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Learning, External Benefits, and Methane Generation From Agricultural Wastes

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External benefits from learning by doing exist for methane generation from agricultural wastes. Under a variety of circumstances, these external benefits exceed in magnitude the program costs necessary to induce the experience necessary to make such generation a viable activity.



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# LEARNING, EXTERNAL BENEFITS, AND METHANE GENERATION FROM AGRICULTURAL WASTES

Poultrymen are currently faced with a number of pressing problems not the least of which is energy availability, reliability, and cost. The possibility of on-farm production of methane gas from poultry manure may contribute to the solution of this problem. The engineering feasibility of this activity is not in question -- see for example (Fry and Merrill; Singh; and Smith). At present, however, the economic feasibility of such a commercial operation is doubtful. For example, Slane, et. al. use budgeting procedures to estimate that the net additional annual cost associated with the methane operation for flocks ranging from 20 to 80 thousand birds is roughly five to ten thousand dollars. It is not likely, then, that poultrymen will institute this sort of operation unless induced by a public subsidy or transfer somewhere near the magnitude of this expected additional cost.

As with other relatively new technologies, one can reasonably expect the experience gained in the construction and operation of such methane producing digesters to result in more efficient production in the future. If the information regarding the experience generated by the potential operation were widely disseminated, cost reductions would be expected to be enjoyed by a host of future poultrymen who might take advantage of the learning. These cost reductions (benefits) are, of course, external to the decisions of the poultryman who must decide whether to engage in the methane production activity (and hence provide the experience).

In the absence of some means of internalizing the externality (future cost reductions), private decisions based solely upon internal benefits and costs may be erroneous from a societal viewpoint. One means of internalizing these benefits is by provision of public subsidies. We provide below an economic analysis designed to suggest whether public provision of transfers on the basis of external benefits is likely to be successful in methane production on commercial poultry farms. In this regard, the second section provides an overview of the literature on learning functions and the third develops a measure of external benefits. The subsequent section presents the estimations of both the learning function and the measure of external benefits. Next, the public cost which would be required to induce sufficient experience to make methane generation economically feasible is approximated. Some conclusions are drawn and limitations are suggested in the final section.

LEARNING FUNCTIONS

A generally accepted relation in the learning literature is:

(1) $Z = FW^b$

where Z is a measure of learning (usually unit costs of production), W is a surrogate for accumulated experience, and F and b denote unknown parameters. The relation has been established and applied primarily in standardized production line processes,[1/] where $Z_i$ generally represents unit costs for the $i^{th}$ unit of production and $W_i = i$.

The slope of the learning function, expressed as a percentage, is often represented by $S = 2^b \cdot 100$. A slope of S percent implies a "progress ratio" of (100 - S) percent, such that, e.g., a value of b of -0.32

yields the 80 percent slope familiar to the airframe industry and implies that each doubling of cumulative production brings a 20 percent reduction in unit costs (the progress ratio).

Although learning theory has been applied mainly to production line processes (Fellner; and Rausser, et. al. provide exceptions), it may also apply to somewhat less standardized processes such as methane production from poultry wastes. In this context, a number of alternative proxies for experience are available. For example, Arrow considers cumulative gross investment as the stock of experience which influences factor productivity while Bardhan argues that learning is more dependent upon cumulative volume of industry output. Fellner reasons that in some instances learning is acquired more by "doing it longer" than by "doing more" and hence suggests time as the best surrogate for experience. For methane production, we follow the lead of Rausser, et. al. and Wells who employ cumulative plant capacity as the appropriate proxy. This choice reflects the belief that it is the construction and some (minimum) amount of operations which permits the learning and that after some point (in time and cumulative production) continued production from the same operation provides minimal experience.

## EXTERNAL BENEFITS

Since the learning derived from the experience of a potential methane production operation would directly alter the production functions of a number of succeeding operations elsewhere, an externality is involved. In evaluating the external learning benefits resulting from the experience to be gained from the potential operation, we consider the present value of projected cost reductions as our measure of benefits. If this estimated value ($V_1$) exceeds the value of the public transfer necessary to induce the potential poultryman to produce methane ($V_2$),[2/] then societal net benefits would be associated with internalization of some part of these external benefits by a subsidy less than $V_1$ but greater than $V_2$.

External benefits (B) are, then, defined as the discounted sum over time of unit cost reductions in year t multiplied by the increase in production from new plants in year t multiplied by the expected life of those new plants.

$$(2) \quad B = C \sum_{t=p}^{p+T} \lambda^t (Z_t - Z_t^*) G_t$$

$$= C \sum_{t=p}^{p+T} \lambda^t \left[ FW_t^b - FW_t^{*b} \right] \left[ M(1+g)^t \right]$$

where,

$C = 328.5 \cdot L$,

L denotes the useful life of the digester in years (assumed to be 20 years in the subsequent application),

F, b, $W_t$ and $Z_t$ are as defined above,

$W_t^*$ is ($W_t + \Delta$), where $\Delta$ is the capacity of the proposed digester under consideration in cubic feet per day (cf/d) and $W_t^*$ is therefore cumulative capacity if the potential digester is constructed,

$G_t$ is expected new daily capacity in t (in cf/d) likely to benefit from the learning, M is initial (current) industry-wide daily capacity additions subject to learning,

g denotes expected growth rate in new capacity,

p denotes the number of years by which the realization of benefits from learning is presumed to lag behind the encouragement and provision of experience, and

T is the number of years during which the incremental value of the learning is positive.

For present purposes, $W_t$ is projected by $K(1+g)^t$, where K is 1974 experience, and hence (2) can be rewritten as (2)':

$$(2)' \quad B = 6{,}570 \sum_{t=p}^{p+T} \lambda^t \left[ F\{K(1+g)^t\}^b - F\{K(1+g)^t + \Delta\}^b \right] \left[ M(1+g)^t \right]$$

The multiplication by 328.5 presumes the system is on-line an average of 328.5 days per year (90 percent) and this magnified by the useful digester life of 20 years, produces a constant of 6,570.

LEARNING FUNCTION EMPIRICAL RESULTS

As mentioned previously, the proxy for experience (W) in (1) is cumulative digester capacity. It is unlikely, however, that learning increases proportionally with digester size and there are further indications from research and development areas suggesting that after some point further size increases provide no incremental learning. Accordingly, the case in which experience ($W'$) is proportional to capacity up to a 2,000 cf/d size and none additional is provided by a larger plant is also examined.

As with most new applications, data are scarce. In such instances, Bayesian estimation of the learning function parameters (F and b) is often useful. To recognize the stochastic nature of the learning function, rewrite (1) as:

(3) $Z_i = F\, W_i^b \exp(u_i)$

where $\exp(u_i)$ is the multiplicative log normal disturbance. Transforming this equation into a linear form yields:

(4) $Y_i = a + b\, X_i + u_i$,

where $\ln Z_i = Y_i$, $\ln F = a$, and $\ln W_i = X_i$. Assume further that the $X_i$ are stochastic but distributed independently of $u_i$ and the $u_i$ are independently and identically distributed (iid) with zero mean and unknown variance $\sigma^2$. We can further denote,

(5) $P_0(a, b, \sigma) = K_0\, f_0(a, b, \sigma)$

as the joint prior density function representing information about a, b, and σ obtained from sources other than the sample. In addition,

(6) $\ell(a, b, \sigma \mid Y_1, \ldots, Y_n)$

denotes the likelihood function of a, b, σ given the data. Using Bayes' Theorem, then, the posterior density on a, b, σ is

(7) $P_1(a, b, \sigma \mid Y) = K_1\, f(a, b, \sigma \mid Y)$

$$= K_2\, f_0(a, b, \sigma)\, \ell(a, b, \sigma \mid Y).$$

For some loss function $L(\hat{\delta}, \delta)$, where $\hat{\delta}$ is the estimator and $\delta$ is the parameter, the risk $R(\hat{\delta})$ is defined by

(8) $R(\hat{\delta}) = \int L(\hat{\delta}, \delta)\, P_1(\delta \mid Y)\, d\delta$

A Bayesian estimator minimizes $R(\hat{\delta})$ for a given $P_1(\delta)$. For present analysis a squared error loss function, i.e.

(9) $L(\hat{\delta}, \delta) = (\delta - \hat{\delta})'(\delta - \hat{\delta})$

is used. The Bayesian estimator of $\delta$ will, then, be the expectation of $\delta$ over $P_1(\delta)$.

One form of prior information in this context can be obtained from estimates of b in other learning industries -- see, for example, Alchian, Asher, Hirsh, Fellner, and Rausser, et. al. For example, the prior obtained from these other estimates of b might reasonably take the form of a $\gamma$ density on $-b(= b^*)$ with non-informative, locally uniform priors on a and $\ln \sigma$. It is further assumed *a priori* that a, b, and $\sigma$ are independent.

This particular prior density on b was chosen because it yields a probability measure of zero for values of $b^* \leq 0$ and has a form that agrees with the information derived from non-methane sources. On the basis of this prior information the parameters selected for the density are 1 and 0.3, i.e.,

$$(10) \quad P(b^*) = K\, b^* \exp\left[-\frac{10}{3}(b^*)\right], \quad b^* > 0$$
$$P(b^*) = 0 \qquad\qquad b^* \leq 0$$

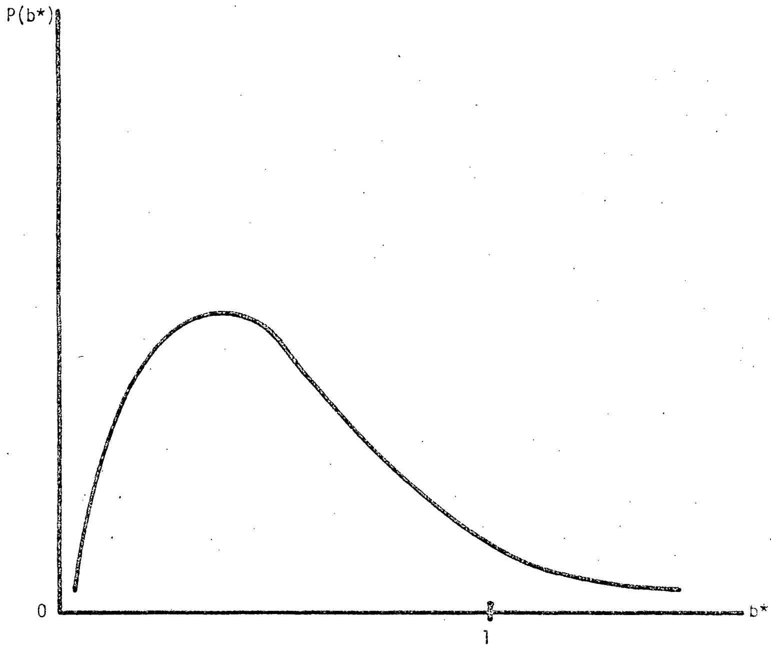
This function gives a probability of approximately 0.85 that $0 < b < 1$. The shape of this $\gamma$ density is illustrated in Figure 1.

The feature that $b < 0$ with a probability of 1 is an example of how restrictions on parameters can be conveniently introduced into Bayesian analysis. We can clearly eliminate $b > 0$, since it implies dislearning.

Given the $\gamma$ prior on $b^*$, the joint prior density[3/] on a, $b^*$, and $\sigma$ is

Figure 1

The Gamma Prior Density



$$(11)\quad P_0(a, b^*, \sigma) = K_3\, \sigma^{-1}\, b^* \exp\left[-\frac{10}{3}(b^*)\right], \quad b^* > 0$$

$$P_0(a, b^*, \sigma) = 0 \qquad\qquad , \quad b^* \leq 0$$

Preliminary estimates for a and b for the cumulative capacity proxy are a = 3.21 and b = -0.422, and for the modified proxy are a = 1.96 and b = -0.388.[4/]

EXTERNAL BENEFITS EMPIRICAL RESULTS

These estimated values of a and b are used to estimate the measure of external benefits (B). Since the posterior density of B requires integrations which are extremely difficult, if not intractable,[5/] the preliminary empirical analysis uses only expected values of a, b, and the other coefficients of (2). Hence, computer solutions to (2) are provided where alternative values of the key parameters (spanning the probable range of uncertainty surrounding these values) are assumed. For example, the values of B are examined where the magnitudes of b under each formulation ranges ten percent in either direction of its estimated value. Likewise, while the expected value of M is 2,500 cf/d, lower and upper limits of 1,000 and 5,000, respectively, were examined. The values of $\Delta$ examined were 200, 1,000 and 2,000 cf/d. These sizes are arbitrary -- a proposed operation of any size can of course be evaluated. The 2,000 cf/d capacity was chosen as the upper limit for our investigation, however, since the second proxy used presumes no incremental learning after this level. The value of K was assumed to be 1,004,615[6/] cf/d and the growth rate (g) was assumed to be .10.[7/] The alternative discount rates (r) of .10 and .15 were employed in the estimation of external learning benefits. Finally, the period of time during which the learning is presumed to be of value (T) was set at forty years. After this period, the combination of moving to the flatter portion of the learning curve and the discounting of future values produces negligible learning benefits. It seems reasonable that some period of time (p) would exist during which the operator of the digester in question gains the experience, is able to disseminate this information, and the new operations are able to capture (internalize) this learning in the form of lower production costs. The selection of p = 3 is judgmental and hence the implications of extreme assumptions (p = 0, 5) were also examined.

Some preliminary results are depicted in Table 1. Using the ten percent discount rate, construction and operation of the largest digester is estimated to generate a stream of external benefits of roughly 2.7 million dollars. This could be as low as approximately 1 million dollars under the least favorable assumptions regarding the parameter M. Assuming a fifteen percent discount rate, the range becomes 565 thousand to 1.4 million dollars. For the smaller (Δ = 1,000 and 200) operations these benefits are, of course, somewhat less. For the most optimistic values of b and M, the estimated benefits are even greater.

PROGRAM COSTS

The empirical results set out above represent estimates of a measure of external benefits only. These benefits further presume that experience will be sufficient to reduce costs to such a point that on-farm production of methane will be a viable operation without further public subsidy. In this section we provide estimates of the amount of experience

Table 1

External Benefits Associated With Various Size (Δ) Methane Generating Units With Learning Lagged Three Years Under Alternative Discount Rates and Learning Parameters for the U. S. (in millions of dollars)

| Experience Proxy | Size of Unit (Δ) | Learning Parameter (b) | Discount Rate (r) and Additions to New Capacity (M) .10 | | .15 | |
|---|---|---|---|---|---|---|
| | | | 1,000 | 2,500 | 1,000 | 2,500 |
| Cumulative Capacity | 2,000 | -0.4215 | 1.0869 | 2.7173 | 0.5645 | 1.4114 |
| | 1,000 | -0.4215 | 1.0503 | 2.6258 | 0.5494 | 1.3735 |
| | 200 | -0.4215 | 0.9156 | 2.2891 | 0.4927 | 1.2317 |
| Modified Cumulative Capacity | 2,000 | -0.3883 | 1.0802 | 2.7005 | 0.5545 | 1.3864 |
| | 1,000 | -0.3883 | 1.0390 | 2.5975 | 0.5375 | 1.3438 |
| | 200 | -0.3883 | 0.8933 | 2.2332 | 0.4762 | 1.1905 |

and public subsidy ($V_2$) which would be anticipated to be required to achieve this result.

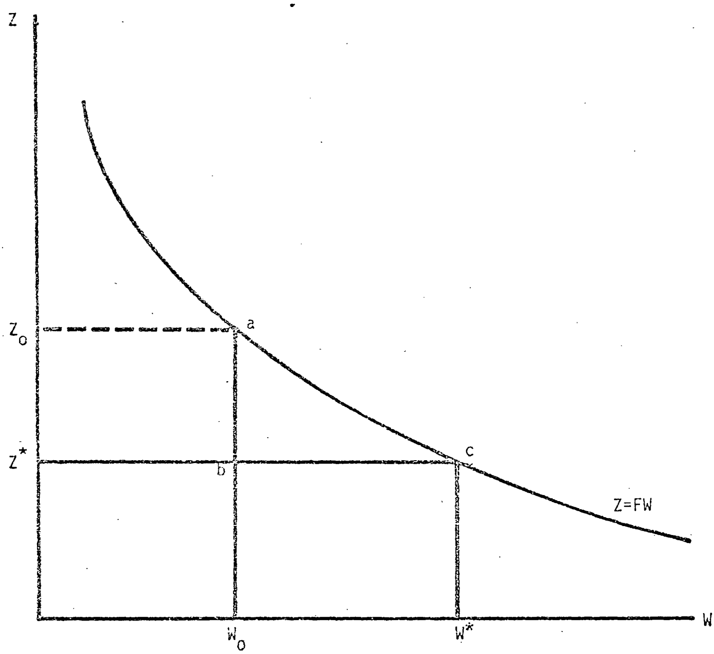
The budgeting analysis which underlies the research reported here suggests a break-even price for commercial electricity of 6.17 cents/kw-hr as compared to the assumed average price of 2.3 cents. Indeed, since the budgeting and estimation was performed, the price of electricity has fluctuated at values somewhat above this average level.

The first step, then, in estimating the subsidy required to generate sufficient learning is to determine the level ($Z^*$) to which methane generation unit costs must fall to make the operation economic. Second, using the estimates of a and b to estimate the level of cumulative experience ($W^*$) corresponding to $Z^*$. The required subsidy ($V_2$) to achieve $W^*$, $Z^*$ then depends upon the magnitude of the area under the learning curve, above $Z^*$, and to the right of present experience. In Figure 2 this corresponds to the area abc, where $W_0$, $Z_0$ denote present methane experience and unit costs, respectively.

Area abc can be considered an extreme (highest) estimate of this magnitude. First, it presumes the current rate of 2.3 cents for commercial electricity will remain into the future until $W^*$ obtains. In practice one would expect this charge to increase over time so that segment bc would be positively sloped to the right such that the intersection $Z^*$ would occur earlier and at a higher unit methane cost. Second, due to uncertainty and to alternative firm and individual goals, one would expect some of this experience ($W^* - W_0$) to be acquired without subsidy. That is, for example, some poultrymen are likely to begin experimentation

Figure 2

Subsidy Required to Make Methane Generation
Economically Feasible



as Z approaches $Z^*$ even though it is not yet economic. For these reasons, then, area abc overstates the quantity $V_2$ and hence serves as a conservative estimate.

Finally, since experience (W) is measured in cubic feet per day capacity, the unit costs in abc must be multiplied by the number of days per year and number of years during which this subsidy is required for these experience-generating operations. Assuming as above that the systems are on-line an average of 328.5 days per year (90 percent), then the present value of the required subsidy ($V_2$) is given as the discounted value of 328.5 (abc) T, where T is the number of years during which the subsidy must be continued. Initially, one could set T at 20 years since this is the useful life presumed above. A more reasonable value is 5 years, since there is substantial evidence that learning becomes negligible after several years of experience.

For the present investigation, $W_0$ is 754,615 cf/d and $Z_0$ is 0.0107 dollars per cubic foot. The highest value of $V_2$ under these assumptions obtains then when the unit methane generation costs are forced to a level equivalent to a reduction in commercial electricity price of 6.17 cents to 2.3 cents, or 62.7 percent [(6.17 - 2.3) ÷ 6.17]. Hence $Z^*$ falls from $Z_0$ to (1 - .627) .0107 = .0040. Using the estimates of a, b of 3.2132 and -.4215, this value of $Z^*$ obtains when $W^*$ = 7.85 million cf/d capacity. The value of $V_2$ in this case is then approximately 32.6 million dollars (assuming a 10 percent discount rate and T = 5). A more reasonable estimate of $V_2$ in light of the reasoning of the previous paragraph would require that costs decline by about half this magnitude.

In this event, the relevant value of $V_2$ is roughly 2.5 million dollars (again, for r = .1 and T = 5). Finally, perhaps the opposite extreme case would be that methane generation costs would need to decline by only ten percent in order to create a self-sustaining flow of experience. This represents the combination of sharply rising future costs of commercial electricity, the anticipation of such on the parts of substantial numbers of poultrymen, and innovative preferences on the parts of others. In this event the value of $V_2$ is roughly 58 thousand dollars. Estimated values of $V_2$ for alternative assumptions within these extremes are tabulated in Table 2 below.

CONCLUSIONS

Estimates in Table 1 suggest that under most combinations of assumptions, the present value of external learning benefits derivable from increasing experience in on-farm methane generation is somewhere in the one to two and one-half million dollar vicinity. Table 2 provides a far greater range of possible program costs depending on a variety of assumptions. Under what is considered the most realistic situation -- viz., unit methane costs would have to fall by only one-half of the 62.7 percent required to make methane generation economically equivalent to commercial electricity sources -- with poultry operators subsidized an average of five years, the estimated present value of program costs is in the 2.5 million dollar range. These estimates are extremely sensitive to assumptions made regarding future commercial energy costs (as reflected in the "percent reduction in Z needed" assumptions). Clearly, movement of energy costs toward the 6.17 cents/kw-hr break-even level reduces

Table 2

Estimated Values of Break-even Costs ($Z^*$), Required Experience ($W^*$) and Present Value of Required Subsidy ($V_2$)

| Experience Proxy | Percent* Reduction of Z Needed | Break-even Unit Methane Costs ($Z^*$) in $/cf | Required Experience ($W^*$) in $10^6$ cf/d Capacity | Required Subsidy ($V_2$) in Millions of Dollars for Alternative Discount Rates and Payment Periods** .10, 5 | .10, 20 | .15, 5 | .15, 20 |
|---|---|---|---|---|---|---|---|
| Cumulative Capacity | 100 | .0040 | 7.8467 | 32.5996 | 73.2141 | 30.1379 | 56.2752 |
| | 70 | .0060 | 2.9760 | 7.1476 | 16.0526 | 6.6079 | 12.3386 |
| | 50 | .0073 | 1.8437 | 2.5030 | 5.6213 | 2.3140 | 4.3208 |
| | 30 | .0087 | 1.2330 | 0.6666 | 1.4972 | 0.6163 | 1.1508 |
| | 10 | .0100 | 0.8804 | 0.0578 | 0.1299 | 0.0535 | 0.0998 |
| Modified Cumulative Capacity | 100 | .0040 | 8.5204 | 36.0882 | 81.0490 | 33.3630 | 62.2973 |
| | 70 | .0060 | 2.9744 | 7.4170 | 16.6576 | 6.8569 | 12.8036 |
| | 50 | .0073 | 1.7688 | 2.5269 | 5.6750 | 2.3361 | 4.3621 |
| | 30 | .0087 | 1.1480 | 0.6601 | 1.4825 | 0.6102 | 1.1395 |
| | 10 | .0100 | 0.7929 | 0.0568 | 0.1276 | 0.0525 | 0.0981 |

*The 100 percent figure provides the extreme value of $V_2$ discussed in the text. For reasons expressed above, the 50 percent value is considered the most likely.

**The discount rates (r) are the .10 and .15 values and the payment periods (T) considered are 5 and 20 years.

program costs significantly. For example, if commercial electricity prices were projected to rise to within 30 percent of the difference between the current 2.3 cents/kw-hr and the break-even 6.17 cents/kw-hr (Table 2), so that the break-even unit methane cost ($Z^*$) would be .0087 dollars per cubic foot as compared with the present cost of .0107 per cubic foot, then the program costs would be in the 600 to 667 thousand dollar range.

In short, these results suggest that an R and D program of this sort is economically justified for the situation in which present energy supply prices are expected to grow at a moderate to rapid pace, but not for the situation where these prices are unlikely to increase substantially.

In more general terms, the poultryman's decision based on internal net benefits alone is not favorable to activities associated with on-farm generation of methane. If externalities are recognized, however, public decision-makers also have a responsibility. Unfortunately, such "yes-no" (subsidy) decisions are generally made without a useful framework for analysis. It is felt that the framework provided here can be of use in a rather broad range of public decision-making contexts.

To be sure, the analysis has been performed on the basis of currently available (and limited) data. Since the estimates of external benefits B obviously depend critically on the estimates of the learning parameter b (and other values), the operational use of this measure in such public decisions requires the estimates of b (and the other parameters) to be modified as additional information unfolds.

Finally, a factor leading to a possible overstatement of benefits is inherent in the learning function formulation. When cost reductions are related to a measure of experience, a part of those cost reductions must be attributed to non-methane generation research and development, such as improved pumps, new digester technology, microbiology technology, new displacement technology, etc. To some extent this may be offset by the omission of by-product benefits or new technologies made in the methane generating industry that are disseminated outside the industry.

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APPENDIX

Recall the $\gamma$ density on $b^*$, (= -b) may be specified in terms of the joint prior on a, b, and $\sigma$ as:

$$P_0(a, b, \sigma) = K_5 \left(\frac{1}{\sigma}\right)\left(b^*\right) \exp\left[-\frac{10}{3}(b^*)\right]; \; b^* > 0$$

$$P_0(a, b, \sigma) = 0 \qquad ; \; b^* \leq 0$$

By the use of the sample likelihood function (presumed multivariate normal), the posterior density function for this case is:

$$P_1(a, b, \sigma \mid X_2, Y_2) = K_5 \left(\frac{1}{\sigma}\right)^{n_2+1} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n_2} \cdot \left(Y_{2i} - a - b\, X_{2i}\right)^2\right]$$

$$(b^*) \exp\left[-\frac{10}{3}(b^*)\right]; \; b < 0,$$

$$P_1(a, b, \sigma \mid X_2, Y_2) = 0; \; b \geq 0.$$

If we integrate out $\sigma$, we have

$$\int_0^\infty P_1(a, b, \sigma \mid X_2, Y_2) = K_6\, (b^*) \exp\left[-\frac{10}{3}(b^*)\right] \int_0^\infty \left(\frac{1}{\sigma}\right)^{n_2+1} \cdot$$

$$\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n_2} u_i^2\right) d\sigma$$

$$= K_7\, (b^*) \exp\left[-\frac{10}{3}(b^*)\right] \left(1 + \frac{(\beta - \hat{\beta}_2)' X_2' X_2 (\beta - \hat{\beta}_2)}{e_2' e_2}\right)^{-n_2/2} \cdot$$

Further, we can integrate out a analytically since the factor containing it is of the multivariate t form. (It can be shown that the marginal distribution for a random variable from a multivariate t is a univariate t.) That is,

$$P_1(b \mid X_2, Y_2) = \int_{-\infty}^{\infty} P_1(a, b \mid X_2, Y_2)\, da$$

$$= K_7 \,(b^*) \exp\left[-\frac{10}{3}(b^*)\right] \cdot$$

$$\left(1 + \frac{\left(\sum X_{2i}^{\;2} - n_2 \overline{X}_2^2\right)\left(b - \hat{b}_2\right)^2}{e_2' e_2}\right)^{-[(n_2-1)/2]}$$

where $\overline{X}_2 = \sum_{i=1}^{n_2} X_{2i} \div n_2$ and $\hat{b}_2$ is the second element of $\hat{\beta}_2$. This expression for $P_1(b \mid X_2, Y_2)$ can be integrated numerically to obtain $\overline{b}$ and var (b).

## Footnotes

1/ Alchian and Asher review much of the work in the learning function literature.

2/ The quantity $V_2$ is discussed in a separate section below.

3/ The posterior density function is set out in the Appendix.

4/ Refer to Slane, et. al. for the data used. Additional data are currently being compiled for this Bayesian estimation.

5/ The derivations of this posterior density function are available upon request.

6/ This figure is simply the sum of the cumulative system capacity associated with the 1972 system (754,615 cubic feet per day) and an estimated addition to cumulative system capacity from 1972 through 1973 of 250,000 cubic feet per day.

7/ This rate represents a best subjective judgment. It implies, for example, that the current capacity presumed, subject to learning, of 2,500 cubic feet per day would grow to 113,125 cubic feet per day in a period of 40 years.