ECONOMIC GROWTH CENTER

YALE UNIVERSITY

P.O. Box 208269
27 Hillhouse Avenue
New Haven, CT 06520-8269

CENTER DISCUSSION PAPER NO. 782

THREE LECTURES ON THE WALRASIAN HYPOTHESES FOR EXCHANGE ECONOMIES

Donald J. Brown

Yale University

August 1997

Note: Center Discussion Papers are preliminary materials circulated to stimulate discussions and critical comments. This is a revised version of the paper originally written in August 1996.

Three Lectures on The Walrasian Hypotheses for Exchange Economies*

by

Donald J. Brown

Abstract

This paper discusses the testable implications of the Walrasian hypotheses:

H1 *Observed* market demand is the sum of consumer's demands derived from utility maximization subject to budget constraints.

H2 There exists an *observable* (locally) unique equilibrium price system such that the *observable* market demand is equal to the *observable* market supply in every market.

H3 The *observed* equilibrium price system is a (locally) stable equilibrium of tâtonnement price adjustment.

The main results are the Brown–Matzkin Theorem: H1 is testable, and the Brown– Shannon Theorem: H2 and H3 are not testable.

Keywords: Exchange Economies, Testable Restrictions.

*The author wishes to thank the Institute Veneto di Scienze, Lettere ed Arti, Venezia for the opportunity to present his current research on the Walrasian model at the Conference on General Equilibrium Theory in Venice, June 1997. I am also pleased to acknowledge the warm hospitality of the conference organizers Yves Balasko and Piero Gottardi. Finally, let me express my appreciation to G. Ames, T. Keister, and E. Minelli for their editorial assistance in preparing these lectures notes. The comments of T. N. Srinivasan are also appreciated.

LECTURE 1

Notes by E. Minelli (June 4, 1997)

In the introduction to his *Foundations of Economic Analysis*, Paul Samuelson defines meaningful theorems as “hypotheses about empirical data which could conceivably be refuted if only under ideal conditions.” My objective in these three lectures is to answer the question “Are the Walrasian hypotheses meaningful theorems?” How do we give an operationally useful notion of refutation for Walrasian theory? How do we bring the Walrasian model to data? It is important to distinguish between observables and unobservables in the Walrasian model. Utility and production functions are theoretical constructs, that cannot be observed. A meaningful theorem must have implications in terms of observable quantities, prices, total demand, total profits, etc. Our goal is to derive the *testable* restrictions of the Walrasian hypotheses, i.e., their implications in terms of *observables*.

Let me start by giving my definition of the Walrasian hypotheses:

H1 *Observed* market demand is the sum of consumer’s demands derived from utility maximization subject to budget constraints at *observed* market prices.

H2 There exists an *observable* (locally) unique equilibrium price system such that the *observable* market demand is equal to the *observable* market supply in every market.

H3 The *observed* equilibrium price system is a (locally) stable equilibrium of tâtonnement price adjustment.

Notice the difference between H1 and H2. H2 is about market clearing, H1 is not. H1 is a disaggregation assertion, it says that the total demand we *observe* can be disaggregated into individual demands derived by maximization of individual utility functions.

Notice also the difference between H1 and the Sonnenschein–Debreu–Mantel theorem: their theorem asks how stringent must assumptions be at the level of individual demand in order for aggregate demand to behave like a representative agent’s demand? Walras’ proposition goes the other way: when is it true that the observed market demand can be written as the sum of individual demands? There is no claim about aggregate market demand behaving as if derived from some representative individual’s maximization, nor about market clearing. This is a very important point, because if we want to test market clearing, we must observe the aggregate supply. On the other hand, we can test H1 without assuming market clearing. H1 is a statement about our ability to disaggregate market demand into individual demands that result from utility maximization. It is not a statement about demand being equal to supply.

To give the technical content of these hypotheses we use the notion of equilibrium manifold, which was introduced independently by Steve Smale and Yves Balasko in the seventies. The two definitions are different: Balasko defines the equilibrium manifold as the combinations of endowments and prices such that markets clear. Smale considers instead the combinations of endowments and individual consumptions such that markets clear. In principle, both definitions could be used for testing the Walrasian hypotheses. In practice, aggregate data is much easier to collect than individual consumption data. In our analysis, we observe market prices, total demand, and sometimes individual endowments. We never assume observability of individual consumptions.

In this and in the next lecture, I will discuss the testability of H1. The third lecture will be about some recent ongoing research with Chris Shannon. In this work, which was partly motivated by a paper of Piero Gottardi and Thorsten Hens on monotone comparative statics for the Walrasian model, we look more closely at a particular class of economies, *distribution economies*. These are exchange economies in which the income distribution is exogenously specified, for example, exchange economies with collinear individual endowments. In this context we investigate the testability of H2 and H3.

Let us go back to the main question: How do we bring the Walrasian model to the data? In economics there are two different methodologies for deriving testable implications of theories. One method, used often in consumer theory, is to derive comparative statics restrictions. The other methodology is revealed preference theory, originating in Samuelson's *Foundations of Economic Analysis*. By the Sonnenschein–Debreu–Mantel theorem we know that the comparative statics approach cannot be extended from consumer demand to aggregate market demand unless we make very restrictive assumptions on individual demand.

Hence, we will follow the revealed preference approach. The main theorem we shall need is Afriat's theorem. Given a finite number of observations on prices and consumption choices, this theorem states the equivalence between four conditions:

(a) the observations are consistent with maximization of a non-satiated utility function,
(b) the observations satisfy a form of the strong axiom of revealed preferences,
(c) there exists a finite set of numbers that, jointly with the observations, satisfy a set of inequalities called the Afriat inequalities,
(d) the observations are consistent with maximization of a concave, monotone, continuous, non-satiated utility function.

This is a theorem about individual demand, and our question is: Can we extend Afriat's approach to market demand?

What is striking about Afriat's theorem is that these four conditions are equivalent. In particular if you can rationalize the data with any utility function, then you can rationalize it with a utility function that is concave, monotone and continuous. Finite data sets simply cannot distinguish between quasiconcave and concave utility functions. The concave utility function in (d) is actually constructed from the market data and the finite set of numbers that constitute the solutions of the Afriat inequalities.

But how do we interpret the Afriat inequalities? They are the Kuhn–Tucker first-order conditions for maximizing a concave function subject to a budget constraint. These inequalities involve two kinds of variables: market data and unobservables. Afriat assumes one can observe not only prices, but also individual consumption. The other variables, utility levels, and marginal utilities of income are unobservable theoretical constructs. On the other hand, the axiom in (b), a version of the strong axiom of revealed preference, only involves observable market data: prices and consumptions. In going from (c) to (b), Afriat has managed to eliminate all the unobservables! Notice that the Afriat inequalities are linear in the unobservables. In this case there is an algorithm that will decide in polynomial time if this set of inequalities is consistent, i.e., if there exist utility levels and marginal utility levels that satisfy them.

When we extend revealed preference theory from individual demand to market demand, we will lose this linearity. This is the major impediment in the extension of Afriat's and Samuelson's program to the whole market. We will provide an algorithm for deciding if the equilibrium inequalities have a solution, but it will typically be an exponential-time decision procedure.

Let us consider for a moment Afriat's single consumer case. What methods allow us to verify if a given system of linear (or convex) inequalities admits a solution? There are three main decision procedures for these types of inequalities.

Fourier–Motzkin elimination amounts to a generalization of the method of substitution that we learn in high school, and provides an exponential time algorithm for linear inequalities. For convex inequalities we have two types of algorithms, the ellipsoid method and the interior point method, both of which are polynomial-time decision methods.

As an illustration of Fourier–Motzkin elimination, suppose that we have a finite set of linear inequalities in two real variables x_1 and x_2. Applying the Fourier–Motzkin method to eliminate x_2 amounts to projecting on the x_1 axis the admissible set defined by the inequalities. Indeed, if we take x_1 in the projection, we know that there exists an x_2 such that we have a point in the polytope defined by our set of inequalities. If we carry out the Fourier–Motzkin elimination procedure, we can have three mutually exclusive possible outcomes. Either we discover that the inequalities are always satisfied: the projection is the whole domain of the definition of x_1. Or we discover that there is no solution, i.e., the inequalities are inconsistent and the projection is empty. Or lastly, we are in the case we are most interested in, the case in which for *some* values of x_1 the system has a solution: the projection is a nonempty interval on the x_1 axis.

In Brown and Matzkin (1996) the same logic is applied to analyze the testability of the Walrasian hypothesis H1. We look for a set of equilibrium inequalities that characterizes a general equilibrium of the economy and for a finite time algorithm with the property that the general equilibrium hypothesis is refutable if and only if the algorithm does not end in the states 1 = 1, i.e., the inequalities are always satisfied and 1 = 0, i.e., the inequalities are never satisfied. If we guarantee the algorithm must terminate in finite time in one of the three states, and we prove that neither 1 = 1 nor 1 = 0 is possible, then this will mean the hypothesis is refutable.

To argue that the algorithm cannot terminate with 1 = 0, it is sufficient to invoke an existence theorem. Alternatively, we can produce an example in which an equilibrium exists. For the purpose of this methodology these two methods are equivalent. When we are asked why do we care about proving existence, there are usually two answers. One is: proving theorems is fun. The other is: in order to bring the model to data. But we actually do not need an existence theorem to conclude that the system of equilibrium inequalities is consistent.

Similarly, to show that the algorithm cannot terminate with 1 = 1, we only need to produce an example in which all possible equilibrium allocations violate the revealed preference axiom that characterizes an individual utility maximization subject to budget constraints, i.e., the Afriat inequalities.

In the Brown–Matzkin paper, we find necessary and sufficient conditions for rationalizing market data with the Walrasian model. These conditions are the Afriat inequalities, budget constraints for each individual, and the market clearing conditions in each period of observation. Here again notice that when we say “market clearing” we mean that observed total demand can be rationalized as the sum of rational individual demands, not that demand equals supply. We are checking the testability of H1, not of H2.

We use a version of Afriat’s theorem that was proved in Chiappori and Rochet (1987). In this version the strong axiom of revealed preferences is strengthened by requiring that different prices must induce different consumption choices. The conclusion is also stronger: data satisfying the axiom can be rationalized by a utility function that is not only concave and monotonic, but actually smooth.

Our theorem gives a set of polynomial inequalities characterizing Walrasian equilibrium in each observation, and the Tarski–Seidenberg theorem on quantifier elimination provides an algorithm that terminates in finite time in one of three mutually exclusive states. The construction of two examples allows us to exclude the states 1 = 1 and 1 = 0. The only remaining possibility is that the inequalities can be satisfied for some values of the observables, but not for all values, i.e., the Walrasian model is testable. The main difference with Afriat’s work is we do not assume that individual consumption is observable. This implies that the inequalities are no longer convex in the unobservables. Hence, we do not have a polynomial time algorithm to check consistency.

Lack of convexity explains why we cannot use one of the decision methods described above and we must instead invoke the Tarski–Seidenberg theorem on quantifier elimination. This theorem provides an algorithm for reducing a given set of polynomial inequalities to an equivalent set of polynomial inequalities in the coefficients. Applied to our case this means that we can reduce the set of inequalities that characterize market equilibrium to a set of inequalities in the observables: prices, incomes, and market demand . The Tarski–Seidenberg algorithm must terminate in finite time in one of three possible states: the given set of inequalities is never satisfied (1 = 0), the given set of inequalities in always satisfied (1 = 1), or the system of inequalities is reduced to an equivalent one in which only the observables are involved.

As an example, consider the quadratic equation $ax^2 + bx + c = 0$. The equivalence between the existence of a real solution to this equation and the inequality $b^2 - 4ac \geq 0$ is an instance of quantifier elimination.

Like Fourier–Motzkin elimination for linear inequalities, this is not a polynomial time procedure. Happily, to show testability of the market equilibrium inequalities we do not need to carry out quantifier elimination.. It is enough to provide examples that rule out the 1 = 1 and the 1 = 0 states.

This will be demonstrated in the second lecture. The Walrasian hypothesis H1 is therefore testable. We then go on to prove in the third lecture that, in the more restricted domain of distribution economies, H2 and H3 are not testable. Hence only H1 is a meaningful theorem.

LECTURE 2

(June 5, 1997)

This lecture was on the published paper, "Testable Restrictions on the Equilibrium Manifold" by Don Brown and Rosa Matzkin, *Econometrica,* Vol. 64, No. 6 (November 1996), pp. 1249–1262. The Introduction of their paper is given below.

The core of the general equilibrium research agenda has centered around questions of existence and uniqueness of competitive equilibria and stability of the price adjustment mechanism. Despite the resolution of these concerns, i.e., the existence theorem of Arrow and Debreu, Debreu's results on local uniqueness, Scarf's example of global instability of the tâtonnement price adjustment mechanism, and the Sonnenschein–Debreu–Mantel theorem, general equilibrium theory continues to suffer the criticism that it lacks falsifiable implications or in Samuelson's terms, "meaningful theorems."

Comparative statics is the primary source of testable restrictions in economic theory. This mode of analysis is most highly developed within the theory of the household and theory of the firm, e.g., Slutsky's equation, Shephard's lemma, etc. As is well known from the Sonnenschein–Debreu–Mantel theorem, the Slutsky restrictions on individual excess demand functions do not extend to market excess demand functions. In particular, utility maximization subject to a budget constraint imposes no testable restrictions on the set of equilibrium prices, as shown by Mas-Colell (1977). The disappointing attempts of Walras, Hicks, and Samuelson to derive comparable statics for the general equilibrium model are chronicled in Inagro and Israel (1990). Moreover, there has been no substantive progress in this field since Arrow and Hahn's discussion of monotone comparative statics from the Walrasian model (1971).

If we denote the market excess demand function as $F_{\hat{w}}(p)$ where the profile of individual endowments $\hat{w}$ is fixed but market prices p may vary, then $F_{\hat{w}}(p)$ is the primary construct in the research on existence and uniqueness of competitive equilibria, the stability of the price adjustment mechanism, and comparative statics of the Walrasian model. A noteworthy exception is the monograph of Balasko (1988) who addressed these questions in terms of properties of the equilibrium manifold. To define the equilibrium manifold we denote the market excess demand function as $F(\hat{w}, p)$, where both $\hat{w}$ and p may vary. The equilibrium manifold is defined as the set $\{(\hat{w}, p) | F(\hat{w}, p) = 0\}$. Contrary to the result of Mas Colell, cited above, we shall show that utility maximization subject to a budget constraint does impose testable restrictions on the equilibrium manifold.

To this end we consider an alternative source of testable restrictions within economic theory: the nonparametric analysis of revealed preference theory as developed by Samuelson, Houthakker, Afriat, Richter, Diewert, Varian, and others for the theory of the household and the theory of the firm. For us, the seminal proposition in this field is Afriat's theorem (1967), for data on prices and consumption bundles. Recall that Afriat, using the Theorem of the Alternative, proved the equivalence of a finite family of linear inequalities — now called the Afriat inequalities — that contain unobservable utility levels and marginal utilities of income with his axiom of revealed preference, "cyclical consistency" — finite families of linear inequalities that contain only observables (i.e., prices and consumption bundles), and with the existence of a concave, continuous monotonic utility function rationalizing the observed data. The equivalence of the Afriat inequalities and cyclical consistency is an instance of a deep theorem in model theory, the Tarski–Seidenberg theorem on quantifier elimination.

The Tarski–Seidenberg theorem — see Van Den Dries (1988) for an extended discussion — proves that any finite system of polynomial inequalities can be reduced to an equivalent finite family of polynomial inequalities in the coefficients of the given system. They are equivalent in the sense that the original system of polynomial inequalities has a solution if and only if the parameter values of its coefficients satisfy the derived family of polynomial inequalities.. In addition, the Tarski–Seidenberg theorem provides an algorithm which, in principle, can be used to carry out the elimination of the unobservable — the quantified — variables, in a finite number of steps. Each time a variable is eliminated, an equivalent system of polynomial inequalities is obtained, which contains all the variables except those that have been eliminated up to that point. The algorithm terminates in one of three mutually exclusive and exhaustive states: (i) $1 \equiv 0$, i.e., the original system of polynomial inequalities is never satisfied; (ii) $1 \equiv 1$, i.e., the original system is always satisfied; (iii) an equivalent finite family of polynomial inequalities in the coefficients of the original system which is satisfied only by some parameter values of the coefficients.

To apply the Tarski–Seidenberg theorem, we must first express the structural equilibrium conditions of the pure trade model as a finite family of polynomial inequalities. Moreover, to derive equivalent conditions on the data, the coefficients in this family of polynomial inequalities must be the market observables — in this case, individual endowments and market prices — and the unknowns must be the unobservables in the theory — in this case, individual utility levels, marginal utilities of income, and consumption bundles. A family of equilibrium conditions having these properties consists of the Afriat inequalities for each agent; the budget constraint of each agent; and the market clearing equations for each observation. Using the Tarski–Seidenberg procedure to eliminate the unknowns must therefore terminate in one of the following states: (i) $1 \equiv 0$ — the given equilibrium conditions are inconsistent, (ii) $1 \equiv 1$ — there is no finite data set that refutes the model, or (iii) the equilibrium conditions are testable.

Unlike Gaussian elimination — the analogous procedure for linear systems of equations — the running time of the Tarski–Seidenberg algorithm is in general not polynomial and in the worst case can be doubly exponential — see the volume edited by Arnon and Buchberger (1988) for more discussion on the complexity of the Tarski–Seidenberg algorithm. Fortunately, it is often unnecessary to apply the Tarski–Seidenberg algorithm in determining if the given equilibrium theory has testable restrictions on finite data sets. It suffices to show that the algorithm cannot terminate with $1 \equiv 0$ or with $1 \equiv 1$. In fact, as we shall show, this is the case for the pure trade model.

It follows from the Arrow–Debreu existence theorem that the Tarski–Seidenberg algorithm applied to this system will not terminate with $1 \equiv 0$. In the next section, we construct an example of a pure trade model where no values of the unobservables are consistent with the values of the observables. Hence the algorithm will not terminate with

$1 \equiv 1$. Therefore the Tarski–Seidenberg theorem implies for any finite family of profiles of individual endowments $\hat{w}$ and market prices p that these observations lie on the equilibrium manifold of a pure trade economy, for some family of concave continuous, and monotonic utility functions, if and only if they satisfy the derived family of polynomial inequalities in $\hat{w}$ and p. This family of polynomial inequalities in the data constitute the testable restrictions of the Walrasian model of pure trade.

It may be difficult, using the Tarski–Seidenberg algorithm, to derive these testable restrictions on the equilibrium manifold in a computationally efficient manner for every finite data set, although we are able to derive restrictions for two observations. If there are more than two observations, our restrictions are necessary but not sufficient. That is, if our conditions hold for every pair of observations and there are at least three observations, then the data need not lie on any equilibrium manifold. Consequently, we call our conditions the weak axiom of revealed equilibrium or WARE. Of course, if our conditions are violated for any pair of observations, then the Walrasian model of pure trade is refuted.

References

Afriat, S. (1967): "The Construction of a Utility Function from Demand Data," *International Economic Review*, 8, 67–77.

Arnon, D. S. and B.Buchberger (1988): *Algorithms in Real Algebraic Geometry*. New York: Academic Press.

Arrow, K. and F. Hahn (1971): *General Competitive Analysis*. New York: North Holland.

Balasko, Y. (1988): *Foundations of the Theory of General Equilibrium*. Boston: Academic Press.

Inagro, B. and G. Israel (1990): *The Invisible Hand.* Cambridge: MIT Press.

Mas-Colell, A. (1977): "On the Equilibrium Price Set of an Exchange Economy," *Journal of Economic Theory*, 4, 117–126.

Van Den Dries, L. (1988): "Alfred Tarski's Elimination Theory for Real Closed Fields," *Journal of Symbolic Logic*, 53, 7–19.

LECTURE 3

Notes by T. Keister (June 6, 1997)

The work I reported on in the second lecture shows there is data which allows one to either accept or reject the Walrasian hypothesis that observed market demand is a result of individual utility maximization subject to a budget constraint. The purpose of this lecture is to consider the second and third of the Walrasian hypotheses, which are that market data can be rationalized with a market structure with locally unique observed equilibrium prices which are locally stable with respect to tâtonnement price adjustment. I find this third proposition surprising. It is motivated, at least in part, by two results. First, the theorem of Afriat gives four equivalences. The first one says that you can rationalize the data by optimization with any kind of utility function while the fourth one says you then can rationalize the data with a particularly nice utility function, one that is concave and monotone. Then if you go to the Chiappori–Rochet theorem and strengthen the hypothesis on the observed data a bit more, so that no observations of consumption occur at any kinks, then you get a truly striking result. If you can rationalize the data with any function, then you can rationalize it with a function that is concave, monotone and C^{∞}. Rosa Matzkin and I showed there are also conditions on *market* data that allow this kind of rationalization. It may be the case that you can rationalize market data with something better than monotone, concave utility functions; perhaps you can rationalize it with utility functions that give rise to demand functions that generate a locally stable price adjustment mechanism. That is half of the motivation. The other half of the motivation is an unpublished paper by Piero Gottardi and Thorsten Hens called "On Monotone Comparative Statics in Exchange Economies," dated February 1995. Later in this lecture I will describe how this paper influenced us.

A more compelling economic argument for considering tâtonnement price adjustment is provided by an important series of papers by Jerome Keisler, where he gives a price adjustment mechanism that is more informationally decentralized than previous research in the price adjustment literature. He has an auctioneer who has an inventory and there is a random process whereby agents go to the auctioneer, one at a time, and present their demands. Keisler shows that for an arbitrarily small inventory, the auctioneer is able to meet these demands, adjust prices, and estimate the equilibrium price in a relatively short period of time, due to the random sampling. Over a longer period of time he is also able to clear the market, almost. The basic assumption Keisler needs to guarantee that his stochastic price adjustment mechanism has these properties is the existence of an underlying, deterministic locally stable tâtonnement price adjustment. The first step in empirically investigating Keisler's price adjustment processes is to prove local tâtonnement stability.

The propositions that I want to prove in this lecture are theorems due to myself and Chris Shannon. First let me remind you of the Correspondence Principle. Samuelson said in the *Foundations* that when we have systems, as opposed to individuals, we cannot characterize their behavior by saying that some function is being maximized. His idea, essentially following Walras, was to write down a price adjustment mechanism. (Actually, Samuelson was the first person to write down a specific differential equation. Hicks in *Value of Capital* was criticized for providing a lucid discussion but no differential equation.) He then proposed making inferences about the comparative statics by postulating that the observed equilibrium was stable. This is the Correspondence Principle. As I said in the first lecture, Arrow and Hahn point out that when this principle generated comparative statics it was due to some underlying property of the model that guaranteed both the comparative statics and the local stability of tâtonnement. What we are going to show is that if you can rationalize the data by monotone, continuous utility functions, then you can rationalize the data with an indirect utility function having the property that every observed equilibrium is an equilibrium of a locally monotone market demand function. This local monotonicity has three important consequences: local tâtonnement stability; local monotone comparative statics; and local uniqueness.

Now let us return to the Gottardi–Hens paper. I am certain you know the literature on the Law of Demand. Just to remind you, let me quickly tell you what the idea is. The idea is simply that aggregation smooths, so it is conceivable that in the aggregate you can have properties you would not observe at the individual level. Hildenbrand, the leading figure in this literature, was interested (in part) in trying to replicate, in the multi-market setting, downward sloping demand curves. That's what he means by the Law of Demand. Since there is not just a single product, but many of products, we must be clear about what this means. Suppose the demand curve is given by $f(p)$. One way of formalizing the idea of "downward sloping" is the condition

$$(f(p_1) - f(p_2)) \cdot (p_1 - p_2) < 0. \tag{1}$$

It is clear that if you fix the price in every market but one, then you would have the downward sloping property. But condition (1) is much stronger than that. Of course, there is no way that utility maximization subject to a budget constraint will in general generate, individual demand curves with this property. What is original about his research is that by restricting the distribution of agents' characteristics he proves market demand is monotone in cases where individual demand is not monotone.

Gottardi and Hens ask the following intriguing question: "Is there any relationship between the Law of Demand and monotonicity of the Walrasian correspondence?" The Walrasian correspondence, recall, consists of ordered pairs of prices and endowment profiles. Just for a moment think about distribution economies, or economies with collinear endowments, and fix the proportion of income that each type receives. Then the only variable is total endowment. Is it true for any pair of equilibrium prices and pair of total endowments condition (1) holds? If the equilibrium correspondence has this property, then what does this imply about the market demand function, or the excess demand function, defining the correspondence? Hildenbrand states in his seminal paper in 1983 that this is one case where monotonicity of market demand, i.e., the Law of Demand, is equivalent to monotonicity of the Walrasian correspondence. You simply apply the implicit function theorem to the excess demand function at the equilibrium prices. And since income is independent of endowments, the derivative is a product of two matrices.

To be monotone is to assert that one of those matrices is negative definite. Since their product is equal to the identity matrix, then the other matrix must also be negative definite. One matrix is the derivative of equilibrium prices with respect to total endowment. The other matrix is the market demand with respect to equilibrium prices. This is the starting point of the Gottardi–Hens paper. They then consider the relationship between monotonicity of market demand and monotonicity of the Walrasian correspondence for different distributional rules. It is a very nice paper.

After this lengthy introduction, let me begin by laying out the basic notation. We use Δ to represent the price simplex in $\mathbb{R}^{\ell}_{+}$. Agents are indexed by t, and the endowment of agent t is denoted $e_t \in \mathbb{R}^{\ell}_{++}$. The agent's demand function is given by $x_t(p, p \cdot e_t)$, and her excess demand function by

$$f_t(p, p \cdot e_t) = x_t(p, p \cdot e_t) - e_t .$$

We use $\hat{e} = (e_1, \ldots, e_t, \ldots, e_A)$ to denote the vector of endowments and $e = \sum_{t=1}^{A} e_t$ for the aggregate endowment. Then market demand is given by

$$X(p, \hat{e}) = \sum_{t=1}^{A} x_t(p, p \cdot e_t),$$

and $F(p, \hat{e}) = X(p, \hat{e}) - e$ is the market excess demand function.

There are several price adjustment processes in the literature. The definition of tâtonnement that I have in mind is

$$\frac{dp}{dt} = F(p, \hat{e}) .$$

Note that this is price adjustment, not quantity adjustment. Let me also list a few properties of market excess demand functions:

(a) Continuity: $F(p, \hat{e})$ is continuous on $\mathbb{R}^{\ell}_{++}$

(b) Walras' Law: $p \cdot F(p, \hat{e}) = 0$ for all p in $\mathbb{R}^{\ell}_{++}$

(c) Homogeneity: $F(p, \hat{e}) = F(\lambda p, \hat{e})$ for all $\lambda > 0$ and for all p in $\mathbb{R}^{\ell}_{++}$

Now it is time to revisit the Sonnenschein–Debreu–Mantel theorem. What do we know from the discussions in the previous lectures about this particular theorem? My claim is that it has a number of interesting implications, one of the most important being that it put an end to the research program attempting to find reasonable conditions guaranteeing global stability of tâtonnement price adjustment. The dream that tâtonnement is globally stable dies very hard with economists. So even after Scarf's counterexamples and the Sonnenschein–Debreu–Mantel theorem, some economists still pursued this will-of-the-wisp. They hoped that by restricting the preference structure and the income structure, global stability might be guaranteed. Why would one even think this could be a real possibility? Recall that Eisenberg's aggregation theorem says if everyone has homothetic preferences, and if the income distribution is price independent, then there is a representative agent. What does the Sonnenschein–Debreu–Mantel theorem say? It says that if you drop the assumption of price independent income distribution, then you can rationalize any excess demand function with an economy with homothetic preferences. Hence any dynamic on the price simplex can be generated by some exchange economy with homothetic preferences. Eisenberg had two assumptions. Sonnenschein–Debreu–Mantel dropped one. A natural question is then to ask what happens if we drop the other one. It might be the case if we fix the income distribution, i.e., look at collinear endowments, and we restrict everyone to have the same preferences, maybe then we cannot rationalize any excess demand function. In a theorem that I find quite remarkable, Kirman–Koch proved that it is still not true. The statement of their theorem is as follows.

Kirman–Kock Theorem (*RES*, 1986): *Given any function g from* $\mathbb{R}^{\ell}_{++}$ *to* $\mathbb{R}^{\ell}$ *with properties* (a), (b), *and* (c), *and arbitrarily chosen positive numbers* $\alpha_1,\ldots,\alpha_n$ *with* $\sum\alpha_1 = 1$, $\alpha_i \neq \alpha_j$ *for* $i \neq j$, *and given* $n \geq \ell$, *then for every compact subset K of* $\mathbb{R}^{\ell}_{++}$ *there exists an exchange economy* $\mathscr{E}$ *with n agents in which every agent has the same preferences and endowments are collinear, i.e.,* $e_t = \alpha_t e$, *such that the market excess demand function of* $\mathscr{E}$ *is equal to g on K.*

There are three great theorems in aggregation over agents. One is Eisenberg's theorem, which I have stated. The second is Sonnenschein–Debreu–Mantel, and the third is the Kirman– Koch theorem. They summarize what we know about aggregation over agents and about the existence of a representative agent. If you drop either one of the hypotheses of Eisenberg's theorem, then anything can happen in terms of tâtonnement dynamics. This is quite important for what we are going to do, since I will restrict my attention to distribution economies and without this theorem you might suspect that I assumed away the problem.

Let me now recall for you the definition of distribution economies.

Definition: Distribution economies are exchange economies with collinear individual endowments, or, equivalently, exchange economies with price independent relative income distribution.

The earliest references to the monotone demand that I have found in the literature is due to Malinvaud.

Malinvaud's Theorem (*Lectures in Microeconomic Theory*, 1969): If, in a distribution economy, each individual's demand function is locally monotone at an equilibrium price vector $\bar{p}$, then the tâtonnement price adjustment process is asymptotically stable at $\bar{p}$. Moreover, $\Delta e \cdot \Delta p < 0$ in a small neighborhood of $\bar{p}$.

I have been unable to find a copy of the 1969 French edition of his *Lectures in Microeconomic Theory*, so I have assumed there is no difference between the French edition and the first English translation which was 1972, so the date here might be wrong. Either way, his theorem appears at least 10 years before there is mention of monotone demand in the recent literature. Notice what this theorem says. Suppose it just happens that at the equilibrium, market demand is locally monotone. Then you get his results. You might think that Hildenbrand's approach is much more reasonable; you better have monotonicity everywhere so when you get to equilibrium you will know that it is monotone there. In point of fact, it is Malinvaud's theorem that we show cannot be refuted by finite data sets.

I should make a related point here. There has been no discussion in any of these lectures about uniqueness. I have been able to avoid worrying about whether or not the equilibrium correspondence was single valued or not. Ex-post there is no issue, it is only ex-ante that uniqueness is a problem. Either the prices you observe are equilibrium prices, or not. The fact that there may be other prices that could have cleared the market is of no consequence. Therefore, uniqueness and the existence theorem of Arrow and Debreu play no role in our analysis. Of course, local uniqueness of the observed equilibria will follow from my theorem with Chris.

Let me call your attention, for the last time, to Afriat's theorem.

Definition: A utility function $u(x)$ rationalizes the data (p^i, x^i), $i = 1,\ldots,n$, if $u(x^i) \geq u(x)$ for all x such that $p^i \cdot x^i \geq p \cdot x$, for $i = 1,\ldots,n$.

Afriat's Theorem (Varian, *RES*, 1983): *The following conditions are equivalent:*

(a) *there exists a non-satiated utility function that rationalizes the data*

(b) *the data satisfy* GARP

(c) *there exist numbers* U^i, $\lambda^i > 0$, $i = 1,\ldots,n$ *that satisfy the Afriat inequalities*

$$U^i \leq U^j + \lambda^j p^j (x^i - x^j) \text{ for } i,j = 1,\ldots,n$$

(d) *there exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.*

I do this only to point out that Afriat uses the Afriat inequalities to actually construct the concave monotone utility function in condition (d). I will give you his construction in a moment. What we will need, though, is not Afriat's theorem, but its refinement due to Chiappori and Rochet, which gives smooth rationalizations. I want to show you their argument. There is an important lemma in their paper.

Chiappori–Rochet Lemma (*Econometrica*, 1987): *If there exist numbers* V^i *and vectors* $q^i > 0$, $i = 1,\ldots,n$ *that satisfy the strict Afriat inequalities*

$$V^i < V^j + q^j (x^i - x^j) \text{ for any } i \neq j,$$

then there exists a concave, strictly increasing C^∞ *function* $\mathbb{R}^\ell_+ \Rightarrow \mathbb{R}$ *such that for every* i, $V(x^i) = V^i$ *and* $DV(x^i) = q^i$, *where* DV *is the gradient of* V.

Proof: Let $W(x) = \min_i \{V^i + q^i (x - x^i)\}$, the "Afriat" function and $V(x) = \phi_\eta * W = \int_{\mathbb{R}^\ell} W(x - y) \phi_\eta(y) dy$, where η and $W(x) = V^i + q^i (x - x^i)$ for $x \in B(x^i, \eta)$ for $i = 1,\ldots, n$. ■

In the first lecture I mentioned how you can guess the Afriat inequalities. I just wrote down the inequalities characterizing a smooth concave function in terms of the gradient and said that if you maximize subject to a budget constraint then you can replace the gradient by λ times the price vector. That is just a first order condition, and it will give you the Afriat inequalities. Chiappori and Rochet actually take this linear inequality and ask if you can find a concave utility function whose gradient at x^i would be q^i. They showed the answer is "yes." This is no surprise. What is interesting, in part, is how they do it. This function, $W(x)$, which is essentially the minimum of the right hand side of the Afriat inequalities, is what they call the Afriat function. This function is clearly piecewise linear. What they then do is they take a convolution of that function with a "bump" function, which allows them to smooth out the "kinks," and still rationalize the data. We are going to use their argument, but with a twist. Their argument is in the primal space, the space of commodities. We use their argument in the dual space, that is, the space of prices and incomes. This is because we use the indirect utility function, which is in terms of the market variables we observe. To formulate the model in terms of a set of inequalities for indirect utility functions, you need to know the following duality between utility maximization and indirect utility minimization.

$$\text{(P)} \quad \begin{aligned} V(p,I) \;=\; \max\; & U(x) \\ \text{s.t.}\; & p \cdot x \le I \\ & x \in \mathbb{R}^{\ell}_{+} \end{aligned} \qquad \text{(D)} \quad \begin{aligned} V(p,I) \;=\; \min\; & V(p,I) \\ \text{s.t.}\; & p \cdot x \le I \\ & x \in \Delta \end{aligned}$$

Here (P) stands for the primal problem and (D) for the dual. How would I convince you that these two problems are dual? I would have to show you that if you look at the optimal value function defined by the primal, that is the definition of indirect utility function. And if you look at the optimal value function defined by the dual, you go back around the circle. You lose something, though, and it is important to understand what you lose. Suppose you start out in (P) with U being strictly concave, smooth, and monotone. What can you say about the V that would result? There is a two line proof in Mas-Colell that V is quasi-convex in (p,I). It's important to understand that it is quasi-convex in both p and I. In some papers you do not see this because they normalize the income to be 1. In fact, it is more than quasi-convex and this turns out to be important. It is actually pseudo-convex. Pseudo-convex functions are quasi-convex functions such that the first order conditions are sufficient for optimality. So if you start with a strictly concave, smooth monotone function U, the V that results is pseudo-convex and is increasing in I and decreasing in p. Similarly, if you start with a smooth convex function $V(p,I)$ in the problem (D), then the U that results is pseudo-concave. So this is what you lose. Nevertheless, these two problems are equivalent, and I will use this fact in our analysis. Our observables will be the market prices, the income distribution, and market demand. We will not allow observation of individual demands, individual endowments or aggregate endowments. Here is the remarkable result of Quah, which plays a crucial role in our analysis.

Quah's Theorem ("The Monotonicity of Individual and Market Demand," 1995): *Suppose h is a convex indirect utility function satisfying the property*

$$\varepsilon(\bar{p},\bar{y}) = \frac{\bar{y}h_{yy}(\bar{p},\bar{y})}{h_y(\bar{p},\bar{y})} < 2 \ \ at \ \ (\bar{p},\bar{y}).$$

Then h generates a locally monotone demand function in a neighborhood of $\bar{p}$.

This is a very nice result, but how are we going to use this proposition on finite data sets? Quah observes in passing that if you have a homothetic utility function, then you can always construct an indirect utility function that is linear in income and therefore this condition would be trivially satisfied, since the second derivative would be zero. The h_{yy} term is the second derivative of the indirect utility function with respect to income. Hence he obtains the well-known result, that homothetic utility functions give rise to monotone demands. We are going to use the Afriat inequalities for the indirect utility function. How do I know there are Afriat inequalities for indirect utility functions? Because in the dual optimization problem I'm minimizing a "convex" function subject to a linear constraint. There had better be Afriat inequalities. If that bothers you, multiply the objective function by –1. So we know there are Afriat inequalities and they are linear in prices and income. What are the Afriat inequalities? They are:

$$V^i - V^j > q^j \cdot (p^i - p^j) + \lambda^j (I^i - I^j), \text{ for } i,j = 1,...,n.$$

The Afriat inequalities for indirect utility functions, unlike the Afriat inequalities for the utility function, are linear in the unobservables.

I should remind you that we also must consider feasibility, which means you have to add up the implied x's. What are the implied x's? Well, you know that the x's are minus the derivative of the indirect utility function with respect to p divided by the derivative of the indirect utility function with respect to income. So there is our non-linearity again, but for this part of the problem this is not an issue. I don't care about the non-linearity, because I am not trying to solve the inequalities. I am just trying to prove that they have a certain property.

You might be a little bothered by my argument. I am saying that I am going to invoke Quah's theorem by going through Afriat's theorem because that will give me an indirect utility function that is linear in income. What about Quah's assumption that $V(p,I)$ is convex? Well, this is just Afriat's theorem. You always get, depending on whether you are doing a maximization or minimization problem, concavity or convexity. So we are going to get convexity and the only remaining question is, how will we get a demand function? There are all these flats. Remember, however, that to get a demand function, what I really care about is the derivative of the indirect utility function with respect to p. So I am going to add a small quadratic perturbation in p, which gives me the needed strict convexity in p. Let me remind you what convolutions are. First I need to define the "bump" function.

Definition: The "bump" function $\phi_a(x)$:

(a) Let $h(t) = \begin{Bmatrix} 0 & \text{if } t \le 0 \\ \exp(-1/t) & t > 0 \end{Bmatrix}$

(b) $h(1 - |x|^2) = h(1 - x_1^2 - \cdots - x_\ell^2)$
(c) $\phi(x) = d^{-1}h(1 - |x|^2)$ where $d = \int_{B(0,1)} h(1 - |x|^2)dx$
(d) $\phi_a(x) = a^{-1}\phi(x/a)$.

This is a function that comes into zero in an exponential way. It is C^∞ at zero and all the derivatives are zero. A few important properties of this function are:
(i) $\phi_a \in C_c^\infty(\mathbb{R}^\ell)$
(ii) $\phi_a \geq 0$
(iii) $\phi_a(x) > 0 \Leftrightarrow |x| < a$
(iv) $\int_{\mathbb{R}^n} \phi_a(x)dx = 1$.

For a convolution, you simply take this function and you use it in a symmetric way to form a ball.

Definition: The **convolution** of $f(x)$ and $\phi_a(x)$, denoted $f * \phi_a$, defines a function $g(x) = \int_{\mathbb{R}^\ell} f(y)\phi{+}a(x - y)dy$.

Recall that the function $g(x)$ has several nice properties. One being, it is infinitely differentiable. Chiappori and Rochet show that if you convolve the bump function with the Afriat function, it is also concave (in our case, convex). So what does convolution do? You take each point in the graph of the function , draw a little ball around it, and average. Well suppose, in fact, that it is linear. Since I have only observed a finite number of points, I can draw these balls small enough so they do not overlap, and so that the convolution will be linear in these little balls. It will not be linear everywhere, but there will be a small neighborhood of each of the observed points where it is linear. Hence, when I take the second derivative, what do I get? Zero! If it is zero at this point, then I know that there is a neighborhood of the point where it is less than two. This is the argument. The only thing left to do is to guarantee that I get a real demand function rather than a demand correspondence. For this, I add a small quadratic perturbation in p, but the perturbation in p is independent of income.

Now let me state the first of our theorems.

Theorem 1 (Brown and Shannon): *Let $\langle p^r, \{I^r\}, e^r\rangle$, $r = 1,...,n$ be given. Then there exists a smooth, pseudo concave, monotone utility function $u(x)$ rationalizing the data such that the implied demand function is locally monotone at (p^r, I^r), $r = 1,...,n$ if the following family of inequalities is satisfied: there exists numbers V^i, λ^i and vectors q^j such that*
(i) $V^i - V^j > q^j \cdot (p^i - p^j) + \lambda^j(I^j - I^j)$, *for* $i, j = 1,...,n$
(ii) $\lambda^j > 0$, $q^j \ll 0$, $j = 1,...,n$
(iii) $p^j \cdot q^j = -\lambda^j I^j$, $j = 1,...,n$.

When you observe market prices, the income distribution, and total market demand, and you observe them in several periods, then we claim there exists a smooth, pseudo concave, monotone utility function rationalizing the data such that the implied demand function is locally monotone if these inequalities are satisfied. That is just the construction I went through by constructing an indirect utility function consistent with the data having the property that the second derivative is zero at each of the data points and consequently in neighborhoods around those points, the second derivative of the income elasticity of the marginal utility of income is less than two. Note that Malinvaud has exactly the right condition for this kind of result. You only get monotonicity where you need it. You can see how this result depends on finiteness. If I had an infinite number of points, my argument fails.

What is the lesson of this analysis? Theoretical results which depend on local properties of demand functions are not refutable on finite data sets.

Let me proceed to our next theorem.

Theorem 2 (Brown and Shannon): *Let* $\langle p^{r}, \{I^{r}_{t}\}^{A}_{t=1}, e^{r}\rangle$, $r = 1,...,n$ *be given. Then there exists a set of smooth, pseudo concave, monotone utility functions rationalizing the data such that for each* $r = 1,...,n$, p^{r} *is an equilibrium price vector for the distribution economy* $\langle \{u_t\}^{A}_{t=1}, \{I^{r}_{t}\}^{A}_{t=1}, e^{r}\rangle$*, such that market excess demands are locally monotone in a neighborhood of* p^{r} *if the following family of inequalities is satisfied:*

(i) *The inequalities in Theorem* 1 *for each agent*

(ii) *Let* $x_j^{r} = -(q_j^{r}/\lambda_j^{r})$. *Then* $\sum_{j=1}^{A} x_j^{r} = e^{r}$ *for* $r = 1,...,n$.

Theorem two is just really an invocation of Malinvaud's theorem. All of the work was done in theorem one. And all of the work in theorem one was done in Quah's theorem. Here we clearly see the implied x's. Finally, we give an analog of Afriat's theorem for distribution economies.

Theorem 3 (Brown and Shannon): *The observed data can be rationalized by agents with smooth characteristics (e.g., Brown–Matzkin) in a distribution economy if and only if it can be rationalized by agents with smooth characteristics such that the observed equilibria are asymptotically stable, and the equilibrium correspondence is locally monotone in the neighborhood of the observed equilibrium price vectors, i.e.,* $\Delta e \cdot \Delta p < 0$ *and the observed equilibrium is locally unique}.*

Theorem 3 says that if you can rationalize the market data in distribution economies with smooth characteristics, which is what Rosa and I did, then you can always rationalize it with smooth characteristics that give rise to local versions of the most important qualitative features of the Walrasian model, i.e., uniqueness, monotone comparative statics and tâtonnement stability. Hence local uniqueness, local monotone comparative statics, and local stability of tâtonnement are not refutable in distribution economies. Only the Walrasian hypothesis H1 is a meaningful theorem in distribution economies.