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**Separability of Stochastic Production
Decisions from Producer Risk Preferences
In the Presence of Financial Markets**

by

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Abstract: This paper presents a unified treatment of the production and financial decisions available to a firm facing frictionless financial markets and a stochastic production technology under minimal assumptions on the firm's stochastic technology and objective function. The specific focus is on separation results for stochastic technologies, that is, on conditions under which the optimal production decision may be determined without regard to the risk preferences of the firm's owners. Necessary and sufficient conditions for separation, which generalize existing results, are presented.

Key words: stochastic production, financial markets, separation

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Separability of Stochastic Production Decisions from Producer Risk Preferences in the Presence of Financial Markets

An important theme in the literature on production under uncertainty concerns the possibility that, in the presence of financial markets, the optimal output choice for a firm may be independent of the risk attitudes of the owner. Such is obviously the case in the presence of complete markets. Results of this type allow the production side of the market to be modelled solely in terms of the physical production technology and the structure of the financial market without regard to the risk preferences of the market participants. Hence such results are particularly convenient for comparative-static analysis and the modelling of equilibrium outcomes because the risk preferences of firm owners enter only through their effects on hedging, speculation, and other activities designed to cope with systemic and idiosyncratic risk.

A set of sufficient, or necessary and sufficient, conditions for such an outcome is generically referred to as a ‘separation result’. One example is the classic finding (Jean-Pierre Danthine, 1978; Duncan Holthausen, 1979; and Ronald Anderson and Jean-Pierre Danthine, 1981, 1983a, 1983b) that, with a single-output, non-stochastic technology and stochastic prices, the output level chosen by an expected-utility maximizing firm with the ability to trade in an active forward market will be independent of the decisionmaker’s risk preferences. The received wisdom seems to be that separation does not apply when output is stochastic, when there is basis risk, or, in an intertemporal setting when there is interest-rate risk.

Similar themes have been addressed, from a somewhat different perspective, in finance theory. In the basic Arrow–Debreu model, with a complete set of state-contingent markets, and assuming that all individual preferences are monotonic in consumption, firms will maximize their profits regardless of their ownership structure. Beginning with Franco Modigliani and Merton Miller (1958), finance theorists have examined whether this unanimous agreement on the firm’s production decision will persist in the presence of incomplete markets. Important contributions include those of Roy Radner (1974) and Sanford Grossman and Joseph Stiglitz (1980).

These two problems are effectively identical when the only security issued by the firm is equity, that is, proportional shares in residual income. More generally, suppose that,

apart from equity, the firm trades only in securities with given state-contingent returns, whose prices are determined in an asset market equilibrium independent of the firm's output decisions. Then, regardless of whether the firm acts as an issuer, buyer or (short) seller of these securities, only holders of equity will be affected by the firm's output decisions. Hence, the problem facing these individuals is the same as that of an entrepreneurial firm owner with the opportunity to trade in financial markets.

The object of this paper is to present a unified treatment of the production and financial decisions available to a firm facing frictionless financial markets and a stochastic production technology under minimal assumptions about the firm's technology and objective function. The specific focus is on separation results for stochastic technologies, that is, on conditions under which the optimal production decision may be determined without regard to the risk preferences of the firm's owners. It is of particular interest to note that a concept of separation is implicit in the standard division of labor between finance theory and microeconomics proper. Where finance theory typically focuses on the determination of asset prices in an endowment economy, thus taking production as given, much of microeconomics takes interest rates and other asset prices as given. Our results characterize conditions under which this division of labor is justified, and, conversely, conditions under which it is likely to be problematic.

The crucial analytical tool is the concept of a derivative-cost function, characterized as the minimum cost of achieving a given vector of state-contingent returns through a combination of production choices and trade in financial assets. The derivative-cost function corresponds to the firm's maximal buying price for a derivative asset. Thus, it is closely related to the asset pricing functionals, which underlie the pathbreaking arbitrage and asset pricing results of Stephen Ross (1976, 1987), Eliezer Prisman (1986), Stephen Clark (1993), and others. The analysis uses the derivative-cost function to generalize previous separation results showing that separation applies when the return vectors associated with financial assets span either the entire state space or a subspace encompassing the firm's 'rational' production choices. It is shown that spanning is sufficient, but not necessary, for separation.

In what follows, we first present our notation and some basic results from convex analysis which are useful to the development of our ideas. We then specify the firm's stochastic envi-

ronment, both in terms of production opportunities and its access to financial markets. Next we characterize the derivative-cost function and briefly develop its most relevant properties. Among other results, we show that firms, regardless of their risk preferences, act as profit maximizers, given an appropriate set of state-claim prices. The degree to which the firm's production decisions are separable from its risk attitudes, therefore, reduces to determining how the firm's internal or virtual equilibrium state-claim prices relate to its risk attitudes. After that we address the issue of separation proper and derive necessary and sufficient conditions for separation. The nonstochastic production problem is briefly considered and existing separation results are generalized. Then the paper concludes.

1 Notation and Preliminaries

Denote the unit vector by $\mathbf{1} \in \mathbb{R}_+^S$. For $\mathbf{m}, \mathbf{m}' \in \mathbb{R}^S$, the notation $\mathbf{m} \cdot \mathbf{m}'$ denotes the componentwise product of the two vectors. That is, $\mathbf{m} \cdot \mathbf{m}' = (m_1 m'_1, \dots, m_s m'_s)$. The notation $\mathbf{m} \mathbf{m}'$ for two conformable vectors denotes the usual inner product. Denote the relative interior of a convex set $A \subseteq \mathbb{R}^S$ by riA .

For a convex function¹ $f : \mathbb{R}^S \rightarrow \mathbb{R}$, its *subdifferential* at $\mathbf{m}$ is the closed, convex set:

$$\partial f(\mathbf{m}) = \{\mathbf{k} \in \mathbb{R}^S : f(\mathbf{m}) + \mathbf{k}(\mathbf{m}' - \mathbf{m}) \leq f(\mathbf{m}') \text{ for all } \mathbf{m}'\}. \quad (1)$$

The elements of $\partial f(\mathbf{m})$ are referred to as *subgradients*. If f is differentiable at $\mathbf{m}$, $\partial f(\mathbf{m})$ is a singleton and corresponds to the usual gradient, which we denote by $\nabla f(\mathbf{m}) = [f_1(\mathbf{m}), \dots, f_s(\mathbf{m})]$ where subscripts denote partial derivatives. Conversely, if $\partial f(\mathbf{m})$ is a singleton, f is differentiable at $\mathbf{m}$.

For f convex, its convex conjugate is denoted

$$f^*(\mathbf{k}) = \sup_{\mathbf{m}} \{\mathbf{k} \mathbf{m} - f(\mathbf{m})\}.$$

If f is proper and closed,² then f^* is also a proper and closed convex function and

$$f(\mathbf{m}) = \sup_{\mathbf{k}} \{\mathbf{k} \mathbf{m} - f^*(\mathbf{k})\}, \quad (2)$$

¹These results on convex functions are all drawn directly from R. Tyrell Rockafellar (1970).

² f is proper if $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x}$, and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$. A proper convex function is closed if it is lower-semicontinuous.

and on the relative interior of their domains

$$\mathbf{k} \in \partial f(\mathbf{m}) \Leftrightarrow \mathbf{m} \in \partial f^*(\mathbf{k}). \quad (3)$$

2 State-Contingent Technologies and the Asset Structure

We model a price-taking firm facing a stochastic environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain. Uncertainty is resolved by ‘Nature’ making a choice from $\Omega = \{1, 2, \dots, S\}$. Each element of Ω is referred to as a state of nature. The only assumption on the firm’s preferences is that they are at least weakly increasing in period 0 consumption.

The firm’s stochastic production technology is represented by a single-product, state-contingent input correspondence.³ To make this explicit, let $\mathbf{x} \in \mathbb{R}_+^N$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathbf{z} \in \mathbb{R}_+^S$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0. If state $s \in \Omega$ is realized (picked by ‘Nature’), and the producer has chosen the *ex ante* input–output combination $(\mathbf{x}, \mathbf{z})$, then the realized or *ex post* output in period 1 is z_s .

The continuous input correspondence, $X : \mathbb{R}_+^S \rightarrow \mathbb{R}_+^N$, which maps state-contingent output vectors into input sets that are capable of producing that state-contingent output vector, is defined by

$$X(\mathbf{z}) = \{\mathbf{x} \in \mathbb{R}_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

Intuitively, $X(\mathbf{z})$ is associated with everything on or above the isoquant for the state-contingent output vector $\mathbf{z}$. At points it will be convenient to consider an alternative, but equivalent, representation, which we refer to as the state-contingent output set,

$$Z(\mathbf{x}) = \{\mathbf{z} \in \mathbb{R}_+^S : \mathbf{x} \in X(\mathbf{z})\}.$$

³For a generalization to the multiple-output case, see Chambers and Quiggin (2000, Chapter 4). Our results extend straightforwardly to that case.

Intuitively, $Z(\mathbf{x})$ can be thought of as all state-contingent outputs on or below a state-contingent product transformation curve. We impose the following properties on $X(\mathbf{z})$:

X.1 $X(\mathbf{0}_{M \times S}) = \mathbb{R}_+^N$ (no fixed costs), and $\mathbf{0} \notin X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$ (no free lunch).

X.2 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

X.3 $\lambda X(\mathbf{z}) + (1 - \lambda)X(\mathbf{z}') \subseteq X(\lambda \mathbf{z} + (1 - \lambda)\mathbf{z}') \quad 0 \leq \lambda \leq 1$.

X.4 X is continuous.

The first part of X.1 says that doing nothing is always feasible, while its second part says that realizing a positive output in any state of nature requires the commitment of some inputs. X.2 says that if an input combination can produce a particular mix of state-contingent outputs then it can always be used to produce a smaller mix of state-contingent outputs. X.3 ensures that the cost function developed below is convex in state-contingent outputs.

Period 0 prices of inputs are denoted by $\mathbf{w} \in \mathbb{R}_{++}^N$ and are non-stochastic. Output price is stochastic, and we denote by $\mathbf{p} \in \mathbb{R}_{++}^S$ the vector of state-contingent output prices corresponding to the vector of state-contingent outputs. Producers take these state-contingent output prices and the prices of all inputs as given. The state-contingent revenue vector, denoted $\mathbf{p} \cdot \mathbf{z} \in \mathbb{R}_+^S$, has typical elements of the form $p_s z_s$.

Financial markets are frictionless, and the *ex ante* financial security payoffs are given by the $S \times J$ non-negative matrix $\mathbf{A}$. The vector of state-contingent payoffs on the j th financial asset is denoted $\mathbf{A}_j \in \mathbb{R}_+^S$, and its price is denoted v_j . Denote the span of the matrix $\mathbf{A}$ by M . The firm's portfolio vector, corresponding to the period 0 purchases of the financial assets, is denoted $\mathbf{h} \in \mathbb{R}^J$.

Dual to $X(\mathbf{z})$ is the cost function, $c : \mathbb{R}_{++}^N \times \mathbb{R}_+^S \rightarrow \mathbb{R}_+$, defined as

$$c(\mathbf{w}, \mathbf{z}) = \min_{\mathbf{x}} \{\mathbf{w}\mathbf{x} : \mathbf{x} \in X(\mathbf{z})\} \quad \mathbf{w} \in \mathbb{R}_{++}^N$$

if there exists an $\mathbf{x} \in X(\mathbf{z})$ and ∞ otherwise. Mathematically, $c(\mathbf{w}, \mathbf{z})$ is equivalent to the multi-product cost function familiar from non-stochastic production theory (Rolf Färe 1988). If the input correspondence satisfies properties X, $c(\mathbf{w}, \mathbf{z})$ satisfies (Robert Chambers and John Quiggin, 2000): $c(\mathbf{w}, \mathbf{z}) \geq 0$, $c(\mathbf{w}, \mathbf{0}_S) = 0$, and $c(\mathbf{w}, \mathbf{z}) > 0$ for $\mathbf{z} \geq \mathbf{0}, \mathbf{z} \neq \mathbf{0}$; $\mathbf{z}^o \geq \mathbf{z} \Rightarrow c(\mathbf{w}, \mathbf{z}^o) \geq c(\mathbf{w}, \mathbf{z})$; and $c(\mathbf{w}, \mathbf{z})$ is convex on $\mathbb{R}_+^S$ and continuous on the interior of

the region where it is finite.⁴

For $\mathbf{q} \in \mathfrak{R}_+^S$, the convex conjugate of c ,

$$c^*(\mathbf{w}, \mathbf{q}) = \sup_{\mathbf{z}} \{\mathbf{qz} - c(\mathbf{w}, \mathbf{z})\},$$

can be interpreted as a profit-function for the ‘price’ vector $\mathbf{q}$. Let $\mathbf{z}' \in \arg \sup \{\mathbf{qz} - c(\mathbf{w}, \mathbf{z})\}$, then

$$\mathbf{qz}' - c(\mathbf{w}, \mathbf{z}') \geq \mathbf{qz} - c(\mathbf{w}, \mathbf{z})$$

for all $\mathbf{z}$, and hence $\mathbf{q} \in \partial c(\mathbf{w}, \mathbf{z}')$. By (3), we then obtain $\mathbf{z}' \in \partial c^*(\mathbf{w}, \mathbf{q})$, which restates Hotelling’s lemma in terms of subdifferentials.

Dual to the financial asset structure is the ‘minimal investment function’ (for example, Prisman, 1986; and Ross, 1987) defined by the linear program

$$c^i(\mathbf{y}) = \min \{\mathbf{vh} : \mathbf{Ah} \geq \mathbf{y}\},$$

if $\{\mathbf{h} : \mathbf{Ah} \geq \mathbf{y}\}$ is nonempty and ∞ otherwise. $c^i(\mathbf{y})$, which is the maximal price the firm facing only the financial asset structure $\mathbf{A}$ would be willing to pay to acquire the state-contingent income vector $\mathbf{y}$, is mathematically equivalent to a multiple-output cost function for a linear production technology. It is positively linearly homogeneous and convex in $\mathbf{y}$, and it is linear on M (Prisman, 1986; Ross, 1987; Clark, 1993).

2.1 The stochastic production function

It is worthwhile to briefly compare the Arrow–Debreu state-space representation with the technology most commonly considered in the literature on production under uncertainty as well as in many financial applications, the stochastic production function (John Cochrane, 1991; Rahi, 1995). We do this for two reasons: First, it connects our specification to the one most typically encountered in the literature. And second, it highlights the need for the use of notions of subdifferentiability in place of the more familiar notion of differentiability in

⁴Because it is a cost function, c , also satisfies monotonicity and curvature properties in $\mathbf{w}$. We do not use these in what follows, and they are therefore not discussed. Chambers and Quiggin (2000) contains a complete discussion.

the formal arguments that follow. A stochastic production function is typically specified

$$z_s = f(x, \varepsilon_s)$$

where x is a scalar input, typically capital, chosen by the producer and ε is a random input taking the value ε_s in state s .

Chambers and Quiggin (1998, 2000) show that the cost function for this technology⁵ is given by

$$\begin{aligned} c(\mathbf{w}, \mathbf{z}) &= \inf \{wx : f(x, \varepsilon_s) \geq z_s \forall s\} \\ &= \sup_s \{w f^{-1}(z_s; \varepsilon_s)\} \end{aligned}$$

Its output set can be written as

$$Z(x) = \{\mathbf{z} : f(x, \varepsilon_s) \geq z_s, s \in \Omega\}.$$

As is apparent, this technical specification leads to a cost function which is not generally smoothly differentiable. It is this nonsmoothness in this important special case that motivates the use of subdifferentials and subgradients in place of the more usual derivatives and gradients.⁶ Under our assumptions, these subdifferentials always exist even when the cost structure is not smoothly differentiable as above.

Figure 1, drawn from John Cochrane (2001), illustrates by comparing a general state-contingent output set and that associated with a stochastic production function that agrees with the general set at a given equilibrium point. Notice, in particular, that $Z(x)$ maintains that state-contingent outputs are not substitutable for one another. Chambers and Quiggin (2000) offer a three-dimensional illustration of the same technology, which they refer to as output-cubical. In particular, if the technology does not allow X.2, and

$$Z(x) = \{\mathbf{z} : f(x, \varepsilon_s) = z_s, s \in \Omega\},$$

⁵Chambers and Quiggin (2000) consider the more general multiple input case and show that

$$c(\mathbf{w}, \mathbf{z}) \geq \max \{c_1(\mathbf{w}, z_1), \dots, c_S(\mathbf{w}, z_S)\}$$

where $c_i(\mathbf{w}, z_i)$ is the cost function corresponding to $f(\mathbf{x}, \varepsilon_i)$.

⁶At an intuitive level, little is lost by thinking of the subdifferential as the gradient of a smooth cost function.

the associated state-contingent output set is a single point. The range of state-contingent outputs, which can be feasibly produced, is then even more severely circumscribed and corresponds to a manifold emanating from the origin.⁷ This specification, thus, severely limits the ability of the firm to use its physical technology in conjunction with operations in financial markets to construct derivative assets. And, in some instances, for example, those of additive or multiplicative uncertainty, this specification can render the physical technology effectively redundant, thereby giving rise to separation results. Because a nonstochastic output with a stochastic output price is isomorphic to multiplicative production uncertainty, the best-known separation results can thus be viewed as a consequence of this nonsubstitutability.

3 Definition and Properties of the Derivative-Cost Function

Define the *derivative-cost function* $C : \mathbb{R}_+^S \rightarrow \mathbb{R}$, by

$$\begin{aligned} C(\mathbf{y}) &= \min_{\mathbf{h}, \mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + \mathbf{v}\mathbf{h} : \mathbf{A}\mathbf{h} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y}\} \\ &= \min_{\mathbf{r}, \mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + c^i(\mathbf{r}) : \mathbf{r} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y}\}. \end{aligned} \quad (4)$$

$C(\mathbf{y})$, thus, represents the minimal cost (maximal buying price) to the firm of constructing the derivative financial asset, $\mathbf{y}$, either through operations in financial markets or through its production operations. As these definitions indicate, by the principle of stage-wise optimization, the derivative cost problem can be split into a production problem and a financing problem. For example, a firm could have a production division charged with assembling $\mathbf{p} \cdot \mathbf{z}$ at minimal cost and a separate financial division charged with assembling $\mathbf{r}$ at minimal cost. The role of central management would then be to allocate $\mathbf{r}$ and $\mathbf{p} \cdot \mathbf{z}$ to minimize overall cost by eliminating any arbitrage opportunities between its production and financial divisions.⁸

We define $\mathbf{z}(\mathbf{y}; \mathbf{w}, \mathbf{p}, \mathbf{A})$ as the set of outputs consistent with cost-minimizing production of the net income vector $\mathbf{y}$:

$$\mathbf{z}(\mathbf{y}; \mathbf{w}, \mathbf{p}, \mathbf{A}) = \arg \min_{\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + \mathbf{v}\mathbf{h} : \mathbf{A}\mathbf{h} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y}\}.$$

⁷This remains true for the multiple input case.

⁸We are indebted to Dan Primont for this interpretation.

At points in the following discussion, it will be useful to assume that no $\mathbf{y} \in \mathfrak{R}_+^S$ can have a cost that is arbitrarily negative.⁹ Formally, the role of this assumption is to ensure that $C(\mathbf{y})$ is a proper convex function, which allows us to invoke standard results on subdifferentials and conjugate duality for convex functions (R. Tyrell Rockafellar, 1970).

Assumption 1 $C(\mathbf{y}) > -\infty, \quad \mathbf{y} \in \mathfrak{R}_+^S$.

We list some basic properties of C .¹⁰ (Proofs not included in the text are in an appendix.)

Theorem 1 C satisfies:

1. $C(\mathbf{y})$ is a nondecreasing, convex function that is continuous on the interior of the region where it is finite. If $\mathbf{z}' \geq \mathbf{z}, \mathbf{z}' \neq \mathbf{z} \Rightarrow c(\mathbf{w}, \mathbf{z}') > c(\mathbf{w}, \mathbf{z})$, then $\mathbf{y}' \geq \mathbf{y}, \mathbf{y}' \neq \mathbf{y} \Rightarrow C(\mathbf{y}') > C(\mathbf{y})$.
2. If C is subdifferentiable at $\mathbf{y} \in \text{ri}\mathfrak{R}_+^S$, $\partial C(\mathbf{y}) \subset \mathfrak{R}_+^S$. If C is strictly monotonic and C is subdifferentiable at $\mathbf{y} \in \mathfrak{R}_{++}^S$, $\partial C(\mathbf{y}) \subset \mathfrak{R}_{++}^S$.
3. $C(\mathbf{0}) \leq 0$.

Theorem 1.1 ensures that we can invoke standard methods from convex analysis in analyzing C . Theorem 1.2 gives consequences of monotonicity for subdifferentials of C . Property 3 exploits the absence of fixed costs for the production technology to show that the firm would never pay a strictly positive price for a non-stochastic zero payoff in each state of Nature. It also demonstrates that the firm can never make a negative profit in its construction of the $\mathbf{0}$ asset, presuming of course that the latter is priced at zero.

Our next theorem provides a conjugate dual representation of C that relates the firm's virtual state-claim prices to the asset structure and to $c(\mathbf{w}, \mathbf{z})$. To motivate this result, notice that the convex conjugate of C ,

$$C^*(\mathbf{q}) = \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{y})\},$$

⁹Assumption 1 is always satisfied if there are no arbitrage opportunities open to the firm (Chambers and Quiggin, 2002).

¹⁰ C is a cost function, and thus it possesses standard properties in terms of the input prices $(\mathbf{v}, \mathbf{h})$. These can be gleaned from any good microeconomics text, and thus in the interest of notational economy and parsimony, we do not discuss them here.

can be interpreted as the firm's virtual profit function for a set of virtual state-claim prices $\mathbf{q} \in \mathfrak{R}_+^S$. Intuitively, it seems clear that this virtual profit function is unboundedly large if there exist any arbitrage opportunities in financial markets at state-claim prices $\mathbf{q}$, that is $\mathbf{q}\mathbf{A} \neq \mathbf{v}$. When there are no such opportunities, the firm's virtual profit is given by the maximal virtual profit realized from the production of $\mathbf{z}$. More formally, for $\mathbf{q} \in \mathfrak{R}_+^S$

$$\begin{aligned}
C^*(\mathbf{q}) &= \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{y})\} \\
&= \sup_{\mathbf{y}} \left\{ \mathbf{q}\mathbf{y} - \min_{\mathbf{h}, \mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + \mathbf{v}\mathbf{h} : \mathbf{A}\mathbf{h} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y}\} \right\} \\
&= \sup_{\mathbf{y}, \mathbf{h}, \mathbf{z}} \{\mathbf{q}\mathbf{y} - c(\mathbf{w}, \mathbf{z}) - \mathbf{v}\mathbf{h} : \mathbf{A}\mathbf{h} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y}\} \\
&= \sup_{\mathbf{h}, \mathbf{z}} \{\mathbf{q}(\mathbf{A}\mathbf{h} + \mathbf{p} \cdot \mathbf{z}) - c(\mathbf{w}, \mathbf{z}) - \mathbf{v}\mathbf{h}\} \\
&= \begin{cases} \infty & \mathbf{q}\mathbf{A} \neq \mathbf{v} \\ c^*(\mathbf{w}, \mathbf{q} \cdot \mathbf{p}) & \mathbf{q}\mathbf{A} = \mathbf{v} \end{cases}.
\end{aligned}$$

For any $\mathbf{q}^* \in \arg \sup \{\mathbf{q}\mathbf{y} - C^*(\mathbf{q})\}$,

$$\mathbf{q}^*\mathbf{y} - C^*(\mathbf{q}^*) \geq \mathbf{q}\mathbf{y} - C^*(\mathbf{q}),$$

so that $\mathbf{y} \in \partial C^*(\mathbf{q}^*)$ and by (3), $\mathbf{q}^* \in \partial C(\mathbf{y})$ and the solutions to the conjugate problem can be interpreted as the firm's internal or virtual state-claim prices.

We conclude by conjugacy.

Theorem 2 *Under Assumption 1 for $\mathbf{y} \in \text{ri} \mathfrak{R}_+^S$, $C(\mathbf{y}) = \sup_{\mathbf{q}} \{\mathbf{q}\mathbf{y} - c^*(\mathbf{w}, \mathbf{q} \cdot \mathbf{p}) : \mathbf{q}\mathbf{A} = \mathbf{v}\}$.*

If $\mathbf{q} \in \partial C(\mathbf{y})$, then two conditions must hold. First, these virtual state-claim prices must allow no virtual profit from financial operations. Because the asset structure effectively exhibits constant returns to scale between the firm's position in financial markets and payouts, virtual profit in those markets is either zero or infinitely large. If such a profit existed, the firm could always create the asset associated with the position, sell it, and then use the proceeds to finance the purchase of $\mathbf{y}$. Because this could be repeated an infinite number of times, the cost of assembling $\mathbf{y}$ would thus be driven to $-\infty$. Second, the firm's cost minimizing choice of $\mathbf{z}$ in creating the derivative asset, $\mathbf{y}$, must also maximize virtual profit in terms of the virtual state-claim prices, $\mathbf{q}$. By this last observation, if $\mathbf{q} \in \partial C(\mathbf{y})$, then

$\mathbf{q} \cdot \mathbf{p} \in \partial c(\mathbf{w}, \mathbf{z})$ at any $\mathbf{z}$ which solves the derivative-cost problem. Hence, a decentralized firm could operate by specifying state-contingent prices $\mathbf{q}$ for its production division, and a desired net financial position $\mathbf{A}\mathbf{h}$, with $\mathbf{q}\mathbf{A}\mathbf{h} = \mathbf{v}\mathbf{h}$, for its financial division.

Intuitively, therefore, Theorem 2 can be viewed as a generalization of the envelope theorem for smoothly differentiable derivative-cost functions. And its intuitive content is precisely that in equilibrium the firm's marginal cost of y_s must equal the firm's marginal cost of producing z_s normalized by p_s .

4 Separation

We can now use the properties of the derivative-cost function to deduce conditions required for separation for general stochastic technologies and for frictionless financial markets. We start by noting that any $\mathbf{z}$, which the firm produces, must belong to

$$Z' = \{\mathbf{z}' : c^i(\mathbf{p} \cdot \mathbf{z}) \geq c(\mathbf{w}, \mathbf{z})\},$$

because for any $\mathbf{z}$ not in this set, the decisionmaker is always better off assembling $\mathbf{p} \cdot \mathbf{z}$ in financial markets. This observation leads us to the crudest kind of separation result. If Z' is empty, then the decisionmaker's production decisions are always independent of his risk attitudes, because all rational decisionmakers would operate at $\mathbf{z} = \mathbf{0}$, regardless of the magnitude of $\mathbf{y}$. This is the case where the technology is entirely redundant in the presence of asset markets.

We now seek instances other than complete production redundancy where production decisions are at least locally independent of the decisionmaker's risk attitudes. Once a local notion of separation is specified, it can be strengthened to yield a more restrictive global notion. Intuitively, our local notion of separation is intended to allow two different decisionmakers with different risk preferences, but facing the same technology and asset structure, to make the same production choices.

The first step in making this definition is to notice that when considering cost minimizing

production choices, Theorem 2 allows us to restrict attention to the set

$$\begin{aligned} Z^* &= \cup_{\mathbf{q}} \{ \mathbf{z} : \mathbf{z} \in \partial c^* (\mathbf{w}, \mathbf{q} \cdot \mathbf{p}) \} \\ &= \cup_{\mathbf{q}} Z^* (\mathbf{q}), \end{aligned}$$

where

$$Z^* (\mathbf{q}) = \partial c^* (\mathbf{w}, \mathbf{q} \cdot \mathbf{p}).$$

We say that *separation applies over a set* Y , if there exists $\mathbf{q}^M$ such that: (i) $\mathbf{q}^M \mathbf{A} = \mathbf{v}$, and (ii) for all $\mathbf{y} \in Y$ the decisionmaker's cost minimizing production choices $\mathbf{z} (\mathbf{y}; \mathbf{w}, \mathbf{p}, \mathbf{A})$, defined above, satisfy

$$\mathbf{z} (\mathbf{y}; \mathbf{w}, \mathbf{p}, \mathbf{A}) \subset Z^* (\mathbf{q}^M). \quad (5)$$

In other words, for all $\mathbf{y} \in Y$, $\mathbf{q}^M \in \partial C (\mathbf{y})$. Hence, regardless of the $\mathbf{y} \in Y$ that a firm may face, its production decisions are always guided by maximization of virtual profit for the state-claim prices $\mathbf{q}^M$. Similarly, any two different firms operating at two distinct elements of Y would make their resource allocation decisions using this same principle of virtual profit maximization for the state-claim prices $\mathbf{q}^M$. A trivial example of such a set is a set of the singleton form $\{\mathbf{y}\}$.

Consider the set obtained by translating the span of the market by the *ex ante* values of the cost minimizing production choices:

$$Y^* = M + \mathbf{p} \cdot Z^*.$$

This set contains all undominated vectors $\mathbf{y}$, that is those for which there exists no $\mathbf{y}' > \mathbf{y}$, $C (\mathbf{y}') \leq C (\mathbf{y})$.

By the basic properties of c^i , $c^i (\mathbf{r}) = \mathbf{q}^M \mathbf{r}$, $\mathbf{r} \in M$. Thus, for $\mathbf{y} \in Y^*$:

$$\begin{aligned} C (\mathbf{y}) &= \min_{\mathbf{z} \in Z^*, \mathbf{r}} \{ c (\mathbf{w}, \mathbf{z}) + c^i (\mathbf{r}) : \mathbf{p} \cdot \mathbf{z} + \mathbf{r} \geq \mathbf{y} \} \\ &= \min_{\mathbf{z} \in Z^*} \{ c (\mathbf{w}, \mathbf{z}) + c^i (\mathbf{y} - \mathbf{p} \cdot \mathbf{z}) \} \\ &= \min_{\mathbf{z} \in Z^*} \{ \mathbf{q}^M \mathbf{y} - \mathbf{q}^M (\mathbf{p} \cdot \mathbf{z}) + c (\mathbf{w}, \mathbf{z}) \} \\ &= \mathbf{q}^M \mathbf{y} - c^* (\mathbf{w}, \mathbf{q}^M \cdot \mathbf{p}), \end{aligned}$$

implying separation over $M + \mathbf{p} \cdot Z^*$.

Suppose that separation applies over Y . Then by Theorem 2 , for all $\mathbf{y} \in Y$

$$C(\mathbf{y}) = \mathbf{q}^M \mathbf{y} - c^*(\mathbf{w}, \mathbf{q}^M \cdot \mathbf{p})$$

with $\mathbf{q}^M \mathbf{A} = \mathbf{v}$. Together these expressions imply that the optimal choice of $\mathbf{r} = \mathbf{A}\mathbf{h}$ and $\mathbf{z}$ satisfy

$$c^i(\mathbf{r}) = \mathbf{q}^M \mathbf{r}, \quad (6)$$

and (5). Expression (6) implies that c^i is linear in $\mathbf{r}$, and hence by the basic properties of c^i , $\mathbf{r} \in M$. Hence, $\mathbf{y} \in M + \mathbf{p} \cdot Z^*(\mathbf{q}^M)$.

Theorem 3 *Separation applies over Y if and only if, for some $\mathbf{q}^M$, $Y \subset M + \mathbf{p} \cdot Z^*(\mathbf{q}^M)$.*

Several things should be noted here. If $\mathbf{p} \cdot Z^* \subset M$, then separation applies for all $\mathbf{y}$, and the decisionmaker's production decision is completely independent of his or attitudes towards risk. In such cases, all decisionmakers make the same physical resource allocation decisions. This is the more usual notion of separation, and an obvious special case emerges when markets are complete so that M spans the states of Nature. We state this obvious fact as a corollary.

Corollary 4 *Let $c^i(\mathbf{r}) = \mathbf{q}^M \mathbf{r}$, $\mathbf{r} \in M$. If $\mathbf{p} \cdot Z^* \subset M$, separation applies for all $\mathbf{y}$, and*

$$Z^* = \arg \sup \{ \mathbf{q}^M (\mathbf{p} \cdot \mathbf{z}) - c(\mathbf{w}, \mathbf{z}) \}.$$

We refer to the case when separation applies for all $\mathbf{y}$ as *strong separation*. The condition $\mathbf{p} \cdot Z^* \subset M$, which is sufficient for strong separation, is generally described as *spanning* since it implies that any output $\mathbf{z}^*$ consistent with cost minimization can be replicated by asset trades. Note that, by the properties of M , spanning may equivalently be characterized as $M + \mathbf{p} \cdot Z^* \subset M$.

Corollary 4 states that spanning implies strong separation. The converse is not true. Even in the presence of separation, full insurance is not typically available and the income accruing to the firm's owners need not lie in the financial span M . Rather, the owners of the firm may be subject to 'background' risk that cannot be offset either by production choices or by financial transactions. In fact, we have:

Theorem 5 *Strong separation holds if and only if, for some $\mathbf{q}^M$,*

$$\mathbf{p} \cdot Z^* \subset M + \mathbf{p} \cdot Z^* (\mathbf{q}^M),$$

or, equivalently

$$M + \mathbf{p} \cdot Z^* = M + \mathbf{p} \cdot Z^* (\mathbf{q}^M).$$

We close this section with an example which illustrates an application of the basic theorem in the case of strong separation.

Example 6 *Consider costs of the form*

$$c(\mathbf{w}, \mathbf{z}) = \hat{c} \left(\mathbf{w}, \max \left\{ \frac{z_1}{\varepsilon_1}, \dots, \frac{z_S}{\varepsilon_S} \right\} \right),$$

with $\varepsilon_s > 0$ for all s and $\hat{c}$ strictly increasing in its second argument. Suppose that $\left(\frac{p_1}{\varepsilon_1}, \frac{p_2 \varepsilon_2}{\varepsilon_1}, \dots, \frac{p_S \varepsilon_S}{\varepsilon_1} \right) \in M$, then

$$\begin{aligned} C(\mathbf{y}) &= \min_{\mathbf{z}, \mathbf{r}} \left\{ \hat{c} \left(\mathbf{w}, \max \left\{ \frac{z_1}{\varepsilon_1}, \dots, \frac{z_S}{\varepsilon_S} \right\} \right) + c^i(\mathbf{r}) : \mathbf{r} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y} \right\} \\ &= \min_{z_1, \mathbf{r}} \left\{ \hat{c} \left(\mathbf{w}, \frac{z_1}{\varepsilon_1} \right) + c^i(\mathbf{r}) : \mathbf{r} + z_1 \mathbf{p} \cdot \left(\frac{1}{\varepsilon_1}, \frac{\varepsilon_2}{\varepsilon_1}, \dots, \frac{\varepsilon_S}{\varepsilon_1} \right) \geq \mathbf{y} \right\} \\ &= \min_{z_1, \mathbf{r}} \left\{ \hat{c} \left(\mathbf{w}, \frac{z_1}{\varepsilon_1} \right) + c^i \left(\mathbf{y} - z_1 \mathbf{p} \cdot \left(\frac{1}{\varepsilon_1}, \frac{\varepsilon_2}{\varepsilon_1}, \dots, \frac{\varepsilon_S}{\varepsilon_1} \right) \right) \right\} \\ &= \mathbf{q}^M \mathbf{y} - \max \left\{ z_1 \mathbf{q}^M \mathbf{p} \cdot \left(\frac{p_1}{\varepsilon_1}, \frac{p_2 \varepsilon_2}{\varepsilon_1}, \dots, \frac{p_S \varepsilon_S}{\varepsilon_1} \right) - \hat{c} \left(\mathbf{w}, \frac{z_1}{\varepsilon_1} \right) \right\}, \end{aligned}$$

and strong separation applies. The second equality follows because any cost minimizing solution must involve

$$\frac{z_1}{\varepsilon_1} = \frac{z_2}{\varepsilon_2} = \dots = \frac{z_S}{\varepsilon_S}.$$

In this case the value of all ‘rational’ production choices lie in M .

4.1 Nonstochastic production

Our treatment thus far has restricted attention to the case where production is stochastic. However, it is to be expected that results will be applicable in the simplest case of production theory, that of a firm with nonstochastic technology and nonstochastic prices. Moreover, an important literature, beginning with Agnar Sandmo (1971) has developed around the case

where the producer faces price uncertainty but not production uncertainty. We now briefly consider such situations in this framework.

Let $c^c(\mathbf{w}, z)$ denote a cost function for a scalar, nonstochastic output z . This cost function satisfies the same basic properties in $\mathbf{w}$ as $c(\mathbf{w}, \mathbf{z})$. In particular, following Sandmo (1971), we assume that it is nondecreasing and convex in the nonstochastic scalar z .

The firm's problem is

$$C(\mathbf{y}) = \min_{z, \mathbf{h}} \{c^c(\mathbf{w}, z) + \mathbf{v}\mathbf{h} : z\mathbf{p} + \mathbf{A}\mathbf{h} \geq \mathbf{y}\}, \quad (7)$$

if there exists $(z, \mathbf{h})$ such that $z\mathbf{p} + \mathbf{A}\mathbf{h} \geq \mathbf{y}$ and ∞ otherwise.

For the case when both prices and technology are non-stochastic, separation occurs if and only if there exists a riskless asset. Let this be a bond with price 1 and payoff $1 + i$. Then a firm will engage in production if and only if the set

$$Z' = \left\{ z : c^c(\mathbf{w}, z) \leq \frac{z}{1+i} \right\}$$

is non-empty. Moreover, if $c^c(\mathbf{w}, z)$ is differentiable the condition $\mathbf{q} \in \partial C(\mathbf{y})$ is simply the usual optimality condition for investment, requiring that the marginal efficiency of capital equal the bond rate. That is,

$$\frac{\partial c^c(\mathbf{w}, z)}{\partial z} = \frac{1}{1+i}.$$

We now turn to separation results in the presence of nonstochastic production and stochastic prices. A fundamental result from the expected utility literature on futures and forward markets (Danthine, 1978; Holthausen, 1979; and Anderson and Danthine, 1981, 1983a, 1983b) is that a firm facing stochastic spot prices, $\mathbf{p}$, and a single forward market for the commodity produced, offering a sure price of v^* for the amount hedged, should have its production decisions independent of its risk attitudes. This result may be recognized as an immediate corollary of Corollary 4.

If the amount hedged is denoted h , then the firm creates the income stream $\mathbf{y} = \mathbf{p}(z - h) + qh$ at a cost to itself of $c^c(\mathbf{w}, z) + v^*h$. Making the simple substitution of $a = (z - h)$, the firm creates an income stream $\mathbf{p}a + q(z - a)$ at a cost of $c^c(\mathbf{w}, z) + v^*(z - a)$. Production of the physical commodity is now financially redundant in the sense that its associated state-contingent return can always be replicated by taking an appropriate position in the forward

market. Thus, regardless of the level of a chosen, the firm always perceives its marginal cost of varying z as $\partial c^c(\mathbf{w}, z) + v^*$, and it thus minimizes this cost by choosing the same z regardless of the choice of $\mathbf{y}$.

5 Concluding comments

The idea that, under appropriate conditions, production and financial decisions may be separated has been presented in many different guises. In this paper, we have used the notion of the derivative-cost function to present a unified treatment, including necessary and sufficient conditions for separation for perhaps the most general class of preferences considered in the literature. A number of well-known results may be derived as simple corollaries of our basic results.

We have derived conditions under which the production decision is completely independent of the preferences of the firm's owners given frictionless financial markets. A promising avenue for generalization of these results, therefore, is to consider conditions under which the production decision is the same for all decision-makers even in the presence of frictions in financial markets.

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Appendix:Proofs

Proof of Theorem 1: Continuity follows by the theorem of the maximum (Claude Berge, 1997). Let $(\mathbf{h}', \mathbf{z}')$ be optimal for $\mathbf{y}' \geq \mathbf{y}$, then C nondecreasing follows because $(\mathbf{h}', \mathbf{z}')$ is feasible for $\mathbf{y}$. Let $\mathbf{y}' \geq \mathbf{y}, \mathbf{y}' \neq \mathbf{y}$. There must exist a $\mathbf{z} \leq \mathbf{z}', \mathbf{z} \neq \mathbf{z}'$ such that $\mathbf{h}'\mathbf{A} + \mathbf{p} \cdot \mathbf{z} \geq \mathbf{y}$, but for which $c(\mathbf{w}, \mathbf{z}) + \mathbf{v}\mathbf{h}' < C(\mathbf{y}')$ if c is strictly increasing. To demonstrate convexity, let $(\mathbf{h}', \mathbf{z}')$ and $(\mathbf{h}'', \mathbf{z}'')$ be optimal for $\mathbf{y}'$ and $\mathbf{y}''$, respectively. By the linearity of the constraint sets $(\lambda\mathbf{h}' + (1 - \lambda)\mathbf{h}'', \lambda\mathbf{z}' + (1 - \lambda)\mathbf{z}'')$ is feasible for $\lambda\mathbf{y}' + (1 - \lambda)\mathbf{y}''$. By C.5,

$$c(\mathbf{w}, \lambda\mathbf{z}' + (1 - \lambda)\mathbf{z}'') + \mathbf{v}(\lambda\mathbf{h}' + (1 - \lambda)\mathbf{h}'') \leq \lambda[c(\mathbf{w}, \mathbf{z}') + \mathbf{v}\mathbf{h}'] + (1 - \lambda)[c(\mathbf{w}, \mathbf{z}'') + \mathbf{v}\mathbf{h}''] .$$

Taking the minimum of the left-hand side yields convexity. This establishes 1.

To establish 2, consider $\mathbf{q} \in \partial C(\mathbf{y})$. By definition,

$$C(\mathbf{y}) - C(\mathbf{z}) \leq \mathbf{q}(\mathbf{y} - \mathbf{z}) .$$

Let $\mathbf{y} = \mathbf{z} + \delta\mathbf{e}_j$ for $\delta > 0$ but suitably small. Weak monotonicity guarantees that $q_j \geq 0$, and strong monotonicity guarantees a strict inequality.

$C(\mathbf{0}) \leq 0$ follows by C.3 and the definition of the cost function because $\mathbf{h} = \mathbf{0}, \mathbf{z} = \mathbf{0}$ is feasible for $\mathbf{y} = \mathbf{0}$.

Proof of Theorem 2: Under Assumption 1 and by Theorem 1, $C(\mathbf{y})$ is a proper and closed function with the convex conjugate identified in the text. By conjugacy, then

$$C(\mathbf{y}) = \sup_{\mathbf{q}} \left\{ \mathbf{q}\mathbf{y} - \sup_{\mathbf{z}} \{ \mathbf{q}(\mathbf{p} \cdot \mathbf{z}) - c(\mathbf{w}, \mathbf{z}) \} : \mathbf{q}\mathbf{A} = \mathbf{v} \right\} .$$

Figure 1: Stochastic production function and general technology