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CENTER DISCUSSION PAPER NO. 931

On the Conditions that Preclude the Existence of the Lerner Paradox and the Metzler Paradox

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February 2006

Notes: Center Discussion Papers are preliminary materials circulated to stimulate discussions and critical comments.

We are much indebted to Masanao Aoki, Donald Brown, Kazuharu Kiyono, T.N. Srinivasan, Makoto Yano, and in particular to Koji Shimomura for valuable suggestions

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Masahiro Endoh and Koichi Hamada

Abstract

The Lerner paradox is the possibility that a tariff on an import good might worsen a country's terms of trade, and the Metzler paradox is the possibility that a tariff on an import good might reduce a country's import price. In a general equilibrium framework with multiple goods, this paper shows that the combination of the invertibility of the Slutsky matrix for the world economy and its similarity across countries will preclude both of the paradoxes, and that the combination of the gross-substitutes assumption for the world demand and the substitute assumption for the demand of an import country property of goods will preclude the Lerner paradox. A modified condition for the Slutsky matrix combined with the gross substitute for the world demand will do the same for the Metzler paradox. A concept of non-surpassed diagonal is used in deriving the result.

Key Words: Lerner paradox, Metzler paradox, tariffs, terms of trade, gross substitutes, dominant diagonal matrix.

JEL Codes: C20, F02, F11

I. Introduction

The Lerner paradox and the Metzler paradox have appealed to the intellect of economists for a long time both in their theoretical curiosity as well as their practical policy significnace. The Lerner paradox (Abba Lerner, 1936) refers to the possibility that a tariff on an import good might worsen a country's terms of trade. The Metzler paradox (Lloyd Metzler, 1949a) refers to the possibility that a tariff on an import good might reduce a country's import price. [1] Metzler examined thoroughly the effect of tariff on domestic relative price in a general setting, showed his "paradox," and investigated what conditions might cause it.

"[I]f the world demand for the tariff-imposing country's exports is inelastic and if the tariffs reduce the demand for imports in the tariff-imposing country to a considerable extent, the fall in world prices of imports, relative to world prices of exports, may be so large that domestic prices of imports are relatively lower than before the tariffs were imposed, even after the tariffs are added to the world prices." (Metzler, 1949b, p.345)

This paper shows that, if one starts from the state of no tariffs in the world, the Metzler Paradox is precluded by two set of conditions: (1)the invertibility of the Slutsky matrix for the world demand and a condition of similarity of the matrix across countries, and (2) the combination of the gross-substitutes for the world demand and the substitutes condition of the demand of the country that impose a tariff on its import. The Lerner paradox is precluded by contition (1) as well, and by a condition imposed on the importing country as (2).

Indeed, the concept of gross-substitutability itself was noticed by Metzler (1945). Here it is found to be a useful tool for analyzing the paradox proposed by himself, as Mundell (1964) explored the link of the tariff question to the concept of gross-substitutability. The similarity condition to be required here concerns the similarity

[1] Lerner (1936) already mentioned this situation but only in a particular context where the government spends the most of tariff revenues.

of trade structures in terms of the Slutsky matrix. When the income effects are moderate in each country, this condition is satisfied. This seems to be one of the reasons that Chipman (1990) considered the Metzler paradox along with the transfer paradox, because transfers may generate different income effects across countries.

II. Model

Suppose the world consists of two countries, A and B, and there are $n+1$ commodities, $j = 0, 1, 2, ..., n$. The non-negative vector of domestic prices for country K is expressed as $p^K = (p_0^K, p_1^K, ..., p_n^K)'$ for $K = A$ and B, with $'$ being the symbol for transposition. Good 0 is the numeraire so that $p_0^K = 1$ for all K. W referes to the total world.

$E^K\left(p^K, u^K\right)$ is the expenditure function of country K ($K = A$ and B), $R^K\left(p^K\right)$ is the revenue function of country K, and u^K is country K's utility level, all defined on price vector in country K, where $p^K \equiv (p_1^K, p_2^K, ..., p_n^K)'$, and $t^K \equiv (t_1^K, t_2^K, ..., t_n^K)'$. Good 0 is a numeraire. Let $S^K\left(p^K, u^K\right) \equiv E^K\left(p^K, u^K\right) - R^K\left(p^K\right)$, then each country's budget constraint and the market clearing condition are as follows:

$$S^K\left(p+t^K, u^K\right) - t^{K\prime}(S^K\left(p+t^K, u^K\right) = 0, \qquad K = A, B,$$
$$\textstyle\sum_{K=A,B} S_p^{K\prime}\left(p+t^K, u^K\right) = 0,$$

where $S_p^K = (\frac{\partial S^K}{\partial p_1^K}, \frac{\partial S^K}{\partial p_2^K}, ..., \frac{\partial S^K}{\partial p_n^K})$. We start from the situation where not tariffs exist for country A and B,and suppose Country B increses t^B from 0 to dt^B. Then, $p^A$is equal to the international price p.

$$p^A + t^B = p^B \tag{1}$$

Noting that $t^A = t^B = dt^A = 0$, we obtain,

$$\left[\Omega\right]\begin{bmatrix} du^A \\ du^B \\ dp \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -S_{pp}^B dt^B \end{bmatrix},$$

where

$(n+2)\times(n+2)$ matrix $\left[\Omega\right] \equiv \begin{bmatrix} S_u^A & 0 & S_p^A \\ 0 & S_u^B & S_p^B \\ S_{pu}^A & S_{pu}^B & S_{pp}^A + S_{pp}^B \end{bmatrix}$, $S_u^K = \frac{\partial S^K}{\partial u^K}$, and $S_{pu}^K = \left[\frac{\partial^2 S^K}{\partial p^K \partial u^K}\right]$.

If we eliminate the du^A and du^B, solving from the first two equations and substitute to the latter n equations, we obtian the relationship between dp (international vector) and dt (vector) such that,

$$[\textstyle\sum_{K=A,B}(S_{pp}^K - \frac{S_{pu}^K S_p^{K\prime}}{S_u^K})]dp = -S_{pp}^B dt^B \tag{2}$$

$\sum_{K=A,B}\left[S_{pp}^K - \frac{S_{pu}^K(S_p^{K\prime})}{S_u^i}\right]$, which we will denote later as X^W is the Slutsky matrix of the total world demand, that is the combination of substitution effect and the income effect. If this Slutsky matrix has the negative entries in the diagonal elements and negative elements in the off-diagonal elemtnts, then goods are called gross substitutes

The reader may wonder why dt^B is multiplied not by the Slutsky matrix but by the pure substitution matrix S_{pp}^B. The reasonis that the income effect is cancelled by the envelope property in the duality approach.

Excess demand, import minus export, of those countries and of the world for i'th commodity are defined as the functions of domestic prices,

$$x_i^K(p_0^K, p_1^K, ..., p_n^K), \qquad K = A \text{ and } B, \qquad x_i^W = x_i^A + x_i^B \tag{3}$$

with price derivatives with the following familiar properties of gross substitutes explicitly defined for n goods except the numéraire. It is ture that x_{ij}^K and x_{ij}^W may be defined for all commodities from 0 to n, but X^K and X^W are defined for commodities 1 to n, since commodity 0 is numéraire. Define the Slutsky matrix, or the matrix of the changes in demand in response to changes in prices, X^K and X^W.

$$X^K = \left[\frac{\partial x_i^K}{\partial p_j^K}\right] = \left[x_{ij}^K\right] \quad i,j = 1,...,n, \; K = A \text{ and } B, \tag{4}$$

$$X^W = \left[\frac{\partial x_i^W}{\partial p_j^K}\right] = \left[x_{ij}^W\right] \qquad i,j = 1,...,n \tag{5}$$

The concept of gross substitutability in this paper can be related to the (now) standard analysis utilizing the duality properties of using expenditure and revenue functions above. One can see, for example,

$$X^W = \left[\frac{\partial x_i^W}{\partial p_j}\right] = [\textstyle\sum_{K=A,B}(S_{pp}^K - \frac{S_{pu}^K S_p^{K\prime}}{S_u^K})] \qquad i,j = 1,...,n, \; K = A \text{ and } B. \tag{6}$$

Here we introduce the concept of gross substitutability on the Slutsky Matrix.

Definition (Gross Substitutability: GS): The goods are gross substitutes if and only if

$$x_{ij}^K > 0, \quad x_{ij}^W > 0, \qquad \text{for } \; i \neq j, \; K = A \text{ and } B.$$

These conditions imply, as well known, that

$$x_{ii}^K < 0, \quad x_{ii}^W < 0, \qquad \text{for } \; i = 1,2,...,n, \; K = A \text{ and } B.$$

Assumption I (Gross Substitutes): All the goods are gross substitutes.

According to studies by Arrow-Hurwicz (1958), Hahn (1958), and Negishi (1958), matrices X^K and X^W are non-singular and their inverses are with all negative elements. We call this property "positively invertible." Needless to say, Assumption I implies Assumption IA below but not vice versa. The following condition of invertibility of X^W is implied by the Walrasian stability condition of the world market.

Assumption *IA (Invertibility of* X^W): X^W *is non-singular.*

For simplicity, we assume that the transportation costs are negligible, and that the specific duties are used as tariffs. Moreover, in order to obtain the insight into the simplest case, we assume that at the beginning no tariffs are levied in the world. We examine the conditions for the absence of the Lerner as well as the Metzler paradox when we start the situation where no tariffs exist. Suppose good k is an import of country B. If a tariff on good k is imposed by country B, that is, $t_k^B = dt_k^B > 0$, then under what conditions does it *increase* the international or exporter country's price, that is, $dp_k^A > 0$ [Lerner paradox] or under what conditions does it *decrease* the importer country's price, that is, $dp_k^B < 0$ [Metzler paradox]?

One can see below that this analysis may be extended to the world where transportation costs exist but constant, where subsidies exist, and where the tariffs take the ad-valorem tariffs. One can also analyze similarly the world where the initial levels of tariffs in both countires are not zero, except that the similarity condition of gross substitute martix in equation (15) should be in terms of international prices.

Definition: the Lerner Paradox occurs if, $\frac{dp_k^A}{dt_k^B} > 0$; the Metzler Paradox occurs if $\frac{dp_k^B}{dt_k^B} < 0$. A Strong Form of the Lerner paradox occurs if the relative international price of the import to any other commodity increases (Lerner paradox), or the price of the import relative to any other commodity decreases (Metzler paradox).

Since

$$dp_k^B = dp_k^A + dt_k^B, \qquad \text{and} \qquad dt_k^B > 0, \tag{7}$$

both paradoxes never coexistent at the same time. Please note that, in the model of multiple goods, the terms of trade is not simply defined as the price ratio of a single import and a single export, but defined as the price of import or export in terms of the numéraire. The strong definition comes to counter this problem. Fortunately, the gross-substitute assmuption among all the goods and the similarity assumption to be defined below is invariant to the choice of a numéraire.

III. The Case with Identical or Similar Net Demand Structure

In this section, we examine the effect of tariff reduction on terms of trade in the case with identical substitute structure as a preliminary analysis. As we have started, we consider the commodity k being exported from country A to country B, and suppose only the tariff on good k from country A to country B is increased from zero.

An element k of t^B is changed. Let dt^B indicate the vector of changes in specific tariffs on goods imported from counrty A to country B. Specifically, we concentrate on the kth commodity on which the tariff rate is reduced. $dt^B = \left(0, \cdots, 0, dt_k^B, 0, \cdots, 0\right)'$. In the two countries we are taking account of, by the introduction of tariff concession, a wedge is introduced between the price of k between country A and country B. $dp_k^A + dt_k^B = dp_k^B$ always holds, then equation (2) implies,

$$X^W dp^A = -S_{pp}^B dt^B \tag{8}$$

or,

$$X^W dp^B = (X^W - S_{pp}^B) dt^B \tag{9}$$

Suppose that X^W and S_{pp}^B are proportionate. then the comparative statics of the model are simple. Assume $S_{pp}^B = \alpha X^W$ and $X^W - S_{pp}^B = (1-\alpha)X^W$. Then, equations (9) and (10) equal to, respectively,

$$dp^A = -\alpha dt^B \tag{10}$$

$$dp^B = (1 - \alpha)\, dt^B \tag{11}$$

For the increase in the tariff in such a form that $dt^B = \left(0, \cdots, 0, dt_k^B, 0, \cdots, 0\right)'$, these equations indicate that $\frac{dp_k^A}{dt_k^B} = -\alpha < 0$ and $\frac{dp_i^A}{dt_k^B} = 0$ for $i \neq k$,and that $\frac{dp_k^B}{dt_k^B} = 1 - \alpha > 0$ and $\frac{dp_i^B}{dt_k^B} = 0$ for $i \neq k$. The domestic price of the $k'th$ goods rises and the international price falls. Thus neither the Metzler paradox nor the Lerner paradox do emerge. When α is close to 1, that is, the country A play the role of a large country and country B absorbs the most of a shock. The increase in import tariff on commodity k by country B increases its domestic price in country B by almost the same amount. On the othr hand, the increase in import tariff on commodity k by country B decreases very little the price in country A, which is equal to the international price and accordingly the terms of trade for country A, only a little.ncreases its domestic price in country B by almost the same amount. A contrasting story can be told when α is close to zero. The international price takes care of most of the shock. This characteristic of the results for the case of the identical excess supply among countries is interesting, since the change of tariff on commodity k does not affect the price of other commodities.

Proposition I: Suppose the excess supply structure of each country is identical. Then, under Assumption I A (invertibility of world trade structure matrix), neither the Metzler paradox or the Lerner paradox will occur. One unit of tariff increase for a commodity by country B will increase the price of the commodity in country B by the relative weight of country A, α, and decrease its price in country A by the relative weight of country B, $\alpha - 1$.

Our analysis can be extended to the case where trade structures S_{pp}^{A} and S_{pp}^{B} are not exactlly similar, with employing an approximation. The effect of t_k^B on export country A without the assumption of propotionate substitution matrix above. From equation (8), it can be written as

$$\begin{aligned} dp^A &= -(X^W)^{-1} S_{pp}^{B} dt^B \\ &= -(X^W)^{-1} \left[\lambda_B X^W + \left(S_{pp}^{B} - \lambda_B X^W \right) \right] dt^B, \end{aligned} \tag{12}$$

where $\lambda_B = augmin_\lambda \left\| S_{pp}^{B} - \lambda X^W \right\|$ and the norm of square matrix $A_{ij} = (a_{ij})$ of order $n \times n$, and the norm of a vector p with n-element are defined in this paper as (See, Bellman, 1960, p.165)

$$\begin{aligned} \|A\| &= n \cdot \max_{i,j} \left(|a_{ij}| \right), \\ \|p\| &= \max_{j} \left(|p_j| \right). \end{aligned}$$

This definition of the norm is easily shown to satisfy regular triangle inequalities attached to the norm[2]. Then, equation (12) is rewritten as

[2]In particular, $\|A + B\| \leq \|A\| + \|B\|$ and $\|AB\| \leq \|A\| \, \|B\|$.

$$dp^A + \lambda_B dt^B = -(X^W)^{-1}(S^B_{pp} - \lambda_B X^W)dt^B. \tag{13}$$

Considering the norm of equation (13),

$$\left\| dp^A + \lambda_B dt^B \right\| \leqq \left\| (X^W)^{-1} \right\| \left\| (S^B_{pp} - \lambda_B X^W) \right\| \left\| dt^B \right\|. \tag{14}$$

Here we introduce a Similarity Condition I.

Similarity Condition I: For a positive value of ε, $\left\| (S^B_{pp} - \lambda_B X^W) \right\| \leqq \varepsilon / \left\| (X^W)^{-1} \right\| \leqq \varepsilon / n(\max x^*_{ij})$, where x^*_{ij} is the (i,j) element of $(X^W)^{-1}$.

Then, if Similarity Condition I is satisfied, equation (14) equals to

$$\left\| dp^A + \lambda_B dt^B \right\| \leqq \varepsilon \left\| dt^B \right\|.$$

When only t^B_k is changed, namely

$$dt^B = \left(0, \cdots, 0, dt^B_k, 0, \cdots, 0\right)',$$

then, $\pi^A_{ik} = \frac{dp^A_i}{dt^B_k}$ is expressed as follows.

$$\begin{aligned} \pi^A_{kk} &= \frac{dp^A_k}{dt^B_k} = -\lambda_B + \mu \\ \pi^A_{ik} &= \frac{dp^A_i}{dt^B_k} = \mu \qquad \text{for } i \neq k, \end{aligned}$$

where λ_B is a positive value not exceeding unity and defined by $\lambda_B = augmin_\lambda \left\| S^B_{pp} - \lambda X^W \right\|$, and μ is a scalar that can take either sign, but limited by $|\mu| < \varepsilon$. The range of π^A_{kk} is $(-\lambda_B - \varepsilon, -\lambda_B + \varepsilon)$, and the range of π^A_{ik} $(i \neq k)$ is within $(-\varepsilon, \varepsilon)$. This shows that the Lerner paradox does not occur. Similary, for country B,

$$\begin{aligned} \pi^B_{kk} &= \frac{dp^B_k}{dt^B_k} = \lambda_A + \mu \\ \pi^B_{ik} &= \frac{dp^B_i}{dt^B_k} = \mu \qquad \text{for } i \neq k, \end{aligned}$$

where $\lambda_A + \lambda_B = 1$. As long as μ is constrained by ε, that is smaller than λ_A, then $\frac{dp^B_k}{dt^B_k} > 0$. This is exactly the condition that excludes the Metzler paradox. The range of π^B_{kk} is $(\lambda_A - \varepsilon, \lambda_A + \varepsilon)$, and the range of π^B_{ik} $(i \neq k)$ is within $(-\varepsilon, \varepsilon)$.

Proposition II: Under assumptions IA and Similarity Condition I with a sufficiently small ε, neither can the Lerner paradox nor the Metzler paradox emerge.

The results above indicate the importance of the similarity of excess supply patterns in assessing the possibility of those paradoxes. If the trade structure is similar enough to give a small ε, one can neglect the effect of a tariff change to a commodity on the price of other commodities.

IV. Gross Substitutability and the Absence of the Metzler Paradox

In this section, we will discuss our questions under Assumption I, that is, the assumption of gross substitutes among goods. Also, we will show that if goods are gross substitutes, and another similarity condition prevails, then there will be no Lerner paradox nor Metzler Paradox. Assumption I assures the invertibility of X^W(Assumption IA). Then in order to prove our main result, we utilize a lemma on the inverse of a dominant diagonal matrix which, to the best of our knowledge, is unseen in textbooks or in their exercises.

For the world as a whole, the Warlas law ensures

$$\overline{p}'\overline{x}^W = 0, \tag{15}$$

where the international $n+1$ dimensional price vector is written as $\overline{p}$ including the numéraire, that is, $\overline{p} = (1, p)'$ and the augmented supply vector of $n+1$ dimension as $\overline{x}^W = (\overline{x_0}^W, \overline{x_1}^W, ..., \overline{x_n}^W)'$, $\overline{x_k}^W$ being the net excess supply of good k in the world market as defined at the beginning of the paper. As Hahn (1958) showed in light of the fact that in equilibrium the excess supply equals zero, i.e.,

$$\overline{x}^W = 0, \tag{16}$$

this implies in terms of the $n \times n$ Slutsky matrix for the world excess demand $X^W = [x_{ij}^W]$, $i, j = 1, ..., n$,

$$p'X^W < 0. \tag{17}$$

Thus for the world Slutsky matrix equation (2) holds exactly, and Hahn successfully used this property to prove the stability of general equilibrium market.

On the othe hand, let us posit the following assumption on the pure substituion term of demand function of country B.

Assumption II: In country B, Goods are Pure Substitutes.

Then, starting from zero tariff situation, and utilizing the zero'th homogeneity of the demand function with respect to price vector and the symmetry of pure substitution matrix S^B_{pp}, we obtain,

$$p' S^B_{pp} < 0 \tag{18}$$

Proposition III: Gross Substitutability of the world total demand and the pure substituability of country B sustain the absence of the Lerner Paradox.

Proof: The proof goes with several steps. We start with the fact that the international price weighted column sum of X^W's negative off-diagonal elements has a smaller absolute value than the corresponding (positive) diagonal element (Step I). Next, we prove the lemma stating that the inverse of a simply, and not weighted sum, dominant diagonal matrix in terms of the column has the non surpassed diagonal that the the diagonal element is always equal or larger than its corresponding off-diagonal row element (Step II). Finally, combining this unsurpassed property with the Euler's relation for the compensated demand function, we can state a set of conditions to preclude the Lerner paradox and the Metzler paradox.

Step I

$\overline{x}^W_i$ is homogenious of zero degree in terms of prices including the numèraire. Then, by Euler's equation, $\sum_{i=0}^n x^W_{ji} p_i = 0$, for $j = 0, \cdots, n$. Good 0 is numeraire and $p_0 = 1$, then $\sum_{i=1}^n x^W_{ji} p_i = -x^W_{j0} < 0$, or

$$\left| x^W_{ii} \right| p_i \geqq \sum_{j \neq i} x^W_{ij} p_j, \qquad \text{for } i = 1, \cdots, n \tag{19}$$

Define $Y^W = X^W P$ and $Y^K = S^K_{pp} P$, where P is a diagonal matrix whose diagonal elements correspond to the element of international price vector p.

$$P = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ & & & \\ 0 & 0 & \dots & p_n \end{bmatrix}$$

Accordingly, This gross-substitutability property of the world excess demand response matrix assures the stability of the market, invertibility of X^W and the fact that $(X^W)^{-1}$ is a negative matrix. Then, equation (20) is rewritten as the row sum condition for the element of $V = [v_{ij}] = [x_{ij}p_j]$

$$\left|v_{ii}^W\right| \geqq \sum_{j \neq i} v_{ji}^W, \qquad \text{for } i = 1, \cdots, n$$

In terms of $Y = [y_{ij}] = [p_i x_{ij}]$

$$y_{jj}^W \geqq \sum_{i \neq j} \left|y_{ij}^W\right|$$

By those preparations, we will show below that the diagonal elements of $\left(X^A + X^B\right)^{-1} S_{pp}^B$ are all positive by utilizing the following curious lemma.

Step II (Lemma)

If $Y^W = \left[y_{ij}^W\right]$ *is a matrix* $y_{ii}^W < 0$, $y_{ij}^W > 0 \quad (i \neq j)$, *and* $\left|y_{ii}^W\right| > \sum_{i \neq j} y_{ij}^W$ *(or* $\sum_{i=1}^n y_{ij}^W < 0$*), for all* i*, then* Y^W *is negatively invertible, i.e.* $(Y^W)^{-1} = Y^{W*} = \left[y_{ij}^{W*}\right] < 0$*, where the asterisk indicates the elements of the inverse matrix. And each non-diagonal element of any row is smaller than its diagonal element, that is,* $\left|y_{ii}^{W*}\right| > \left|y_{ij}^{W*}\right|$ *for* $i \neq j$.

The first statement on the negativity of the inverse is generally known as the result under the column-sum condition, a special case of the *dominant diagonal condition.* The second condition that the diagonal of the matrix is larger than each off-diagonal matirix is perhaps new. We call the property of the inverse matrix as the matrix of "unsurpassed" *diagonal.*

Proof of the Lemma

We prove this lemma by induction on the order n of the matrix, and by partitioned multiplication of the matrix. Since the gross substitutability ensures the existence of a negative inverse Y_{n-1}^{W*} of Y_{n-1}^{W}. Here asterisks indicate the fact that the element is a component of the inverse matrix.

$$Y^{W*} = \begin{bmatrix} & & & y_{1n}^{W*} \\ & Y_{n-1}^{W*} & & \vdots \\ & & & y_{n-1,n}^{W*} \\ y_{n1}^{W*} & \cdots & y_{n,n-1}^{W*} & y_{nn}^{W*} \end{bmatrix} = \begin{bmatrix} Y_{n-1}^{W*} & y_n^{W*+} \\ & \\ y_n^{W*} & y_{nn}^{W*} \end{bmatrix}$$

where $y_n^{W*} = \left(y_{n1}^{W*}, \cdots, y_{n,n-1}^{W*}\right)$ and $y_n^{W*+} = \left(y_{1n}^{W*}, \cdots, y_{n-1,n}^{W*}\right)^{Tfact}$. Then, $Y^{W*}Y^{W} = I$, or,

$$\begin{bmatrix} Y_{n-1}^{W*} & y_n^{W*+} \\ & \\ y_n^{W*} & y_{nn}^{W*} \end{bmatrix} \begin{bmatrix} Y_{n-1}^{W} & y_n^{W+} \\ & \\ y_n^{W} & y_{nn}^{W} \end{bmatrix} = \begin{bmatrix} I & 0 \\ & \\ 0 & 1 \end{bmatrix}$$

where Y_{n-1}^{W} is partitioned matrix. $y_n^{W*} = -y_{nn}^{W*} y_n^{W} \left(Y_{n-1}^{W}\right)^{-1}$.

$$-y_n^{W} \left(Y_{n-1}^{W}\right)^{-1} = -\left(y_{n1}^{W}, y_{n2}^{W}, \cdots, y_{n,n-1}^{W}\right) \begin{bmatrix} y_{11}^{W} & y_{12}^{W} & \cdots & y_{1,n-1}^{W} \\ y_{21}^{W} & y_{22}^{W} & \cdots & y_{2,n-1}^{W} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1,1}^{W} & y_{n-1,2}^{W} & \cdots & y_{n-1,n-1}^{W} \end{bmatrix}^{-1}$$

$$< \left(\sum_{k=1}^{n-1} y_{k1}^{W}, \sum_{k=1}^{n-1} y_{k2}^{W}, \cdots, \sum_{k=1}^{n-1} y_{k,n-1}^{W}\right) \begin{bmatrix} y_{11}^{W} & y_{12}^{W} & \cdots & y_{1,n-1}^{W} \\ y_{21}^{W} & y_{22}^{W} & \cdots & y_{2,n-1}^{W} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1,1}^{W} & y_{n-1,2}^{W} & \cdots & y_{n-1,n-1}^{W} \end{bmatrix}^{-1} = (1, 1, \cdots, 1)$$

The inequality above came from the assumption $\sum_{i=1}^{n} y_{ij}^{W} < 0$, and the fact that all the element of the accompanying inverse of order $(n-1)$ are negative by the assumption of induction. The last equality comes from the fact that each column of the inverse matrix is orthogonal to the elements in the form of summation in the preceding vector except one item with the synchronized index.

Since all the elements of y_n^{W*} and y_{nn}^{W*} are negative, $\left|y_{nn}^{W*}\right| > \left|y_{nj}^{W*}\right|$ *for* $n \neq j$. (Q.D.E. of the Lemma.)

Step III

We show that the diagonal element of the matrix

$$dp^A = -\left(X^A + X^B\right)^{-1} S_{pp}^B = -(X^W)^{-1} S_{pp}^B$$

is positive under the assumption of gross-substitutes assumption for the world economy and substitutes assumption for country B. Define $Y^W = PX^W$and $Y^B = PS_{pp}^B$. Because of the Lemma, $(Y^W)^{-1}$ has a "unsurpassed" diagonal such that $\left|y_{ii}^{W*}\right| > y_{ij}^{W*}$ for $i \neq j$. Then we show that the increase in tariff rate in country B produces the increase in domestic price in country B. Let $dt^B = \left(0, \cdots, 0, dt_k^B, 0, \cdots, 0\right)'$, $dp^B = (X^W)^{-1} S_{pp}^B \cdot dt^B = (Y^W)^{-1} Y^B \cdot dt^B$.

$$\begin{aligned} (Y^W)^{-1} &= Y^{W*} = \left[y_{ij}^{W*}\right] \\ Y^B &= \left[y_{ij}^B\right] \end{aligned}$$

We are interested in the k'th column of dp^B, which is the k'th element of

$$-(Y^W)^{-1} Y^B dt^B = -\left[\begin{array}{cccc} y_{k1}^{W*} & y_{k2}^{W*} & \cdots & y_{kn}^{W*} \end{array}\right] \left[\begin{array}{c} y_{1k}^B \\ y_{2k}^B \\ \vdots \\ y_{nk}^B \end{array}\right] dt^B,$$

is written

$$-y_{kk}^{W*} y_{kk}^A - \sum_{i=1, i\neq k}^{n} y_{ki}^{W*} y_{ik}^A < -y_{kk}^{W*} y_{kk}^A - y_{kk}^{W*} \sum_{i=1, i\neq k}^{n} y_{ik}^A < 0$$

The first comes from the unsurpassed nature of $y_{kk}^{W*}$and non-positiveness of y_{ik}^{A}, and the last inequality comes from the similarity condition imposed on country A.

This implies $\frac{dp_k^A}{dt_k^B} > 0$, that is, the Lerner paradox does not arise. The Lerner's paradox does not occur when the world demand is gross substitute and the importing country's demand is substitute property.

The same argument applies for the absence of the Metzler paradox if the pure substitutability condition is replaced by the following Assumption.

Assumption III. With the international price vector p

$$p'(X^W - S_{pp}^B) < 0$$

holds.

This condition holds if S_{pp}^A is substitutes and if income effects to the two countries more or less cancell each other.

V. Conclusion

More than a half century has passed since Lerner and Metzler discovered these paradoxes, and since Metzler found the nature of the gross substitutes. Gross substitutes has played an important role in the general equilibrium analysis. In this paper we presented another way of utilizing this analytical concept in order to define the sufficient conditions to guarantee the absence of the Lerner and the Metzler Paradox in a genral equilibrium of a world economy with many goods. It was shown that the assumptions of the stability of the market (or the invertibility of the Slutsky matrix) and the similarity of Slutsky matrices across countries exclude the possibility of the emergence of the Lerner paradox and the Metzler paradox. Also, gross substitutability among goods and substitutability of demand in the import country will exclude the Lerner Paradox if the world economy starts from the zero tariff situation. Intuition of these pioneers is well grounded; paradoxes can happen a consequence of reasonable market forces if an asymmetry in excess demand structures and strong income effects exists between countries.

Conventional assumption like gross substitutabity alone cannot preclude the possibility of these paradoxes, and, though both paradoxies emerge from similar mechanism, the Metzler paradox persists more robustly than the Lerner paradox. Our task has been to seek exactly what conditions are necessary to exclude the occurence of these apparently anomalous situations. A small lemma on the dominant diagonal matrix is shown to clarify the situation. If the excess demand structure expressed by the Slutsky matirix is invertible and similar each other, gthen these paradoxes are harder to occur. Also, the dominance of gross and net substition rather than complementality will help the avoidance of the Meltzer paradox.

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