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Mathematics of Seasonal Price Behavior

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ABSTRACT

Differential equations were used to derive seasonal price patterns for Korean rice under two behavioral assumptions ("market conduct"). These were a competitive price rise equal to storage cost and the joint maximization of crop value plus storage profit. Compared with the competitive model, maximization virtually eliminates private storage and produces wider price fluctuations.

Key words: seasonality, intertemporal equilibrium, pricing, storage, differential equations, calculus of variations.

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Mathematics of Seasonal Price

Behavior*

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Patterns of seasonal price and storage behavior result from the behavioral conduct in a subsector. The purpose of this paper is to demonstrate the effects of two alternative behavioral assumptions on price and storage behavior and to elucidate the mathematics of the solution procedure so that others may use this method to analyze intertemporal behavior in other markets. The empirical analysis studies the market for rice in Korea using data which was available to me in Seoul.

Statement of the Problem

A storable commodity is subject to seasonal pattern of production and relatively stable demand throughout the year. The cost of inventory and the value of the inventory. What are the seasonal price and inventory patterns implied by alternative kinds of

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market conduct (alternative behavioral assumptions)? Specifically, if storage is determined so as to maximize the value of production plus storage profits, how does the seasonal price pattern differ from that when the competitive prices are allowed to rise just enough to cover the variable costs of storage?

The Model

Behavioral Assumptions

Maximizing Assumption: Private storage levels are managed so as to maximize the value of production plus the storage return in the private sector.

Competitive Market Assumption: Market prices rise by the amount of the average variable cost of storage [4, 12].

Structural Equations

Private Storage Level

$$S(t) = S(0) + \int_0^t (H(y) + G(y) - D(y))\, dy$$

Inventory Cost

$$IC(t) = r\, P(t)\, S(t) + k\, S(t)$$

Demand Equation

$$P(t) = a - b\,(D(t) + G(t))$$

Storage Profit Level

$$\pi(t) = \pi(0) + \int_0^t [P(y)\,(-S'(y)) - IC(y)]\, dy$$

Farm Harvest Rate

$$H(t) = (1-L)H_0 \frac{256}{6}\, t^3 e^{-4t}$$

Government Sales Rate

$$G(t) = RP\,(P(t) - PT)$$

Variables and Parameters

Variables at time "t"

$P(t)$ = Price	won/MT*
$D(t)$ = Private sales rate to consumers	MT/Month
$S(t)$ = Private storage level	MT
$IC(t)$ = Inventory cost rate	won/Month
$\pi(t)$ = Accumulated profit level	won
$H(t)$ = Harvest rate	MT/Month
$G(t)$ = Government sales rate	MT/Month

Parameters

r = Interest rate	1.5	Percent/Month
k = Warehousing costs	460	(Won/MT)/Month
a = Price intercept	875000	Won
b = Price slope	1.875	Won
H_0 = Level of farm harvest	4344000	MT/year
L = Pre-consumption loss rate	22	Percent
World price, c.i.f. Korea	$500.00	Per MT
Exchange rate	400	Won per U.S. Dollar
PT = Government price target	16350	Won per 80 kilograms
RP = Government purchase response parameter	20,25	(MT/Month)/(Won/MT)

*Won per metric ton. During most of 1974 the official exchange rate was approximately 400 won per dollar. On December 7, 1974, the won was devalued to 480 per dollar. Sources: [5, 9, 10, 11].

Solution to the Model

The derivation and solution of the differential equation for the model assuming maximization of value of production plus storage profits is described below. The equation in the competitive case is obtained by assumption, rather than derivation, but its manner of solution parallels the maximizing case.

Maximizing Model

This model seeks to determine the storage and price equations which maximize production value plus storage returns.

$$\max \int_{0}^{12} [\, P(y)\, H(y) - P\,(y) S'(y) - rP(y) S(y) - kS(y)]\, dy$$

The Calculus of Variations [6] states that the Euler Equation

$$\frac{\partial}{\partial y} f(x, y, y') - \frac{d}{dx} \frac{\partial}{\partial y} f(x,y,y') = 0$$

is a necessary condition to maximize an integral of the form $\int_a^b f(x,y,y')dx$.

First, we must verify that the objective function is indeed of the form $\int_a^b f(t, s, s')\, dt$ --where t is time and s(x) is storage as a function of time. To do this, price levels (P) need to be related to private storage levels (S), rates of change of storage (S') and exogenous functions of time, like harvest (H). The demand equation, private storage identity and government behavior equation imply that price is given by

$$P = \frac{a + 2\, b\, RP\, PT}{1 + 2\, b\, RP} - \frac{b}{1 + 2\, b\, RP} (H - S')$$, or in simpler

notation, $P = c_0 - c_1 (H - S')$. From this, the integrand of our functional is $f(t, S, S') = [c_0 - c_1 (H - S')]\, [H - S' - r\, S] - k\, S$

Thus the Euler equation defines the following differential equation

$$S'' = r\, c_1\, H - 2\, c_1\, H' - r\, c_0 - k$$

which is a nonhomogeneous, second order differential equation with constant coefficients. The solution to this equation is the sum of a particula solution to the nonhomogeneous equation plus the general solution to the homogeneous part of the equation.

The general solution to the homogeneous part of the equation ($S'' = 0$) is $S(t) = S_0 + S_1\, t$. For $S'' = -\, r\, c_0 + k$, a particular solution of the nonhomogeneous equation is $S(t) = -\, 1/2(r\, c_0 + k)\, t^2$.

To solve $S'' = c_1\, (r\, H - 2\, H')$, we derive $H'(t)$ from the definition of $H(t)$ and apply the variation of parameters [8, p. 72] solution method. For this problem, the equation is

$$S'' = c_1\, H_0\, \frac{(4)^4}{3!}\, (r\, t^3 - 6\, t^2 + 8\, t^2)\, e^{-4t}$$

Under the variation of parameters method, a plausible function is specified, differentiated and substituted into the equation to equate coefficients term by term.

$$S(t) = (a_0 + a_1\, t + a_2\, t^2 + a_3\, t^3)\, e^{-4t} \text{ implies that}$$

$$S''(t) = [(2\, a_2 - 8\, a_1 + 16\, a_0) + (6\, a_3 - 16\, a_2 + 16\, a_1)\, t$$

$$+ (-24\, a_3 + 16\, a_2)\, t^2 + 16\, a_3\, t^3]\, e^{-4t}$$

Equating the coefficients of the differential equation term by term,

$$16\,a_3 = c_1 H_0 \frac{(4)^4}{3!}(r+8) \qquad \text{or } a_3 = c_1 H_0 (r+8)\,8/3$$

$$16\,a_2 - 24a_3 = (4)^4 c_1 H_0 \qquad \text{or } a_2 = c_1 H_0 (4r+16)$$

$$16\,a_1 - 16\,a_2 + + 6\,a_3 = 0 \qquad \text{implies } a_1 = c_1 H_0 (3\,r+8)$$

$$16\,a_0 - 8\,a_1 + 2\,a_2 = 0 \qquad \text{implies } a_0 = c_1 H_0 (r+2)$$

So this part of the particular solution is

$$S(t) = c_1 H_0 e^{-4t}\left[(r+2) + (3r+8)t + (4r+16)t^2 + (8r+64)t^3/3\right]$$

Thus the equilibrium storage level is given by the sum of the general solution and each particular solution to the differential equation:

$$S(t) = S_0 + S_1 t - 1/2(rc_0 + k)\,t^2 + c_1 H_0 e^{-4t}\left[(r+2) + (3r+8)\,t + (4r+16)t^2 + (8r+64)t^3/3\right].$$

The constants S_0 and S_1 depend upon the initial conditions of the system. In this paper the solution is illustrated with equal carry-in and carry-out levels, which are as small as possible consistent with positive inventory.

Specifing that carry-in and carry-out levels are equal lets us determine S_1. $S(0) = S(12)$ implies

$$S_0 + c_1 H_0 (r+2) = S_0 + 12\,S_1 - 72(r\,c_0 + k) + c_1 H_0 e^{-48}(5221\,r + 39268)$$

$$\text{or } S_1 = 6(r\,c_0 + K) + c_1 H_0 (r+2)/12$$

since the last term is less than 10^{-10}

Setting S(0) equal to the carryover, we obtain

$$S_0 + c_1 H_0 (r+2) = \text{Carryover} \quad \text{or}$$

$$S_0 = - c_1 H_0 (r+2) + \text{Carryover}$$

The minimum necessary carryover must be numerically determined because the nonlinearities of the storage derivative and the multiple local optima of the storage function frustrate direct solution. The method I use equates S_0 with $- c_1 H_0 (r+2)$ and scans the storage function over the range [0, 12) to determine the minimum. This minimum (a negative number) is then subtracted from the storage function to result in the desired non-negativity property. 1/

The equilibrium price path is calculated from the harvest and storage derivative equations and is

$$P(t) = c_0 + c_1 S_1 - c_1 (rc_0 + k)\, t - rc_1{}^2 H_0 e^{-4} (1+4t+8t^2+32t^3/3)$$
$$-(2c_1{}^2 + c_1) \frac{(4)^4}{3!} H_0\, t^3 e^{-4t}$$

Competitive Model

The differential equation for the competitive model states that the price rise offsets the average cost of storage:

1/ The numerically calculated carryover is slightly more than one-half the absolute value of C.Ho (r+2). In the competitive case which follows, the carryover is much less than the absolute value of the lgebraic expression which preceeds it in the definition of S_o.

$$P'(t) = IC(t)/S(t) = r\,P(t) + k$$

This can be solved as is, but since the initial conditions are on storage levels, expressing and solving the equation for storage, rather than price is desirable.

$$S'' - r\,S' = H' - r\,H + (r\,c_0 + k)/c_1$$

c_0 and c_1 are the same as above.

Precisely the same mathematical processes are followed in this case as were used for the maximizing version of the model. So, the reader may verify that the solution here is

$$S(t) = S_0 + S_1\,e^{rt} + \frac{(r\,c_0 + k)}{r\,c_1}\,t - H_0\,e^{-4t}\,(1 + 4t + 8t^2 + 32t^3/3)$$

and

$$P(t) = r\,c_1\,S_1\,e^{rt} - k/r$$

The initial conditions S(0) = S(12) determine these values for the S_0 and S_1 constants:

$$S_0 = (H_0 e^{12r} - 12\frac{(r\,c_0 + k)}{r\,c_1} - 19633\,H_0\,e^{-48})\,/(e^{12r} - 1) +$$

$$S_1 = (12\frac{(r\,c_0 + k)}{r\,c_1} - H_0 + 19633\,H_0 e^{-48})\,/\,(e^{12r} - 1)$$

The two-step process which calculates the minimum working carryover is also used in the competitive version of the model.

Empirical Results

The numerical results for four runs of the model are presented in Table 1. These correspond to two levels of government intervention each under the competitive and maximizing version of the model.

The intrayear patterns of storage, price and consumption levels are presented in Figures 1 through 4.

The differences between the alternative behavioral assumptions are striking. Adequate private storage occurs when prices rise to offset the cost of storage. However, when producers maximize the value of production plus storage profits virtually no private storage (other than pipeline stocks) results--contrary to prior expectations under this hypothesis.

The lack of private storage demand under the maximizing assumption causes large quantities of grain to flow directly into consumption, depressing prices and farm income. The effect on aggregate consumption expenditure is minimal although the effects on the time paths of prices and consumption are dramatic.

The private and government sector cash flows determine the credit requirements of each and the profit at the end of the year. A quirk in the accounting in this model charges the full cost of the carryover against the private sector. For this reason the yearend balance of the private sector is negative.

Some Aspects of Control

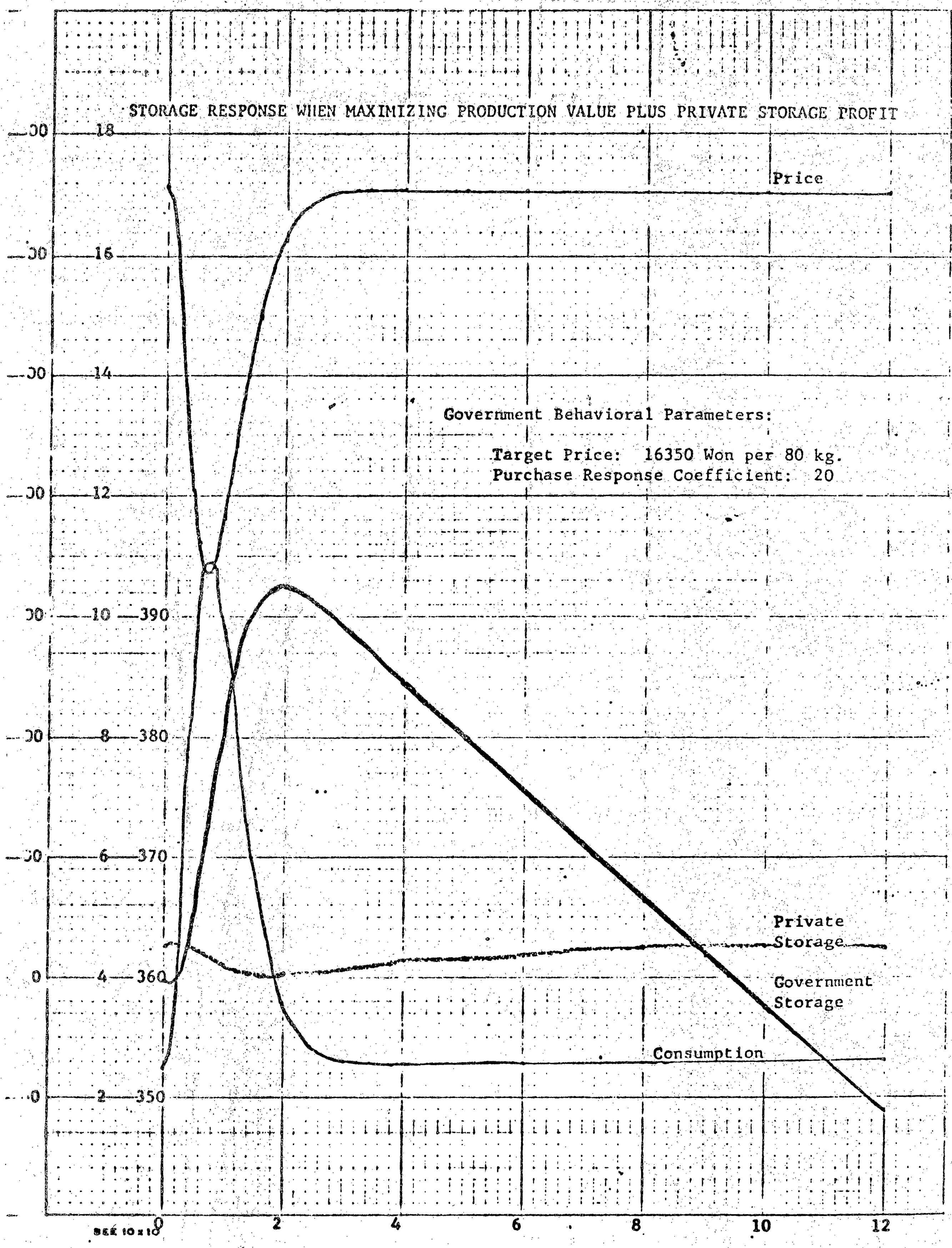
The purpose of public intervention in any market is to change the performance in that market, whether measured by price levels, price patterns, income, marketing costs, or whatever. One means of effecting these desired changes is the rule governing public purchase and storage behavior. In this paper a rule in which public purchases were proportional to prices was analyzed.

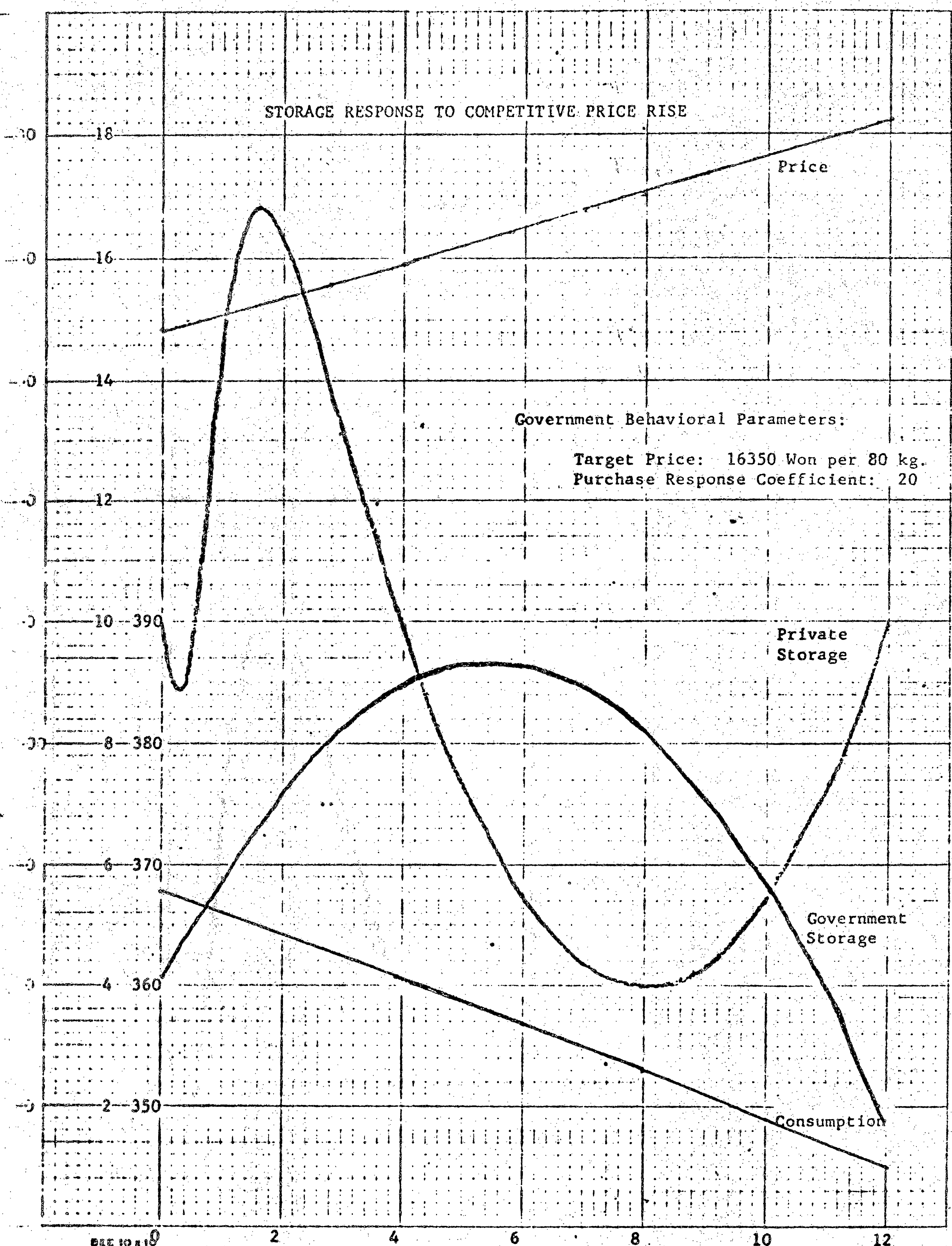
Table 1 showed that the greater the governmental response speed, the less the resulting variation in market prices. In the maximizing model, a 25 percent increase in the speed of government response reduced the range of price variation from 57.5 to 41.4 percent of the minimum price. The effect under the competitive assumption is similar, but less dramatic (23.0 to 22.9 percent).

Control mechanisms can be proportional to the controlled variable (in this case, price), proportional to the the change in (derivative of) that variable, or proportional to the integral of the difference between the controlled variable and its targeted time path.

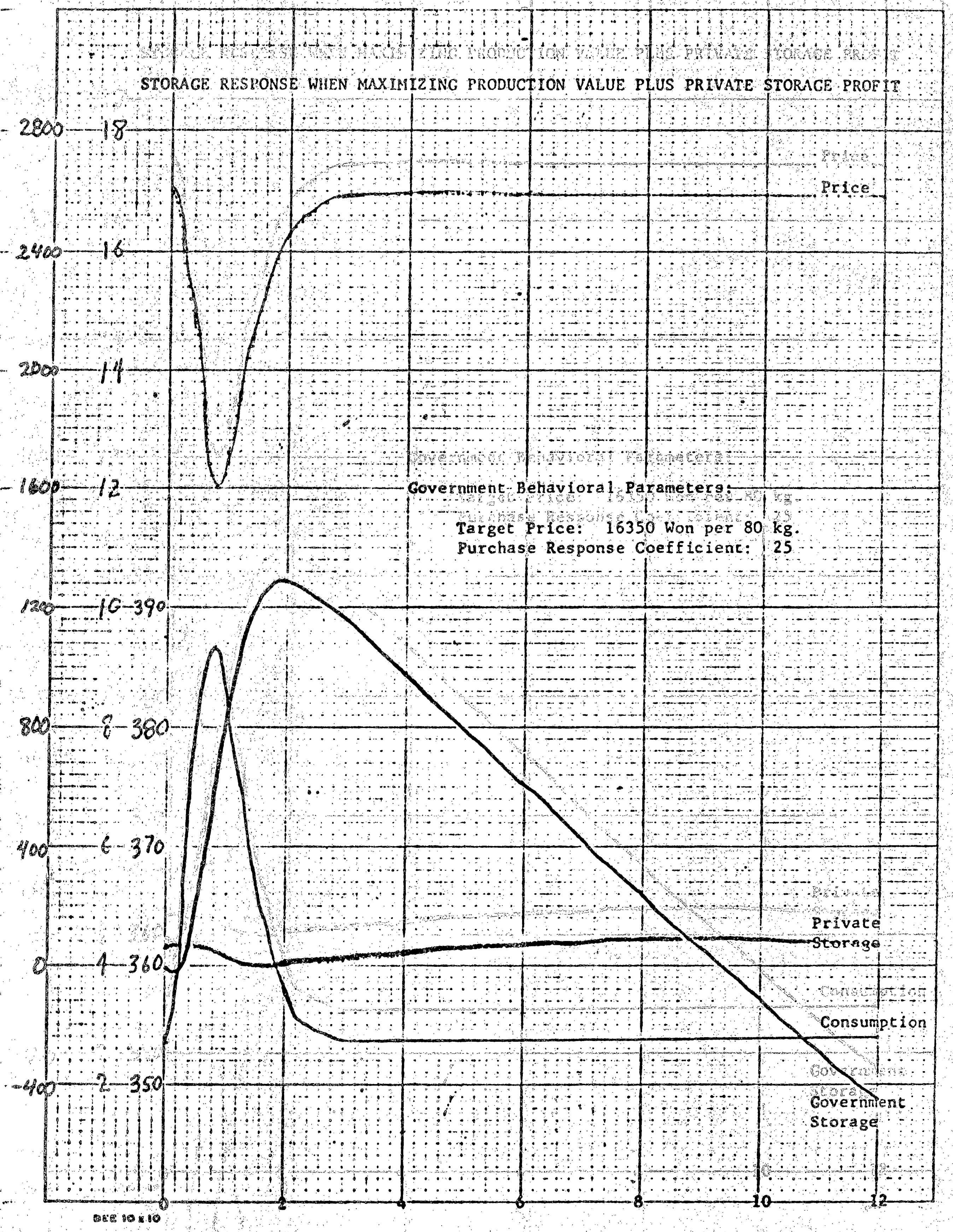
As shown here, proportional rules are only partially effective in stabilizing prices at the targetted levels. Adding an integral controller which in this case would be proportional to storage levels, would achieve the targeted prices at the expense of increasing the seasonal price oscillation (possibly to an explosive extent). To counteract the added oscillation, a derivative control mechanism must also be added. [2, p. 278 paraphrased]. A general governmental purchase rule can thus be formulated and parameterized from these types of controllers to achieve whatever price, consumption, or storage patterns are desired.

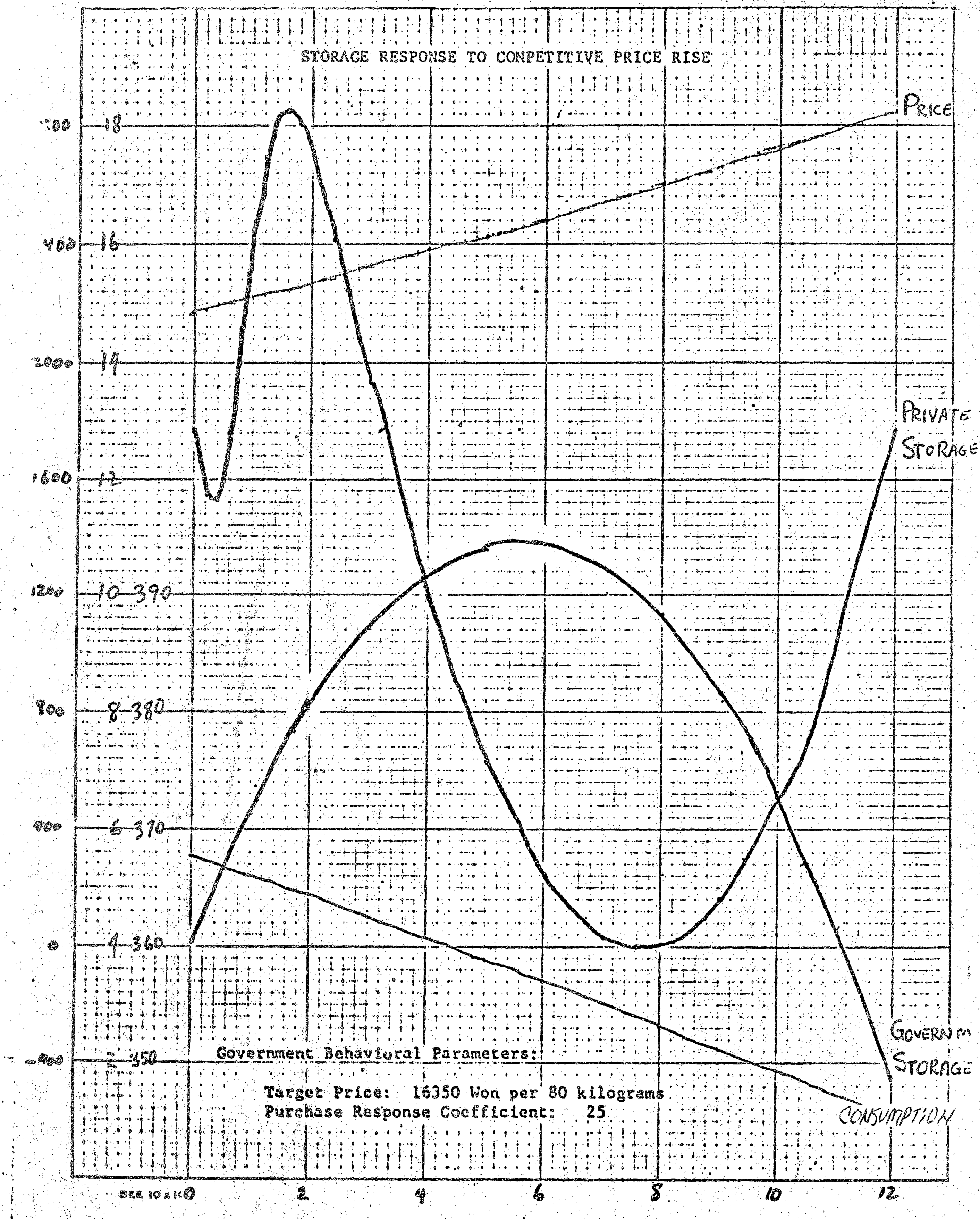
Other storage policies could be derived by assuming that the price rise in the market is zero, or some other percentage, and the storage pattern necessary to achieve it analytically derived from the model. In these situations it would be necessary to deal with total storage, rather than private and government separately.


STORAGE RESPONSE WHEN MAXIMIZING PRODUCTION VALUE PLUS PRIVATE STORAGE PROFIT
Price
Government Behavioral Parameters:
Target Price: 16350 Won per 80 kg.
Purchase Response Coefficient: 20
Private Storage
Government Storage
Consumption
18
16
14
12
10
8
6
4
2
390
380
370
360
350
0
2
4
6
8
10
12



STORAGE RESPONSE TO COMPETITIVE PRICE RISE
Price
Government Behavioral Parameters:
Target Price: 16350 Won per 80 kg.
Purchase Response Coefficient: 20
Private Storage
Government Storage
Consumption
18
16
14
12
10 390
8 380
6 370
4 360
2 350
0
2
4
6
8
10
12


STORAGE RESPONSE WHEN MAXIMIZING PRODUCTION VALUE PLUS PRIVATE STORAGE PROFIT
2800
18
2400
16
2000
14
1600
12
1200
10
390
800
8
380
400
6
370
0
4
360
-400
2
350
Price
Government Behavioral Parameters:
Target Price: 16350 Won per 80 kg.
Purchase Response Coefficient: 25
Private Storage
Consumption
Government Storage
0
2
4
6
8
10
12



STORAGE RESPONSE TO COMPETITIVE PRICE RISE
PRICE
PRIVATE STORAGE
GOVERNM STORAGE
CONSUMPTION
Government Behavioral Parameters:
Target Price: 16350 Won per 80 kilograms
Purchase Response Coefficient: 25
400
2000
1600
1200
800
0
18
16
14
12
10 390
8 380
6 370
4 360
350
0
2
4
6
8
10
12

Effects of Alternative Behavioral Assumptions

on Storage and Price Response 1/

	Unit	Competitive Market Behavior Assumption		Maximize Value of Production Plus Storage Returns	
Gov. Purchase Response					
Parameter	2/	20	25	20	25
Target Price	Won/80kg	16350	16350	16350	16350
Imports	1000 MT	446.0	447.1	445.8	446.9
Import Timing	months	10.9	11.1	9.4	9.3
Foreign Exchange	$. mil.	223.0	223.55	222.9	223.45
Max. Gov. Inventory	1000 MT	1062.	1377	1296.	1286
Max. Pvt. Inventory	1000 MT	2559.	2852	106.1	87.16
Carryover	1000 MT	1197.	1777	97.54	71.83
Gov. Stor. Credit					
Reqt. 3/	Bil. Won	217.7	282.3	205.7	218.0
Pvt. Stor. Credit					
Reqt. 3/	Bil. Won	267.0	214.1	10.53	8.784
Total Stor. Credit					
Reqt. 3/	Bil. Won	374.0	346.0	191.8	297.3

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	Units	Competitive Market Behavior Assumption		Maximize Value of Production Plus Storage Returns	
Consumer Expenditure	Bil. Won	881.7	880.6	880.5	880.2
Gov. Storage Costs	Bil. Won	27.39	36.16	22.56	22.37
Pvt. Storage Costs	Bil. Won	36.36	638.6	3.067	2.540
Gov. Storage Profit 3/	Bil. Won	9.397	8.609	58.75	38.67
Pvt. Storage Profit	Bil. Won	-51.11	-75.76	-9.424	-6.335
Value of Production	Bil. Won	639.8	636.7	539.3	570.8
Max. Cons. Rate	1000MT/Mo.	367.7	367.9	394.2	386.7
Min. Cons. Rate	1000MT/Mo.	345.0	345.2	352.5	353.5
Min. Price Level	Won/80kg	14840	14820	10870	12000
Max. Price Level	Won/80kg	18260	18220	17120	16970

1/ Assumes demand elasticity is 0.4 at the point P = 20000 won/80kg and Q = 333333MT/Mo. Price (in Won/MT) = 875000. - 1.875 (D + G).

2/ For every 1000 Won/MT the market price is less than the target price that the government buys (this parameter times 1000) metric tons of grain per month.

3/ Excludes the cost of storing imports.

References

1. Allen, R. G. D., Macroeconomic Theory; New York: St. Martins Press, 1968
2. Allen R. G. D., Mathematical Economics, New York: St. Martins Press, 1959
3. Bellman, Richard, Modern Elementary Differential Equations, Reading Addison and Wesley Publishing Company, 1968.
4. Bressler, R. G. Jr. and R. A. King, Markets, Prices and Interregional Trade; New York: John Wiley & Sons, 1970.
5. Economic Planning Board, Korea Statistical Yearbook, 1972; Seoul: R.O.K. E.P.B., September 1972.
6. Gelfand, I.M. and S. V. Fomin Calculus of Variations; Englewood Cliffs: Prentical Hall Inc., 1963.
7. Gustafson, R. L. Carryover Levels for Grains: USDA Tech. Bull. 1178 October, 1958.
8. Kreyszig, Erwin, Advanced Engineering Mathematics; New York: John Wiley and Sons, 1962.
9. Ministry of Agriculture and Forestry. Yearbook of Agriculture and Forestry Statistics, 1972; Seoul: R.O.K., M.A.F., December 1972.
10. Ministry of Agriculture and Forestry. Yearbook of Agriculture and Forestry Ststistics - Grain Statistics, 1971-72, Seoul; R.O.K. MAF., November 1972.
11. Rossmiller, G. E. etal. Korean Agricultural Sector Analysis and Recommended Developmental Strategies 1971-1985; East Lansing, Michigan State University, 1972.
12. Samuelson, P.A., Intertemporal Price Equilibrium: A prologue to Theory of Speculation, Collected Scientific Papers; Cambridge

MIT Press, 1966, pp. 946-984.

13. Sharples, J.A. and R. L. Walker, Reserve Stocks of Grain Research Status Report, Comm. Econ. Div. ERS, 1975.

14. Takayama, T. and G. G. Judge. Spatial and Temporal Price and Allocation Models; Amsterdam: North-Holland Publishing Company, 1971.

15. Teigen, L. D. A Model for Private Storage Behavior under Competition and Monopoly with an Application to Korean Rice Storage KASS Working Paper 73-5; Seoul: Agricultural Economics Research Institute, December, 1973.

16. Teigen, L. D. An Intertemporal Price Equilibrium Model for Korean Rice KASS Working Paper 74-74; Seoul: National Agricultural Economics Research Institute, December 1974.

17. Tweeten, L.,D. Kalbfleisch, and Y.C. Lu An Economic Analysis of Carryover Policies for the U.S. Wheat Industry; Stillwater: O.S.U. Tech. Bull. T-132, October 1971.