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A Dynamic Minimum Variance Hedge

by

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Most theoretical models of hedging are static, in the sense that they assume the decision maker to be myopic (Johnson; Stein; Holthausen; Benninga, Eldor, and Zilcha). A myopic decision maker is concerned only about two points in time: the present and some future "terminal" date. In other words, a myopic agent is such that his decision horizon equals his planning horizon, which is equal to one period.[1] Static hedging models have been widely used to estimate optimal hedges, but they implicitly impose the stringent restriction that the decision maker cannot revise either his cash (i.e., physical) or his hedging position between the time of placing the hedge and the time when it is liquidated. Decision makers in real-world situations, however, face a far more complex scenario. After the hedge is originally placed, the decision maker generally receives new information and has the opportunity of modify his cash position and adjust (possibly liquidate) the standing hedge. In addition, it may well be the case that when the original terminal date arrives it is no longer optimal to liquidate the cash and/or the futures position.

Corn storage can be used as an illustration. Assume that at harvest (November) a farmer must decide whether to store or not, and how much to hedge. A static model will then postulate a certain fixed decision horizon, say eight months, at the end of which (i.e., July) the farmer will liquidate both the cash and the futures position. Then, according to the mean, variability, and possibly higher moments of the random returns in July, the farmer will decide how much to store and hedge in November. But such a model ignores the opportunities to adjust the cash and futures positions between harvest and July. An opportunity may arise in April to liquidate both the cash and futures positions at a much higher return than could be expected if carried through July. Alternatively, a profitable opportunity to roll over the hedge from July to September may arise.

In summary, the static hedging model is a very restrictive representation of the actual storage-hedging decision problem, because this process is dynamic in nature. A priori, models allowing the decision maker to revise his cash and futures positions after making his initial decision should perform better than the static paradigm. Anderson and Danthine, Karp, Hey, Martinez and Zering, Mathews and Holthausen, and Howard and D'Antonio are among the authors who have relaxed the static assumption. But the models by Anderson and Danthine, Karp, and Martinez and Zering are too complex to use in practical applications. Hey, on the other hand, makes the unreasonable assumption that prices follow a constant distribution from period to period. The models by Mathews and Holthausen, and Howard and D'Antonio avoid the shortcomings of the other studies, but instead they are overly restrictive in that they allow updating in the futures position but not in the cash position. The agent revises his original hedge between the current and the terminal dates, but his cash position is assumed to either stay at the original level during all that period (Mathews and Holthausen), or to grow at a nonstochastic rate (Howard and D'Antonio).

The previous discussion highlights the need for a dynamic hedging model which allows for updates of both cash and futures positions, but which is still tractable enough for use in practical applications. Such a model is derived and discussed in the following section. Next, this model is applied to corn storage, and compared to static hedges and to the Mathews-Holthausen hedge: In the final section we summarize the main conclusions of the study.

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[1]Merton (p. 656) defines decision horizon as "the length of time between which the investor makes successive decisions, and it is the minimum time between which he would take any action," and planning horizon as "the maximum length of time for which the investor gives any weight in his utility function."

The Theoretical Model

We will derive the optimal size of the current hedge for a decision maker who wants to optimize an objective function dependent upon his wealth at a future terminal date, and who can revise both his cash and futures positions at least once between now and the terminal date.[2] We will denote the current date by $t = 0$, the terminal date by $t = T$, and the "revision" dates by $t = 1$ through $t = T-1$. Note that static models are characterized by the absence of updates between times $t = 0$ and $t = T$; therefore, myopic agents act as if the current date were the next-to-terminal time, i.e., static models assume $t = 0 = T-1$.

To avoid the problems that arise from random input prices and/or stochastic output in a dynamic framework (Batra and Ullah, Hartman, Ratti and Ullah, Wright, Stewart, Perrakis), we will consider the case of a competitive firm involved in speculative storage and futures trading. By speculative storage we mean that the firm stores a particular commodity with the purpose of making a profit from the possible rise in its cash price.[3] Hence, from the perspective of the current date $t = 0$, we can define wealth at the terminal date $t = T$ as

$$(1) \qquad W_T = r^T W_0 + r^{T-1} \pi_1 + r^{T-2} \pi_2 + \ldots + r^2 \pi_{T-2} + r \pi_{T-1} + \pi_T$$

where W_T denotes monetary wealth at the end of the terminal date, r^{T-t} is the per-period interest factor raised to the power T-t (i.e., r equals one plus the per-period interest rate), and W_0 represents initial monetary wealth. π_t is the realized profit from having stored and hedged respectively I_{t-1} and $H_{t-1} I_{t-1}$ commodity units in the period spanning between dates t-1 and t.

$$(2) \qquad \pi_t = \{p_t - r\,[p_{t-1} + c(I_{t-1})] + (f_{t-1,t} - f_{t,t})\,H_{t-1}\}\,I_{t-1}$$

where: p_t = cash price at date t

I_{t-1} = amount stored from date t-1 to date t, $I_{t-1} \geq 0$

$c(\cdot)$ = variable storage cost per unit of commodity (excluding interest), $c(\cdot) \geq 0$, $c'(\cdot) \geq 0$,

$\qquad c''(\cdot) \geq 0$

$f_{t-1,t}$ = futures price prevailing at date t-1 for delivery at date $\tau \geq t$

$f_{t,t}$ = futures price prevailing at date t for delivery at date $\tau \geq t$

H_{t-1} = proportion of storage (I_{t-1}) hedged in the futures market at time t-1 and liquidated at t, for delivery at date $\tau \geq t$

The quantity H_{t-1} is a hedge ratio because it is the proportion of the physical position being hedged; the actual amount sold in the futures market is given by the product $H_{t-1} I_{t-1}$. Expression (2) implicitly imposes the restriction $\pi_t = 0$ when $I_{t-1} = 0$; therefore, any profit (or loss) from a purely speculative futures position (i.e., a futures position different from zero when $I_{t-1} = 0$) is precluded. This poses no problem, however, because in the derivation of the dynamic hedge we will assume that the only purpose of using futures is to minimize the risk of the cash position, in which case the decision maker will never adopt a purely speculative futures position. Expression (2) does not include fixed costs because these will not affect the solution.

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[2]To maintain consistency, we assume that revisions are made to re-optimize the original objective function of terminal wealth to take advantage of the better information available as time passes by. The hedging updates will generally differ from the initial hedge only because of the arrival of new information, not because of changes in the agent's objective function.

[3]We make this distinction clear, because a firm providing storage services for a known fee would not face the risk of random commodity prices.

A benchmark in the hedging literature is the static minimum variance hedge ratio (SMV).[4] The SMV is the proportion of the cash position to be hedged in order to minimize the variance of terminal wealth, for a given cash position. The SMV is important because it represents the optimum hedge ratio for myopic agents who are extremely risk-averse (Ederington; Kahl; Schwarz, Hill, and Schneeweis; Witt, Schroeder, and Hayenga). Another reason for the relevance of the SMV is that it is the optimal static hedge ratio when futures prices are unbiased, regardless of the risk attitude of the myopic decision maker (Benninga, Eldor, and Zilcha). This case is very important because there is evidence that futures are unbiased (Baillie and Myers, Martin and Garcia). But perhaps the most important reasons for its widespread acceptance are that the SMV is easy to estimate empirically, and that it is tractable enough to use in real-world situations.

Given the desirable characteristics of the SMV discussed above, and that our goal is to obtain a tractable dynamic hedge, we will derive a dynamic minimum variance hedge ratio (DMV) as an analogue of the SMV ratio. A fundamental difference between the SMV and the DMV models is that in the latter it is necessary to postulate a decision rule to calculate the cash position. The SMV's objective function is to choose the hedge that minimizes the variance of terminal wealth, given a certain cash position; therefore, most of the SMV literature does not address the problem of how the decision maker chooses his cash position. Mathews and Holthausen's dynamic minimum variance hedge ratio (MHMV) does not require a cash decision rule either, because it is a hedge ratio for a given cash position that is left unmodified between the current and the terminal dates. In our notation, Mathews and Holthausen impose the restriction $I_0 = I_1 = \ldots = I_{T-1}$, where I_0 is given. Similarly, Howard and D'Antonio assume $I_t = \alpha^t I_0$ ($t = 1,\ldots,$ T-1), where α is a known positive constant. In contrast, we want the DMV to specifically allow for I_t/I_0 ($t = 1,\ldots,$ T-1) to be random, and therefore it is necessary to know how the agent updates the cash position (i.e., $I_1,\ldots, I_{T-1}$) even if I_0 is predetermined.

To keep the model simple, we postulate that storage at each decision time $1 \leq t \leq$ T-1 is chosen according to the following scheme:[5]

(3)   a. If $\mathbb{E}_t(p_{t+1}) > r\,[p_t + c(0)]$, then $I_t > 0$ satisfying $\mathbb{E}_t(p_{t+1}) = r\,\{p_t + [c(I_t) + I_t\,c'(I_t)]\}$

      b. $I_t = 0$ otherwise

where $\mathbb{E}_t(\cdot)$ is the expectation operator conditional on the information available at date t. In words, (3) means that the decision maker at date t chooses a cash position (I_t) that maximizes the difference between the discounted expected cash price and the costs of buying and storing the commodity. If the discounted expected cash price is not sufficient to cover the costs of buying and storing, then the best decision is to store nothing. Expression (3) implies that the decision maker chooses storage in order to maximize the expected value of terminal wealth.

The storage decision rule (3) may be too naive in some circumstances because it says that the only requirement to store is to at least "break even" by doing so. However, it is easy to modify (3) to accommodate such circumstances. This modification can be accomplished by requiring the discounted expected price to exceed the costs of buying and storing plus a certain positive threshold margin per unit dependent on the particular preferences of the decision maker.

Another possible objection to the storage decision rule (3) is that it assumes that the decision maker is able to buy the commodity. This assumption is unrealistic in the case of many

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[4]The SMV is generally referred to as the minimum variance hedge, or the risk-minimizing hedge. We add the qualifier "static" to emphasize that the SMV is derived from a static model.

[5]Note that total variable cost is given by $[I_t\,c(I_t)]$ because $c(I_t)$ is per-unit variable cost. Therefore, marginal cost equals $[c(I_t) + I_t\,c'(I_t)]$. When $I_t = 0$, marginal cost reduces to $c(0)$.

farmers, who are primarily concerned with the decision to store their own crop, or whose storage limitations or prohibitively high transfer costs make grain purchasing and storage impractical. In this instance, a preferred storage decision rule for dates $1 \leq t \leq T-1$ is

(3') a. If $I_{t-1} > 0$ and $\max_i[\mathbb{E}_t(p_{T-i}) - r^{T-i} p_t - (r + r^2 + \ldots + r^{T-i}) c(0)] > 0$ for $0 \leq i \leq T-t+1$,
       then $I_t > 0$ satisfying $I_t = \min(I_{t-1}, I^*)$, where
       $I^* = \mathrm{argmax}_I\{\max_i[(\mathbb{E}_t(p_{T-i}) - r^{T-i} p_t - (r + r^2 + \ldots + r^{T-i}) c(I)) I]\}$

       b. $I_t = 0$ otherwise

The scheme (3') simply states that the decision maker stores only if he has beginning stocks and in addition he expects to profit by storing up to sometime later within the planning horizon. If he stores, he will do so up to the point where expected profits are maximized if that storage does not exceed beginning stocks, or up to the amount of beginning stocks otherwise.

To remain consistent with the SMV framework, we postulate that at each decision date $0 \leq t \leq T-1$ the agent decides how much to hedge in order to minimize the variance of the terminal wealth, given a particular cash position. In other words, we hypothesize that at each decision date the decision maker *first* chooses how much to store according to (3) [or (3')], and *then* decides what proportion to hedge in order to minimize the variance of terminal wealth. In summary, we assume that at any decision date the agent chooses the cash position yielding the maximum expected terminal wealth, and immediately after doing so selects the risk-minimizing hedge.

Given the optimum cash position I_0, the objective at the current date $t = 0$ is to choose the hedge ratio H_0 that minimizes the variance of terminal wealth conditional on the information currently available, i.e.,

(4) $\min_{H_0} \mathrm{Var}_0(W_T)$

The solution to this problem is obtained by backward induction. If the agent were at date T-1, the variance of terminal wealth would be

(5) $\mathrm{Var}_{T-1}(W_T) = \mathrm{Var}_{T-1}(\pi_T)$

$$= I_{T-1}^2 [\mathrm{Var}_{T-1}(p_T) + H_{T-1}^2 \mathrm{Var}_{T-1}(f_{T,T}) - 2 H_{T-1} \mathrm{Cov}_{T-1}(p_T, f_{T,T})]$$

by application of (1). The first order condition (FOC) corresponding to the minimization of (5) with respect to H_{T-1} is

(6) $\dfrac{\partial \mathrm{Var}_{T-1}(W_T)}{\partial H_{T-1}} = I_{T-1}^2 [2 H_{T-1} \mathrm{Var}_{T-1}(f_{T,T}) - 2 \mathrm{Cov}_{T-1}(p_T, f_{T,T})] = 0$

which yields

(7) $H_{T-1}^* = \dfrac{\mathrm{Cov}_{T-1}(p_T, f_{T,T})}{\mathrm{Var}_{T-1}(f_{T,T})}$

as the DMV for an agent standing at T-1. Note that expression (7) is also the formula of the SMV ratio when $t = 0 = T-1$.

Consider now the optimal decision for an agent at date T-2. The variance of terminal wealth in this instance is given by

$$(8) \quad \text{Var}_{T-2}(W_T) = \text{Var}_{T-2}(r\,\pi_{T-1} + \pi_T)$$

$$= r^2\,\text{Var}_{T-2}(\pi_{T-1}) + \text{Var}_{T-2}(\pi_T) + 2\,r\,\text{Cov}_{T-2}(\pi_{T-1}, \pi_T)$$

$$= r^2\,I_{T-2}^2\,[\text{Var}_{T-2}(p_{T-1}) + H_{T-2}^2\,\text{Var}_{T-2}(f_{T-1,T-1}) - 2\,H_{T-2}\,\text{Cov}_{T-2}(p_{T-1}, f_{T-1,T-1})]$$

$$+ \text{Var}_{T-2}(\pi_T) + 2\,r\,I_{T-2}\,[\text{Cov}_{T-2}(p_{T-1}, \pi_T) - H_{T-2}\,\text{Cov}_{T-2}(f_{T-1,T-1}, \pi_T)]$$

and the FOC corresponding to the minimization of (8) with respect to H_{T-2} is

$$(9) \quad \frac{\partial \text{Var}_{T-2}(W_T)}{\partial H_{T-2}} = r^2\,I_{T-2}^2\,[2\,H_{T-2}\,\text{Var}_{T-2}(f_{T-1,T-1}) - 2\,\text{Cov}_{T-2}(p_{T-1}, f_{T-1,T-1})]$$

$$- 2\,r\,I_{T-2}\,\text{Cov}_{T-2}(f_{T-1,T-1}, \pi_T) = 0$$

Solving (9) for H_{T-2} we obtain

$$(10) \quad H_{T-2}^* = \frac{\text{Cov}_{T-2}(p_{T-1}, f_{T-1,T-1})}{\text{Var}_{T-2}(f_{T-1,T-1})} + \frac{\text{Cov}_{T-2}(f_{T-1,T-1}, \pi_T)}{r\,I_{T-2}\,\text{Var}_{T-2}(f_{T-1,T-1})}$$

is the DMV for a decision maker standing at date T-2.

By repeating the above procedure, we finally obtain[6]

$$(11) \quad H_0^* = \frac{\text{Cov}_0(p_1, f_{1,1})}{\text{Var}_0(f_{1,1})} + \frac{\text{Cov}_0(f_{1,1}, \pi_2)}{r\,I_0\,\text{Var}_0(f_{1,1})} + \ldots + \frac{\text{Cov}_0(f_{1,1}, \pi_{T-1})}{r^{T-2}\,I_0\,\text{Var}_0(f_{1,1})} + \frac{\text{Cov}_0(f_{1,1}, \pi_T)}{r^{T-1}\,I_0\,\text{Var}_0(f_{1,1})}$$

is the DMV at the current date $t = 0$.

From (11), it can be seen that the DMV at $t = 0$ has two components. The first component is a standard SMV [the first term in the right-hand side of (11)], which follows from the risk reduction attributable to the relationship between cash and futures prices at date $t = 1$. The second component comprises the terms including $\text{Cov}_0(f_{1,1}, \pi_t)$. These terms reflect the contribution of the DMV to risk reduction by accounting for the relationship between next date's futures price $(f_{1,1})$ and posterior profits (π_t). Note that these terms are discounted, the more so the further into the future the profit considered (i.e., the closer is π_t to date T). The rationale for such discounting is that the further into the future is a particular profit, the smaller is its share of terminal wealth, and therefore the smaller is its proportional contribution to the variance of terminal wealth. Finally, all the $\text{Cov}_0(f_{1,1}, \pi_t)$ terms are divided by the current storage (I_0) because, other things equal, a larger current storage increases its share of terminal wealth, and consequently the larger is its contribution to the variance of terminal wealth relative to the storage and hedging decisions taken at dates $t = 1$ through $t = T-1$.

Comparison with Previous Models

As we already mentioned, the motivation for deriving the DMV is that other hedging models used in empirical applications either allow for no updates, or only allow for hedging updates. Given the derivation of the DMV presented in the previous section, it is of interest to

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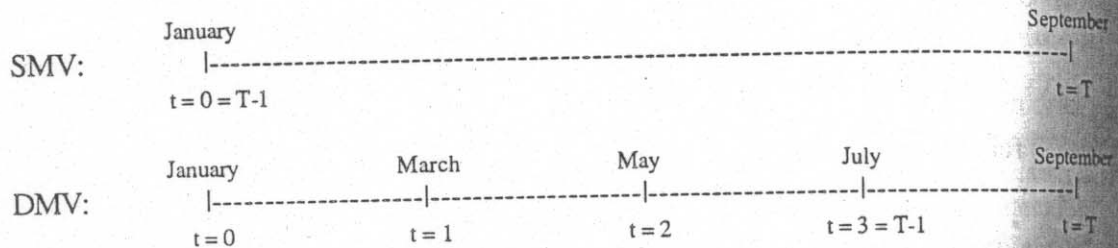[6]See Appendix A for the derivation of expression (11).

compare the DMV with the alternative hedging models to better understand its advantages over them.

The most popular hedging model is the standard SMV, which is represented by

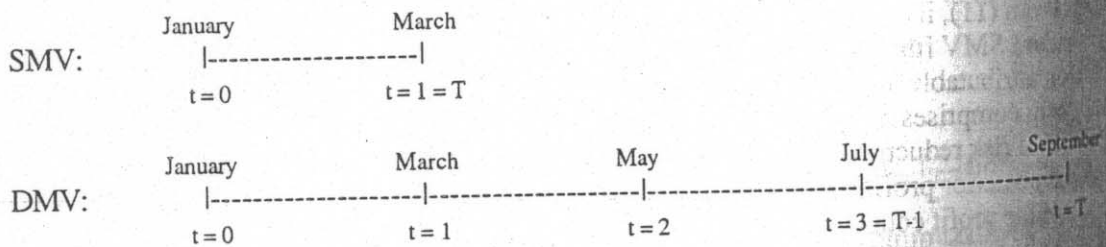$$(12) \quad H_0 = \frac{Cov_0(p_T, f_{T,T})}{Var_0(f_{T,T})}$$

It can be shown that the standard SMV is nested in the DMV when it is assumed that (i) the current date is the next-to-terminal time or (ii) that the first revision date is the terminal time. We will label them as SMV Case (i) and SMV Case (ii), respectively.

In the SMV Case (i) we have, for example,

```
             January                                                          September
SMV:            |--------------------------------------------------------------------|
             t = 0 = T-1                                                        t = T


             January          March            May             July          September
DMV:            |-------------------|-------------------|-------------------|-------------------|
             t = 0             t = 1           t = 2          t = 3 = T-1       t = T
```

Therefore, in this interpretation the standard SMV is like a DMV with the restrictions that there is no "mark to market" and that $I_t = I_0$ and $H_t = H_0$ for t = March, May, and July. In other words, the standard SMV does not allow for storage and hedging updates in March through July, and it ignores "marking to market." Note also that in this instance the SMV hedge has to be placed with the September futures contract or any later contract, whereas the DMV in January can be placed with the March futures contract or any later contract.

Case (ii) of the standard SMV is exemplified by

```
             January          March
SMV:            |-------------------|
             t = 0             t = 1 = T


             January          March            May             July          September
DMV:            |-------------------|-------------------|-------------------|-------------------|
             t = 0             t = 1           t = 2          t = 3 = T-1       t = T
```

In this alternative interpretation, the standard SMV follows from the DMV by imposing the restrictions $I_t = 0$ and $H_t = 0$ for t = March, May, and July. This means that in this instance the standard SMV assumes that the cash and futures positions will both be liquidated in March, and that there will be neither storage nor hedging from March through September.

A more sophisticated type of SMV is the "margin return" SMV. This hedge ratio is static because it does not allow storage and hedging updates, but it does take into account the cash flows that occur as a result of "marking to market." In our notation, this hedge ratio is obtained by imposing the restrictions $I_t = I_0$ and $H_t = H_0$ in expression (2), which yields

$$(13) \quad \pi_t = \{ p_t - r [p_{t-1} + c(I_0)] + (f_{t-1,t} - f_{t,t}) H_0 \} I_0$$

for t = March, May, and July. A close look at the "margin return" reveals that it is like Case (i) of the standard SMV, but without the restriction of no "mark to market."

Finally, the MHMV is a dynamic hedge ratio because it allows hedging updates, but it does not permit storage updates (Mathews and Holthausen). In such instance, $I_t = I_0$, and profits can be represented by

$$(14) \quad \pi_t = \{p_t - r [p_{t-1} + c(I_0)] + (f_{t-1,t} - f_{t,t}) H_t\} I_0$$

instead of (2), for t = March, May, and July. The advantage of the MHMV over the SMVs is that it accounts for hedging revisions, but it is more restrictive than the DMV in that storage updates are precluded.

In summary, well-known applied hedging models are special cases of the DMV because they can be derived from the DMV by imposing specific restrictions. Hence, we can conclude that the DMV is a more general hedging model than any of the alternatives discussed in this section.

Empirical Results

In this section, we will compare the empirical estimates of the DMV, the SMVs, and the MHMV. The DMV is considerably more tractable than other dynamic hedging models, but still is more complex to estimate than either of the alternative applied hedging models. Hence, it is important to know how different the DMV hedge position may be, and what factors may affect such difference in practice, in order to better assess the trade-off between its added complexity and its expected superior performance as a hedging tool.

To compare the hedge ratios from the alternative models, we considered hedging corn with a two-month decision horizon,[7] and two alternative planning horizons (four and six months). The terminal dates were selected to match the delivery months for corn in the Chicago Board of Trade, e.g., March, May, July, September, and December. For example, if the terminal date is December and the planning horizon is four months, then t = 0 = August, t = 1 = T-1 = October, and t = T = December. If the terminal date is December but the planning horizon is six months instead of four, then t = 0 = June, t = 1 = August, t = 2 = T-1 = October, and t = T = December. The futures prices are the settlement prices on the first Thursday of the month, taken from the Wall Street Journal. Cash prices are cash prices for North-Central Iowa on the first Thursday of the month, reported by the Iowa State University Market News. The interest rate is the annual average of the interest rate for one-year T-bills, obtained from the Federal Reserve Bank of Kansas City. The period analyzed comprises the crop years 1978/79 through 1990/91.

To simplify the calculations, we assumed that the decision maker has a fixed storage capacity of Q units of corn. We hypothesize that per-unit variable storage cost (c) is a finite constant up to capacity ($0 \leq I_t \leq Q$), and is infinite above capacity ($I_t > Q$). The specific figure used in the calculations is c = 0.035 cents per bushel over a two-month period.[8]

The results are summarized in Tables 1 and 2 (see Appendix B for a detailed explanation of the estimation procedure). Table 1 contains the hedge ratios for a decision maker with a four-month planning horizon. It can be observed that the values for the DMV are similar to the values of either of the SMVs. This similarity holds when the data were pooled in estimating the hedge ratios or individually estimated for each terminal date ("Aggregate" indicates the pooled estimates). The most dissimilar hedges are those placed in March; in this instance, the SMV Case (i) is 13 percent greater than the DMV and the SMV Case (ii) is 19 percent smaller than the DMV. The MHMV, on the other hand, is the hedge ratio that exhibits greater differences with respect to the others. The most important insight from Table 1, however, is that the major differences are not

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[7]This means that storage and hedging decisions are revised bimonthly.

[8]This figure is an average of the on-farm storage costs provided by Iowa State University extension specialists.

Table 1. Estimated hedge ratios corresponding to a four-month planning horizon

Current Date (0)	Terminal Date (T)	Standard SMV		MHMV	DMV	Percentage Years with Storage at (Date T-1)	
		Case (i)	Case (ii)				
May	September	1.02	0.89	1.15	0.93	8	(Jul)
March	July	0.85	0.61	1.00	0.75	31	(May)
January	May	0.78	0.74	0.80	0.79	92	(Mar)
November	March	0.43	0.38	0.25	0.46	38	(Jan)
August	December	0.92	0.98	0.95	0.95	38	(Oct)
Aggregate	Aggregate	0.88	0.82	1.00	0.86	42	(Agg)

Table 2. Estimated hedge ratios corresponding to a six-month planning horizon

Current Date (0)	Terminal Date (T)	Standard SMV		MHMV	DMV		Percentage Years with Storage at (Date T-1)		
		Case (i)	Case (ii)		Purch.	No Purch.	Purch.	No Purch.	(T-1)
Mar	Sep	1.03	0.56	1.01	0.51	0.51	31	31	(May)
Jan	Jul	0.85	0.78	0.99	0.99	1.01	92	100	(Mar)
Nov	May	0.57	0.37	0.06	0.32	0.69	38	77	(Jan)
Sep	Mar	0.62	0.79	0.51	0.84	0.93	23	46	(Nov)
June	Dec	0.93	0.87	0.82	0.80	0.87	15	31	(Aug)
Aggregate	Aggregate	0.87	0.78	0.76	0.75	0.83	40	57	(Agg)

among the alternative hedge ratios but rather among the different storage periods for each hedge ratio. For example, the DMV equals 0.95 when the current date is August, but only 0.46 when the current date is November.

There are at least two possible explanations for the noticeable seasonal differences among the hedge estimates. The first is that the unpooled data consist of only thirteen observations for each estimated hedge, which leads to less precise estimates. The second is that, in fact, the hedge ratios do change throughout the year. Evidence in this sense is provided by Baillie and Myers, who estimated SMVs for corn by means of a bivariate GARCH model and obtained hedge estimates covering the range of almost zero up to 1.5.

The existence of seasonal differences in the hedge ratios is particularly important in the case of the DMV because the frequency of profitable opportunities to store vary substantially during the year. For example, Table 1 reveals that the decision maker would have stored from July to September in only 8 percent of the years, compared to 92 percent of the years for March to May. Therefore, the expected loss in the objective function from choosing the SMV Case (ii) rather than the DMV in May is minimal because there is a very high probability of liquidating both cash and futures positions in July.[9,10] Similarly, the expected loss in the objective function from choosing the SMV Case (i) rather than the DMV in January is small because most years the agent will store from March to May.[11] In contrast, the expected losses in the objective function from selecting the most appropriate SMV instead of the DMV are highest in March, November, and August, because neither Case (i) nor Case (ii) of the SMV capture well the actual probabilities to continue storing at the first revision date.

Table 2 reports the results for the six-month planning horizon. In this instance there are two types of DMV: one allowing for purchases, and the other not permitting purchases [i.e., storage is determined by expressions (3) and (3'), respectively]. The DMVs are more dissimilar with respect to the SMVs than in the case of a four-month planning horizon. For example, the SMV Case (i) ranges from 102 percent greater to 26 percent smaller than the purchase-allowed DMV (in March and September, respectively). Similarly, the SMV Case (ii) ranges from 16 percent greater to 21 percent smaller than the purchase-allowed DMV (in January and November, respectively). As in Table 1, the differences across terminal dates for each hedge ratio are greater than the differences among hedge ratios for each terminal date. We should note, however, that the relatively small number of observations in the unpooled estimates may exaggerate the actual differences in the hedges, because the number of covariance estimates involved in the DMVs is considerably larger.

Our analysis reveals that a commodity with marked seasonality in prices such as corn may offer quite different opportunities to store during the year, and that the hedge ratios may have a substantial seasonality. This result suggests that there are circumstances in which there are larger potential gains from employing the DMV instead of the most appropriate standard SMV. These situations arise when the probability of continuing storing at the revision dates is close to 0.5. When that probability is close to 1 there is little loss to expect by using the SMV Case (i) rather than the DMV. Similarly, if that probability is close to 0 the potential losses from employing the SMV Case (ii) instead of the DMV are relatively small. Note, however, that employing the SMV Case (ii) when the probability of continuing storing is close to 1 or the SMV Case (i) when the probability is close to 0, leads to the worst decision. Therefore, analyzing the seasonality of the

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[9]Recall that the SMV Case (ii) is like a DMV in which the cash and futures positions are liquidated at the first revision date with probability 1, and that the SMV Case (i) is like a DMV in which the cash and futures positions remain unchanged through the whole planning horizon with probability 1.

[10]But note that the opposite is true for the SMV Case (i), i.e., the expected loss in the objective function is large if the SMV Case (i) is chosen in May.

[11]But the expected loss in the objective function is large if the SMV Case (ii) is chosen in January.

storage opportunities gives an important clue as to the circumstances in which it may be more important to estimate the DMV, and as to which SMV is the most appropriate to use [i.e., Case (i) when the probability to continue storing is close to 1, and Case (ii) when that probability is close to 0].

As a final observation, we should also note that to estimate the DMV it is necessary to carefully analyze the opportunities to profit from storage. This is a very important byproduct of the DMV estimation, as those opportunities constitute the main reason to be involved in storage. However, this point is typically ignored in the standard hedging literature.

Conclusions

A dynamic hedging model is presented. The model is developed in analogy to the traditional static minimum variance hedge ratio (SMV), and for this reason is called the dynamic minimum variance hedge ratio (DMV). The DMV is more complex than the SMV, but it is substantially simpler than other dynamic hedging models available in the literature. The most important characteristics of the DMV are that it allows for updates of both cash and futures positions, and that it is relatively tractable for practical applications.

It is shown that the SMV is a special case of the DMV under two alternative scenarios. First, the SMV equals the DMV if no hedging and storage updates are allowed within the planning horizon [SMV Case (i)]. Second, the SMV is identical to the DMV if there is a zero probability of storing beyond the first revision date [SMV Case (ii)]. The advantage of choosing the DMV instead of the most appropriate type of SMV is greater when (a) the probability to continue storing at the first revision date is neither close to zero nor close to one, and (b) the value of the DMV differs substantially from that of the appropriate SMV.

The estimation of the DMV is illustrated with an application to corn storage assuming four- and six-month planning horizons. In the four-month planning horizon scenario, employing the DMV to hedge corn does not lead to large potential gains in efficiency compared to using the appropriate SMV. This result is attributable to the similar values obtained for the DMV and the appropriate SMV, the only exception being the hedges placed in March. In the six-month planning horizon, in contrast, using the DMV is potentially important because of the larger differences among the DMV and the SMVs, and the greater frequency of probabilities to continue storage at the revision dates differing from near 0 or near 1.

Appendix A: Derivation of Expression (11)

By proceeding in the same way as to obtain (5) and (8), we finally obtain the variance of terminal wealth from the standpoint of the current date $t = 0$ as

$$
\begin{aligned}
\text{(A1)} \quad \text{Var}_0(W_T) &= \text{Var}_0(r^{T-1}\,\pi_1 + \ldots + r\,\pi_{T-1} + \pi_T) \\
&= r^{2T-2}\,\text{Var}_0(\pi_1) + \ldots + r^2\,\text{Var}_0(\pi_{T-1}) + \text{Var}_0(\pi_T) + 2\,r^{2T-3}\,\text{Cov}_0(\pi_1, \pi_2) \\
&\quad + \ldots + 2\,r^T\,\text{Cov}_0(\pi_1, \pi_{T-1}) + 2\,r^{T-1}\,\text{Cov}_0(\pi_1, \pi_T) + 2\,r^{2T-5}\,\text{Cov}_0(\pi_2, \pi_3) \\
&\quad + \ldots + 2\,r^{T-1}\,\text{Cov}_0(\pi_2, \pi_{T-1}) + 2\,r^{T-2}\,\text{Cov}_0(\pi_2, \pi_T) + \ldots + 2\,r\,\text{Cov}_0(\pi_{T-1}, \pi_T)
\end{aligned}
$$

(A1') $\text{Var}_0(W_T) = r^{2T-2} I_0^2 [\text{Var}_0(p_1) + H_0^2 \text{Var}_0(f_{1,1}) - 2 H_0 \text{Cov}_0(p_1, f_{1,1})]$

$$+ \ldots + r^2 \text{Var}_0(\pi_{T-1}) + \text{Var}_0(\pi_T) + 2 r^{2T-3} I_0 [\text{Cov}_0(p_1, \pi_2) - H_0 \text{Cov}_0(f_{1,1}, \pi_2)]$$

$$+ \ldots + 2 r^T I_0 [\text{Cov}_0(p_1, \pi_{T-1}) - H_0 \text{Cov}_0(f_{1,1}, \pi_{T-1})]$$

$$+ 2 r^{T-1} I_0 [\text{Cov}_0(p_1, \pi_T) - H_0 \text{Cov}_0(f_{1,1}, \pi_T)] + 2 r^{2T-5} \text{Cov}_0(\pi_2, \pi_3)$$

$$+ \ldots + 2 r^{T-1} \text{Cov}_0(\pi_2, \pi_{T-1}) + 2 r^{T-2} \text{Cov}_0(\pi_2, \pi_T) + \ldots + 2 r \text{Cov}_0(\pi_{T-1}, \pi_T)$$

The FOC corresponding to the minimization of (A1') with respect to H_0 is

(A2) $\dfrac{\partial \text{Var}_0(W_T)}{\partial H_0} = r^{2T-2} I_0^2 [2 H_0 \text{Var}_0(f_{1,1}) - 2 \text{Cov}_0(p_1, f_{1,1})] - 2 r^{2T-3} I_0 \text{Cov}_0(f_{1,1}, \pi_2)$

$$- \ldots - 2 r^T I_0 \text{Cov}_0(f_{1,1}, \pi_{T-1}) - 2 r^{T-1} I_0 \text{Cov}_0(f_{1,1}, \pi_T) = 0$$

Solving (A2) for H_0 yields expression (11).

Appendix B: Application of the DMV to Corn Storage

To emphasize the operationality of the DMVs, these hedge ratios were estimated by means of the Microsoft Excel spreadsheet except for the ARIMA model (B1). The spreadsheet is available from the authors upon request. The procedure to calculate the DMV corresponding to date T-3 (H_{T-3}^*) under the storage decision rule (3) is as follows:

Step 1.   *Select the futures contracts used for hedging at the current and at the revision dates.*

Step 2.   *Fit models to estimate the conditional forecasts of cash and futures prices.*
       In this application we used monthly data for the K = 13 years from 1978/79 (k = 1) through 1990/91 (k = K). We fitted the following ARIMA model to predict cash prices:

(B1) $(1 - B) (1 - B^{12}) (1 - \alpha_{12k} B^{12}) p_{tk} = (1 - \beta_{24k} B^{24}) a_{tk}, \quad k = 1, \ldots, K$

where B is the "backshift" operator (i.e., $B p_{tk} = p_{t-1k}$), and a_{tk} is n.i.d. $(0, \sigma_{ak}^2)$. The subscript k highlights that the ARIMA model was updated every year, to incorporate the new data available.
       We also fitted several ARIMA specifications to forecast futures prices, but the current futures price performed better than any of them. Hence, we used

(B2) $f_{t,t} = f_{t-1,t} + u_t$

where u_t is n.i.d. $(0, \sigma_u^2)$, to calculate the conditional forecasts of futures prices.

Step 3.   *Estimate the variances of futures prices at t and the covariances between futures and cash prices at t conditional on the information at t-1.*
       Following the procedure advanced by Peck, we employed expressions (B1) and (B2) to obtain the conditional forecasts needed for the estimation of the conditional variances and covariances:

(B3) $\text{Var}_{T-i-1}(f_{T-i,T}) = \dfrac{1}{K} \sum_{k=1}^{K} (f_{T-i,Tk} - f_{T-i-1,Tk})^2$

(B4) $\text{Cov}_{T-i-1}(p_{T-i}, f_{T-i,T}) = \frac{1}{K} \sum_{k=1}^{K} [p_{T-ik} - \mathbb{E}_{T-i-1k}(p_{T-ik})] (f_{T-i,Tk} - f_{T-i-1,Tk})$

for $i = 0, 1, 2$, and where $\mathbb{E}_{T-i-1k}(p_{T-ik})$ is calculated by means of (B1).

Step 4.   *Obtain the ex ante optimum storage decision at each revision date.*
    The *ex ante* optimum storage was obtained by applying expression (3), i.e.,

(B5)   a. $I_{T-1-jk} = Q$ if $\mathbb{E}_{T-1-jk}(p_{T-jk}) > r_k (p_{T-1-jk} + c)$ for $j = 0, 1$
       b. $I_{T-1-jk} = 0$ otherwise

Step 5.   *Obtain the DMV corresponding to T-1.*
    The DMV for T-1 was estimated by employing the variance and covariance conditional on the information in T-1, which were computed in Step 3.

(B6) $H_{T-1}^{*} = \dfrac{\text{Cov}_{T-1}(p_T, f_{T,T})}{\text{Var}_{T-1}(f_{T,T})}$

Step 6.   *Calculate the realized profits at T.*
    The realized profits at the terminal date were found by using the optimal storage and hedging at T-1 (from Steps 4 and 5, respectively):

(B7) $\pi_{Tk} = [p_{Tk} - r_k (p_{T-1k} + c) + (f_{T-1,Tk} - f_{T,Tk}) H_{T-1}^{*}] I_{T-1k}$

Step 7.   *Compute the expected profits at T conditional on the information at T-2.*
    By application of the results in Maddala (p. 365), the expectation of profits at T conditional on the information at T-2 was calculated as[12]

(B8) $\mathbb{E}_{T-2k}(\pi_{Tk}) = Q (1 - r_k) \text{Std}_{T-2k}(p_{T-1k}) [h_{T-2,T-1k} \Phi(h_{T-2,T-1k}) + \phi(h_{T-2,T-1k})]$

where: $h_{T-2,T-1k} = \mathbb{E}_{T-2k}(x_{T-1k})/[(1 - r_k) \text{Std}_{T-2k}(p_{T-1k})]$
       $x_{tk} = \mathbb{E}_{tk}(p_{t+1k}) - r_k (p_{tk} + c)$
       $\Phi(\cdot) = $ cumulative distribution function of the standard normal distribution
       $\phi(\cdot) = $ density function of the standard normal distribution

The values of $\mathbb{E}_{T-2k}(x_{T-1k})$ and $\text{Std}_{T-2k}(p_{T-1k})$ were obtained from the ARIMA model (B1).

Step 8.   *Obtain the covariance between futures at T-1 and profits at T, conditional on the information at T-2.*
    Given the realized profits and the conditional expected profits derived in Steps 6 and 7, respectively, this conditional covariance was obtained as

(B9) $\text{Cov}_{T-2}(f_{T-1,T}, \pi_T) = \frac{1}{K} \sum_{k=1}^{K} (f_{T-1,Tk} - f_{T-2,Tk}) [\pi_{Tk} - \mathbb{E}_{T-2k}(\pi_{Tk})]$

---

[12]The derivation of expression (B8) is omitted to save space, but it is available from the authors upon request.

Step 9.   *Calculate the DMV at T-2.*

The DMV corresponding to date T-2 was computed employing the variance and covariances conditional on the information at T-2 (from Steps 3 and 8, respectively):

$$(B10) \quad H_{T-2k}{}^* = \frac{Cov_{T-2}(p_{T-1}, f_{T-1,T})}{Var_{T-2}(f_{T-1,T})} + \frac{Cov_{T-2}(f_{T-1,T}, \pi_T)}{r_k \, Q \, Var_{T-2}(f_{T-1,T})}$$

Step 10.   *Compute the realized profits at T-1.*

The realized profits at date T-1 were obtained using the optimal storage and hedging at T derived in Steps 4 and 9, respectively:

$$(B11) \quad \pi_{T-1k} = [p_{T-1k} - r_k (p_{T-2k} + c) + (f_{T-2,Tk} - f_{T-1,Tk}) H_{T-2k}{}^*] \, I_{T-2k}$$

Step 11.   *Calculate the expected profits at T-1 and T conditional on the information at T-3.*

Using formulas analogous to expression (B8), these conditional expectations are

$$(B12) \quad \mathbb{E}_{T-3k}(\pi_{T-1k}) = Q (1 - r_k) \, Std_{T-3k}(p_{T-2k}) \, [h_{T-3,T-2k} \, \Phi(h_{T-3,T-2k}) + \phi(h_{T-3,T-2k})]$$

$$(B13) \quad \mathbb{E}_{T-3k}(\pi_{Tk}) = Q (1 - r_k) \, Std_{T-3k}(p_{T-1k}) \, [h_{T-3,T-1k} \, \Phi(h_{T-3,T-1k}) + \phi(h_{T-3,T-1k})]$$

where: $h_{T-3,T-2k} = \mathbb{E}_{T-3k}(x_{T-2k})/[(1 - r_k) \, Std_{T-3k}(p_{T-2k})]$
$\quad\quad\quad h_{T-3,T-1k} = \mathbb{E}_{T-3k}(x_{T-1k})/[(1 - r_k) \, Std_{T-3k}(p_{T-1k})]$

Step 12.   *Obtain the covariances between futures at T-2 and profits at T-1 and T, conditional on the information at T-3.*

Given the expected profits conditional on the information at T-3 (from Step 11), the conditional covariances were calculated as follows:

$$(B14) \quad Cov_{T-3}(f_{T-2,T}, \pi_{T-1}) = \frac{1}{K} \sum_{k=1}^{K} (f_{T-2,Tk} - f_{T-3,Tk}) \, [\pi_{T-1k} - \mathbb{E}_{T-3k}(\pi_{T-1k})]$$

$$(B15) \quad Cov_{T-3}(f_{T-2,T}, \pi_T) = \frac{1}{K} \sum_{k=1}^{K} (f_{T-2,Tk} - f_{T-3,Tk}) \, [\pi_{Tk} - \mathbb{E}_{T-3k}(\pi_{Tk})]$$

Step 13.   *Calculate the DMV at T-3.*

The DMV corresponding to date T-3 was computed by means of the variance and covariances conditional on the information at T-3 (from Steps 3 and 12, respectively):

$$(B16) \quad H_{T-3k}{}^* = \frac{Cov_{T-3}(p_{T-2}, f_{T-2,T})}{Var_{T-3}(f_{T-2,T})} + \frac{Cov_{T-3}(f_{T-2,T}, \pi_{T-1})}{r_k \, Q \, Var_{T-3}(f_{T-2,T})} + \frac{Cov_{T-3}(f_{T-2,T}, \pi_T)}{r_k{}^2 \, Q \, Var_{T-3}(f_{T-2,T})}$$

To obtain the DMV at date T-3 under the storage decision rule (3'), it is necessary to replace Steps 4 and 11 with Steps 4' and 11' below.

Step 4'. *Obtain the ex ante optimum storage decision at each revision date.*
   Application of expression (3') yields

(B5') a. $I_{T-1k} = Q$ if $\mathbb{E}_{T-1k}(p_{Tk}) > r_k (p_{T-1k} + c)$ and $I_{T-2k} > 0$
   b. $I_{T-1k} = 0$ otherwise

(B5") a. $I_{T-2k} = Q$ if $\mathbb{E}_{T-2k}(p_{T-1k}) > r_k (p_{T-2k} + c)$ or $\mathbb{E}_{T-2k}(p_{Tk}) > r^2 p_{T-2k} + (r + r^2) c$
   b. $I_{T-2k} = 0$ otherwise

instead of (B5).

Step 11'. *Calculate the expected profits at T-1 and T conditional on the information at T-3.*
   Under the storage decision rule (3'), expressions (B12') and (B13') must be employed
instead of (B12) and (B13), respectively:[13]

(B12') $\mathbb{E}_{T-3k}(\pi_{T-1k}) = Q (1- r_k) \, \text{Std}_{T-3k}(p_{T-2k}) \, [h_{T-3,T-2k} \Phi(h_{T-3,T-2k}) + \phi(h_{T-3,T-2k})]$

$$\text{if } h_{T-3,T-2k} \geq k_{T-3,T-2k}$$

$$= Q (1- r_k) \, \text{Std}_{T-3k}(p_{T-2k}) \, [k_{T-3,T-2k} \Phi(k_{T-3,T-2k}) + \phi(k_{T-3,T-2k})]$$

$$\text{if } h_{T-3,T-2k} < k_{T-3,T-2k}$$

(B13') $\mathbb{E}_{T-3k}(\pi_{Tk}) = Q (1- r_k) \, \text{Std}_{T-3k}(p_{T-1k}) \, [h_{T-3,T-1k} \Phi(h_{T-3,T-1k}) + \phi(h_{T-3,T-1k})] \, \Phi(h_{T-3,T-2k})$

$$\text{if } (h_{T-3,T-2k} + k_{T-3,T-2k}) \geq 0$$

$$= Q (1- r_k) \, \text{Std}_{T-3k}(p_{T-1k}) \, [h_{T-3,T-1k} \Phi(h_{T-3,T-1k}) + \phi(h_{T-3,T-1k})] \, \Phi(-k_{T-3,T-2k})$$

$$\text{if } (h_{T-3,T-2k} + k_{T-3,T-2k}) < 0$$

where: $k_{T-3,T-2k} = \mathbb{E}_{T-3k}(y_{T-2k})/[(1- r_k^2) \, \text{Std}_{T-3k}(p_{T-1k})]$
   $y_{tk} = \mathbb{E}_{tk}(p_{t+2k}) - r^2 p_{tk} - (r + r^2) c$

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[13]The derivation of expressions (B12') and (B13') can be obtained from the authors upon request.

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