and reduces the no-arbitrage risk-free rate if, in an obvious notation, $C(\mathsf{y}; \mathsf{A}_j) \leq C(\mathsf{y}; \mathsf{0})$. But this is trivially true because the range of feasible choices has grown. Similarly, it is trivially true that the introduction of a new physical technology with zero fixed costs always increases the no-arbitrage prices of assets and the no-arbitrage risk-free rate.

6. Concluding Comments

The primary purpose of this paper has been to demonstrate that a state-contingent model of production under uncertainty can fruitfully be integrated with the standard finance-theoretic model of asset pricing, giving rise to significant extensions of basic arbitrage arguments. A number of observations, made in passing, could potentially be developed further.

First, the model provides a natural framework for treating frictions, transactions costs and taxes. Rather than distinguishing, as we have here, between a financial sector characterized by a conical span M, and consuming no real resources, and a more general state-contingent production technology, it would be possible, and appropriate, to treat both non-financial and financial firms as being jointly engaged in real economic activity, consuming resources and producing state-contingent outputs.

Second, as Cochrane (2001) observes, the limitations of the endowment-economy assumption commonly used in finance theory are most evident in relation to comparative static analysis. For example, despite the central role of claims about technological change in recent discussions of the behavior of financial markets, there has been little formal analysis of the impact on asset prices of changes in the technology of production. Changes in the actual manner in which the physical technology is employed arise from a variety of sources including technical change and evolving market conditions. Moreover, as has been emphasized above, the manner in which a technology is employed depends upon both financial and resource markets. Although we have only superficially addressed these issues here, $C(\mathsf{y})$ affords a natural and well-understood means for examining such issues. $C(\mathsf{y})$ is mathematically equivalent to a cost function for a multi-product state-contingent production technology. Just as $c(\mathsf{w}, \mathsf{z})$ has characteristics that make it particularly suitable for comparative-static analysis, so does $C(\mathsf{y})$. In particular, $C(\mathsf{y})$ can be shown to nondecreasing, positively linearly homogeneous, and concave in (w, v) while also satisfying a version of Shephard's lemma in (w, v). We have demonstrated that asset pricing can be conducted solely in terms of the directional derivatives of $C(\mathsf{y})$. Each of these properties can be used to examine the effect of exogenous changes in technology (via technical change), marketed asset prices, input prices, and state-contingent output prices on the no-arbitrage prices derived above specifically, and more generally on the no-arbitrage equilibrium itself.

The presentation of financial and nonfinancial technology in terms of underlying production sets provides a natural way of modelling both financial and technological innovation. In particular, it is possible to distinguish between exogenous technological change, modelled as an induced shift in $c(\mathsf{w}, \mathsf{z})$, and induced technical change, modelled as a shift in the production-finance vector in response to changing factor or state-claim prices. The ability to do this for a no-arbitrage equilibrium is a first and necessary analytic step in doing the same in a truly general-equilibrium setting. Moreover, the approach advanced in this paper promotes the further integration of the theory of the financial and productive firms rather than their continued segregation into specialized sub disciplines.

Third, asset pricing rules can always be recaptured from both $C(\mathsf{y})$ and $c(\mathsf{w}, \mathsf{z})$. Thus, once procedures for estimating non-stochastic production structures are extended to the estimation of cost functions for state-contingent technologies, estimated cost structures can be used to price derivative assets.

7. Reference List

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8. Appendix: Proofs

Proof of Theorem 1: Continuity follows by the theorem of the maximum (Berge, 1963). Let $(\mathsf{h}', \mathsf{z}')$ be optimal for $\mathsf{y}' \geq \mathsf{y}$, then C nondecreasing follows because $(\mathsf{h}', \mathsf{z}')$ is feasible for y. Let $\mathsf{y}' \geq \mathsf{y}, \mathsf{y}' \neq \mathsf{y}$, there must exist a $\mathsf{z} \leq \mathsf{z}', \mathsf{z} \neq \mathsf{z}'$ such that $\mathsf{h}'\mathsf{A} + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y}$, but for which $c(\mathsf{w}, \mathsf{z}) + \mathsf{v}\mathsf{h}' < C(\mathsf{y}')$ if c is strictly increasing. To demonstrate convexity, let $(\mathsf{h}', \mathsf{z}')$ and $(\mathsf{h}'', \mathsf{z}'')$ be optimal for y' and y'', respectively. By the linearity of the constraint sets $(\lambda \mathsf{h}' + (1-\lambda)\mathsf{h}'', \lambda \mathsf{z}' + (1-\lambda)\mathsf{z}'')$ is feasible for $\lambda \mathsf{y}' + (1-\lambda)\mathsf{y}''$. By C.5,

$$c(\mathsf{w}, \lambda \mathsf{z}' + (1-\lambda)\mathsf{z}'') + \mathsf{v}(\lambda \mathsf{h}' + (1-\lambda)\mathsf{h}'') \leq \lambda [c(\mathsf{w}, \mathsf{z}') + \mathsf{v}\mathsf{h}'] + (1-\lambda)[c(\mathsf{w}, \mathsf{z}'') + \mathsf{v}\mathsf{h}''].$$

Taking the minimum of the left-hand side yields convexity. This establishes 1.

To establish 2, consider $\mathsf{q} \in \partial C(\mathsf{y})$. By definition,

$$C(\mathsf{y}) - C(\mathsf{z}) \leq \mathsf{q}(\mathsf{y} - \mathsf{z}).$$

Let $\mathsf{y} = \mathsf{z} + \delta \mathsf{e}_j$ for $\delta > 0$ but suitably small. Weak monotonicity guarantees that $q_j \geq 0$, and strong monotonicity guarantees a strict inequality.

To establish 3, suppose that $C(\mathsf{y})$ is finite at y, then by Assumption 1, it is a proper convex function. Theorem 23.4 of Rockafellar (1970) then yields the result.

$C(\mathsf{0}) \leq 0$ follows by C.3 and the definition of the cost function because $\mathsf{h} = \mathsf{0}, \mathsf{z} = \mathsf{0}$ is feasible for $\mathsf{y} = \mathsf{0}$.

To establish 5,

$$\begin{aligned} C\left(\mathsf{y} + \boldsymbol{\delta}\mathsf{A}_j\right) &= \min_{\mathsf{h},\mathsf{z}} \left\{ c(\mathsf{w}, \mathsf{z}) + \mathsf{v}\mathsf{h} : \mathsf{A}\mathsf{h} + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y} + \boldsymbol{\delta}\mathsf{A}_j \right\} \\ &= \min_{\mathsf{h},\mathsf{z}} \left\{ c(\mathsf{w}, \mathsf{z}) + \mathsf{v}_{-j}\mathsf{h}_{-j} + v_j(h_j - \delta) : \mathsf{A}_{-\mathsf{j}}\mathsf{h}_{-j} + (h_j - \delta)\mathsf{A}_j + \mathsf{p} \cdot \mathsf{z} \geq \mathsf{y} \right\} + v_j \delta \\ &= C(\mathsf{y}) + \delta v_j. \end{aligned}$$

Proof of Theorem 3: Under Assumption 1 and by Theorem 1, $C(\mathsf{y})$ is proper and closed function with the convex conjugate identified in the text. By conjugacy, then

$$C(\mathsf{y}) = \sup_{\mathsf{q}} \left\{ \mathsf{q}\mathsf{y} - \sup_{\mathsf{z}} \{ \mathsf{q}(\mathsf{p} \cdot \mathsf{z}) - \mathsf{c}(\mathsf{w}, \mathsf{z}) \} : \mathsf{q}\mathsf{A} = \mathsf{v} \right\}.$$

Proof of Corollary 8: By Theorem 3, $\mathsf{q} \in \partial C(\mathsf{y})$ requires

$$\mathsf{v} = \mathsf{q}\mathsf{A},$$

and thus under the maintained condition

$$(\boldsymbol{\pi} - \mathsf{q})\,\mathsf{A} = 0$$

so that $(\boldsymbol{\pi} - \mathsf{q})$ is orthogonal to A and thus to M. q is thus the projection of $\boldsymbol{\pi}$ on M. In this case,

$$\begin{aligned}
C^*(\mathsf{q}) &= \sup_{\mathsf{y}} \{\mathsf{qy} - C(\mathsf{y})\} \\
&= \sup_{\mathsf{y}} \left\{\mathsf{qy} - \min_{\mathsf{h,z}} \{c(\mathsf{w},\mathsf{z}) + \mathsf{vh} : \mathsf{Ah} + \mathsf{p}\cdot\mathsf{z} \geq \mathsf{y}\}\right\} \\
&= \sup_{\mathsf{h,z}} \{\mathsf{q}[\mathsf{Ah} + \mathsf{p}\cdot\mathsf{z}] - c(\mathsf{w},\mathsf{z}) - \boldsymbol{\pi}\mathsf{Ah}\} \\
&= \sup_{\mathsf{h,z}} \{(\mathsf{q} - \boldsymbol{\pi})\ \mathsf{Ah} + \mathsf{q}(\mathsf{p}\cdot\mathsf{z}) - c(\mathsf{w},\mathsf{z})\} \\
&= \begin{cases} \infty & \mathsf{q} - \boldsymbol{\pi} \notin M^{\perp} \\ \sup_{\mathsf{z}} \{\mathsf{q}(\mathsf{p}\cdot\mathsf{z}) - c(\mathsf{w},\mathsf{z})\} & \textit{otherwise} \end{cases}.
\end{aligned}$$

If $\boldsymbol{\pi} - \mathsf{q} \notin M^{\perp}$ then $\partial C^*(\mathsf{q}) = \emptyset$, hence $\boldsymbol{\pi} - \mathsf{q} \in M^{\perp}$ for there to exist a y satisfying $\mathsf{q} \in \partial C(\mathsf{y})$.

Proof of Theorem 9: Apply Theorem 22.3 of Rockafellar (1970) to the condition that

$$\mathsf{Ah}' + \mathsf{p}\cdot\mathsf{z}' \geq \mathsf{Ah} + \mathsf{p}\cdot\mathsf{z} \Rightarrow \mathsf{vh}' + c(\mathsf{w},\mathsf{z}') \geq \mathsf{vh} + \mathsf{c}(\mathsf{w},\mathsf{z}).$$

Proof of Theorem 17:

$$\begin{aligned}
C^*(\mathsf{q}) &= \sup_{\mathsf{y}} \{\mathsf{qy} - C(\mathsf{y})\} \\
&= \sup_{\mathsf{y}} \left\{\mathsf{qy} - \min_{z,\mathsf{h}} \{c^c(\mathsf{w},z) + \mathsf{vh} : z\mathsf{p} + \mathsf{Ah} \geq \mathsf{y}\}\right\} \\
&= \sup_{\mathsf{y},z,\mathsf{h}} \{\mathsf{qy} - c^c(\mathsf{w},z) + \mathsf{vh} : z\mathsf{p} + \mathsf{Ah} \geq \mathsf{y}\} \\
&= \sup_{\mathsf{h},z} \{\mathsf{q}[\mathsf{Ah} + \mathsf{p}z] - c^c(\mathsf{w},z) - \mathsf{vh}\} \\
&= \begin{cases} \infty & \mathsf{qA} \neq \mathsf{v} \\ \sup_z \{z\sum q_s p_s - c^c(\mathsf{w},z)\} & \mathsf{qA} = \mathsf{v} \end{cases}.
\end{aligned}$$

The result follows by conjugacy.

Proof of Theorem 18: Suppose that $\mathsf{p} \in M$, then there exists an a such that

$$\mathsf{Aa} = \mathsf{p},$$

and the minimization problem becomes

$$\begin{aligned}
C(\mathsf{y}) &= \min_{z,\mathsf{h}} \{c^c(\mathsf{w},z) + \mathsf{vh} : \mathsf{A}(\mathsf{h} + \mathsf{a}z) \geq \mathsf{y}\} \\
&= \min_{z,\mathsf{h}} \{c^c(\mathsf{w},z) + \mathsf{v}(\mathsf{h} + \mathsf{a}z) - \mathsf{va}z : \mathsf{A}(\mathsf{h} + \mathsf{a}z) \geq \mathsf{y}\} \\
&= \min_{z,\mathsf{h}^*} \{c^c(\mathsf{w},z) + \mathsf{vh}^* - \mathsf{va}z : \mathsf{Ah}^* \geq \mathsf{y}\} \\
&= \min_{z} \{c^c(\mathsf{w},z) - \mathsf{va}z\} + \min_{\mathsf{h}^*} \{\mathsf{vh}^* : \mathsf{Ah}^* \geq \mathsf{y}\}.
\end{aligned}$$

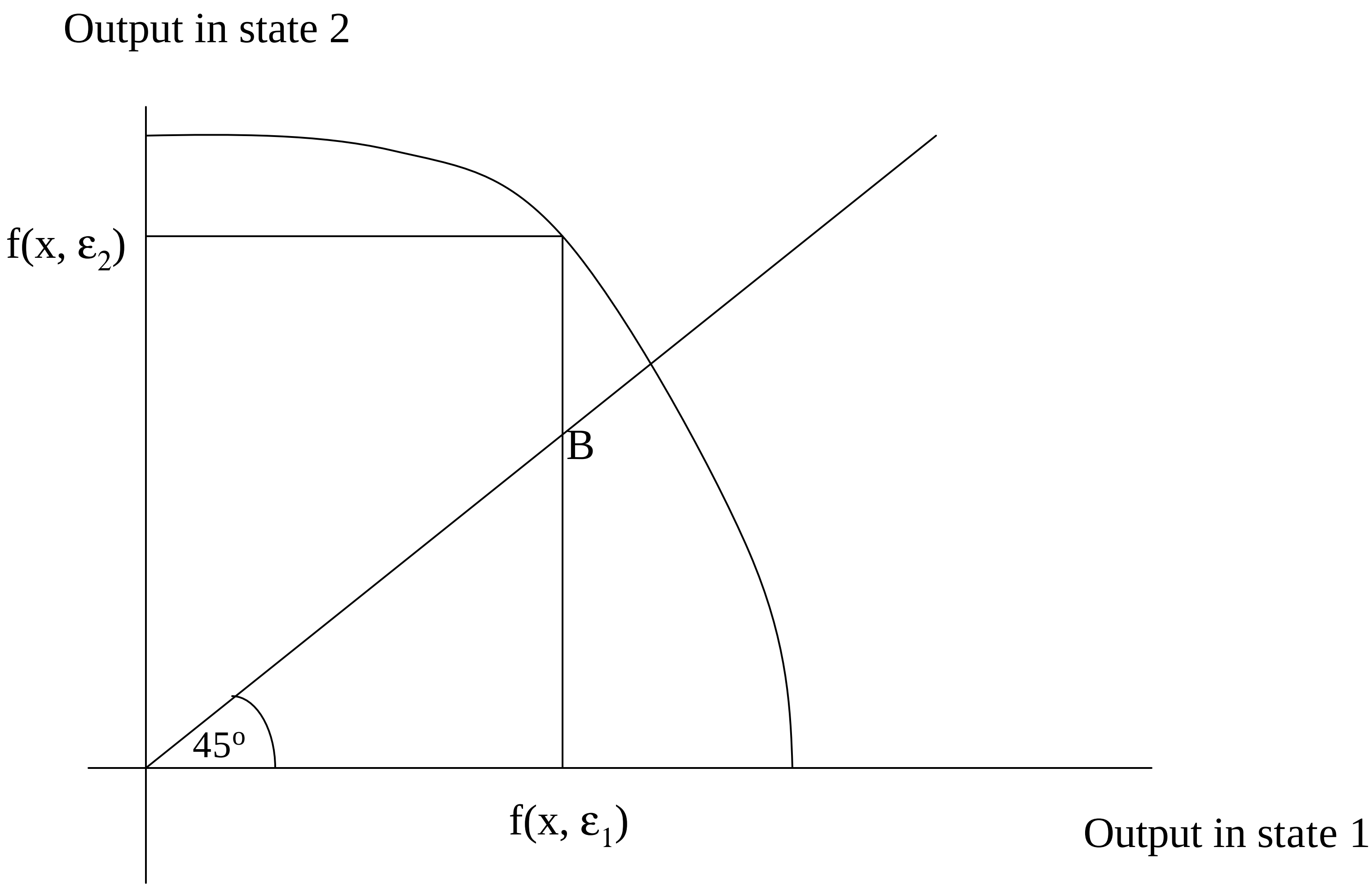

Figure 1: Stochastic production function and general technology

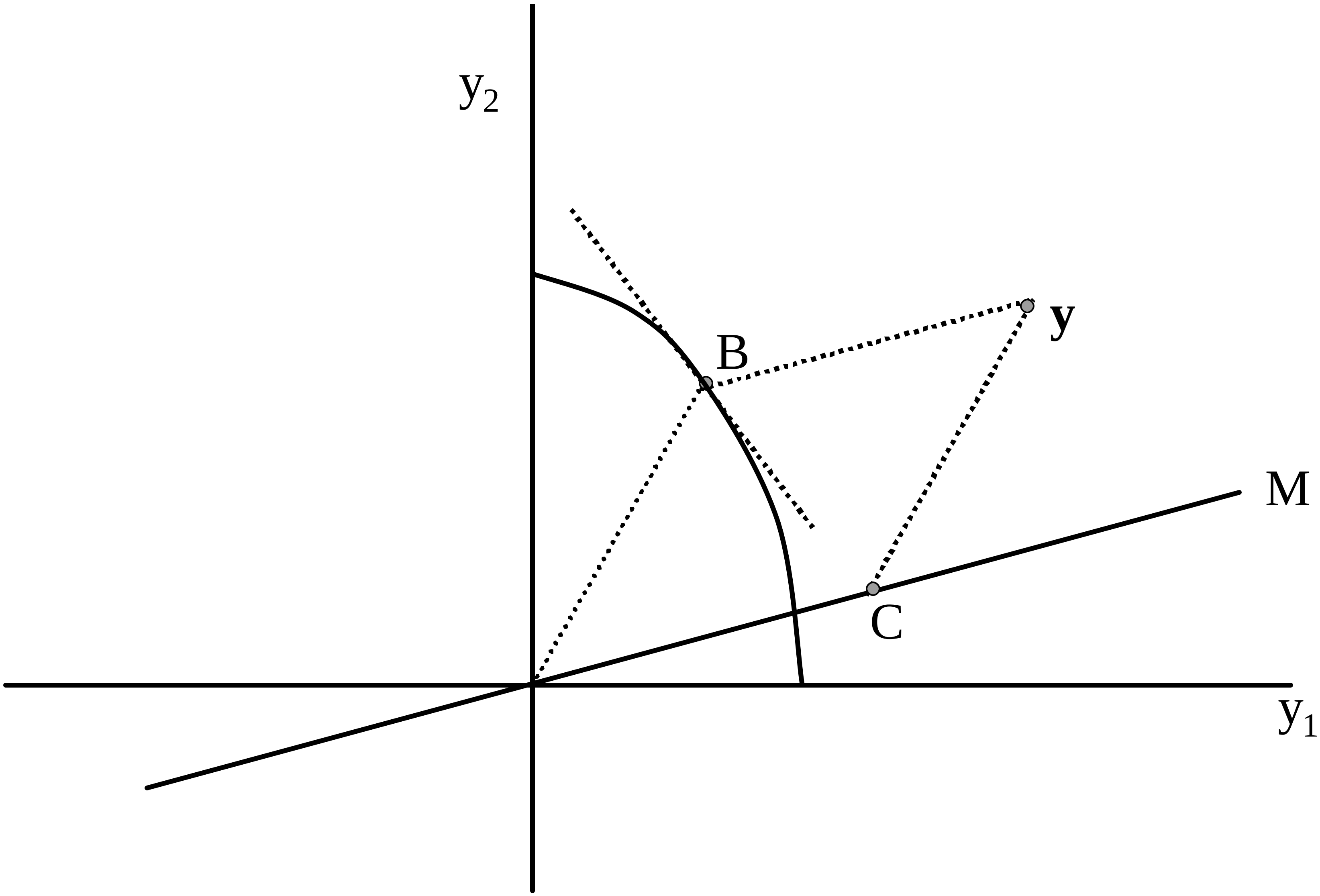

Figure 2a: Equilibrium with Incomplete Financial Markets

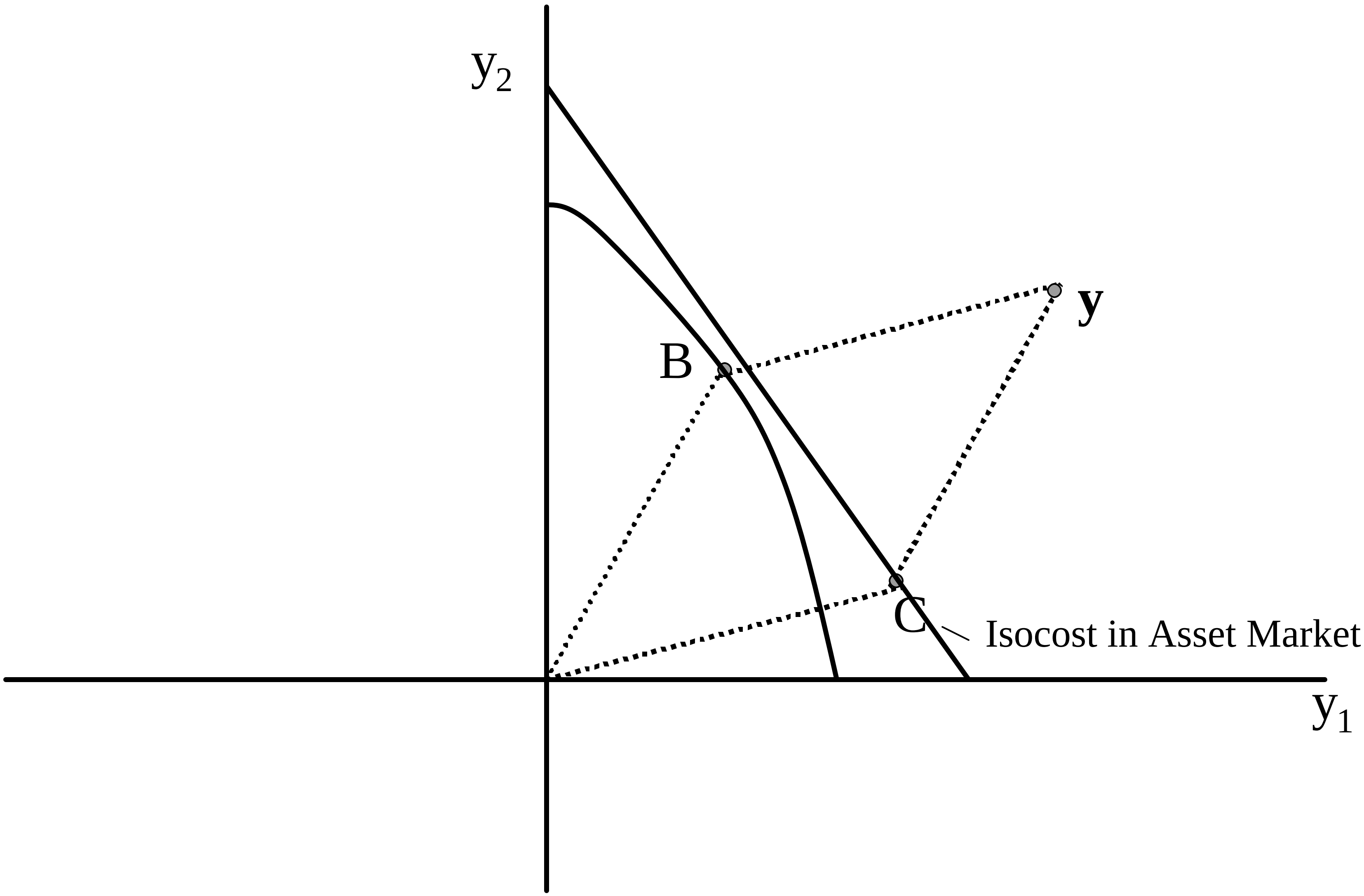

Figure 2b: Equilibrium with Complete Financial Markets

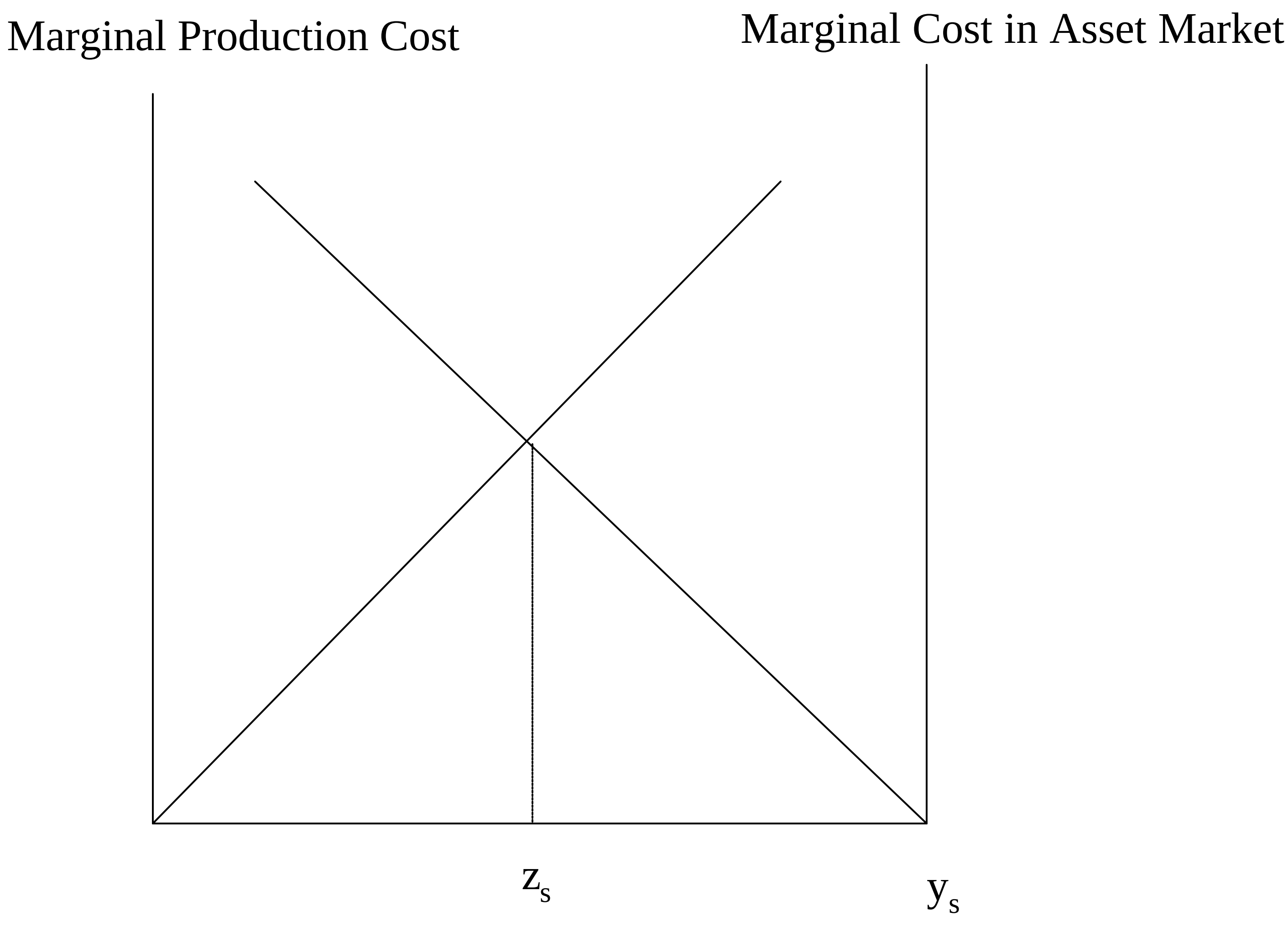

Figure 3: Equilibrium as Output Nonjoint Technology

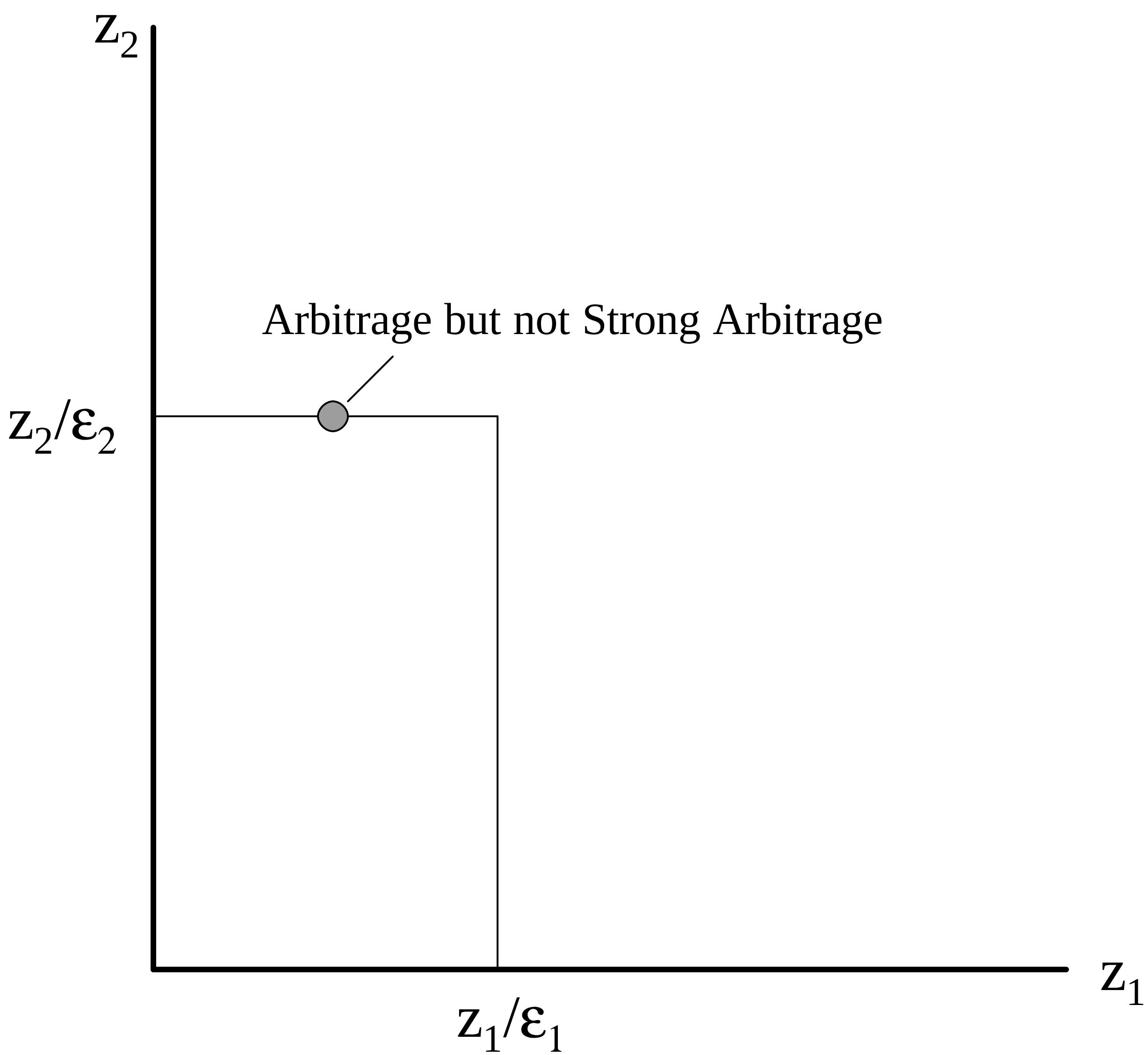

Figure 4: Strong Arbitrage and Arbitrage