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PRODUCTION FUNCTION, COST OF PRODUCTION AND ASSOCIATED OPTIMALITY LINKAGES: A TEXT BOOK SUPPLEMENT

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ABSTRACT

The elementary algebra and geometry of the intimate relationship between the production function and cost curves, and between the associated input and output optimality conditions, is developed for the single- and two-variable factor case. The presentation is intended to supplement a major deficiency of most textbook treatments.

July '75

PRODUCTION FUNCTION, COST OF PRODUCTION AND ASSOCIATED OPTIMALITY LINKAGES: A TEXT BOOK SUPPLEMENT*

Bruce R. Beattie and Wade L. Griffin**

One of the most important concepts of production economics, viz, the linkage of costs of production (expressed in terms of output) to the production function is poorly treated in most intermediate level micro-theory and production economics texts. In several texts the linkage is so sketchy it is of limited pedagogical value (e.g., Ferguson; Heady; Henderson and Quandt; Leftwich). (The best available treatments are provided by Doll, Rhodes and West and Buse and Bromley, both of which are introductory texts.) Consequently many students not only do not understand this linkage and the resulting consistency of the input profit-maximizing conditions and the output profit-maximizing conditions, but they also become confused as to the meaning of and relationship between marginal revenue and marginal value productivity and between marginal cost and marginal factor cost.

It is the purpose of this paper to lay out in geometric and elementary mathematical terms the connection between these concepts for the single product case. We procede first with the mathematical and geometric development in terms of a single variable factor. The analysis is then extended to the two-factor case using a Cobb-Douglas example. We make no pretense here of originality of concept. It is rather our purpose to contribute a consistent and clearly developed framework that will be useful to production economics teachers and students.

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Notation

For this purpose the following notation is introduced:

Y = product (output)

P = product price

X_i = ith factor (input)

r_i = price of ith factor

TPP = total physical product

MPP_i = marginal physical productivity of the ith factor

TVP = total value product

AVP_i = average value productivity of the ith factor

MVP_i = marginal value productivity of the ith factor

C = cost expressed in terms of factors and factor prices, i.e.,

$$C = \sum_i r_i X_i$$

MFC_i = marginal factor cost of the ith factor

TC = cost expressed in terms of output[1]

AC = average cost

MC = marginal cost

TR = total revenue

MR = marginal revenue

π = profit (may be expressed either in terms of inputs, output, or both when constrained by the production function)

$i = 1, 2$

Single Factor Case

This section is organized into three parts. First, the elementary algebra of getting from the factor side (production function) to the

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[1]Throughout the paper fixed costs are ignored for simplicity.

output side (cost curves) is developed. This is followed by a geometric interpretation and a discussion of additional relevant linkages such as optimality conditions, etc. The section concludes with a discussion of the mechanics of how to most effectively present the material to students. For simplicity, perfect competition in all product and factor markets is assumed.

Algebra

For the single factor case the algebra of deriving the cost function from the production function is simple and direct. Given the production function

(1) $Y = f(X)$

and the cost equation

(2) $C = rX$

it follows immediately that costs may be expressed as a function of the product by solving the production function for X in terms of Y (taking an inverse),

(3) $X = g(Y)$

and substituting into the cost equation. Thus, we obtain

(4) $TC = C = r\,g(Y)$

A simple algebraic example is provided by the power production function,

(1a) $Y = 2x^{\frac{1}{2}}$

Taking the inverse we obtain

(3a) $X = \frac{Y^2}{4}$

Assuming an input price of 4, cost in terms of product is

(4a) $TC = 4\,\frac{Y^2}{4} = Y^2$

By plotting 1a and 4a the student can verify for himself the "mirror-image" phenomenon so often referred to but not often communicated in our teaching.

Geometry

The geometry of this basic linkage is presented in Figure 1. In panel a is the production function (corresponding to eq. 1); beneath it (panel b) is the cost equation (eq. 2). In panel a' is a 45° line permitting the translation of output from the vertical to the horizontal axis; beneath it (panel b') is the resulting cost-product relationship (eq. 4) obtained by "closing the circle" and connecting the locus of intersection points. From this kind of visual presentation the student can immediately see the correspondence of inflection, ray-line tangency and maximum output points between the production function and the total cost curve; these points are noted by the connecting lines in Figure 1. Also the implication of a three-stage production function for the shape of cost curves is made clear, including the often omitted Stage III as noted by Tangri.

By adding TVP to panel b and TR to panel b' (Figure 2), the consistency of optimal levels of Y, $\overline{Y}$, and X, $\overline{X}$, is seen when viewed from either the factor or product sides (see dashed lines on panels a, a', b and b'). The student may verify that the horizontal distance to $\overline{Y}$ in panel b' equals the vertical distance in panel a. The addition of panels c, c', d, d', e and e' permit the completion of the visual linkage in terms of the factor marginal conditions with the product marginal conditions. Panels a, a', e and e' are the geometric equivalents of the mathematical inverses which permitted the "completion of the circle" in the algebraic section.

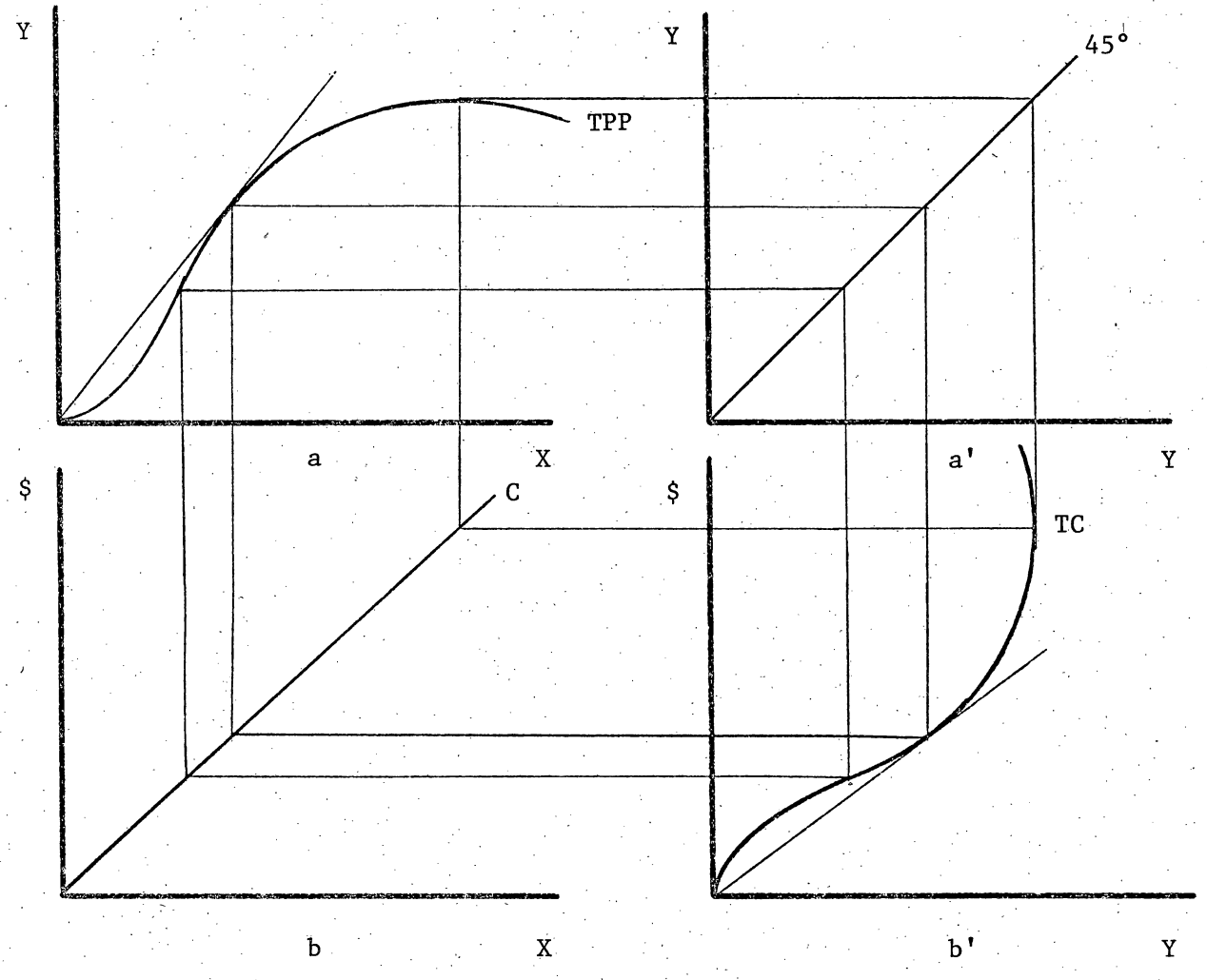

Figure 1. Linkage between production function and cost-output relationship.

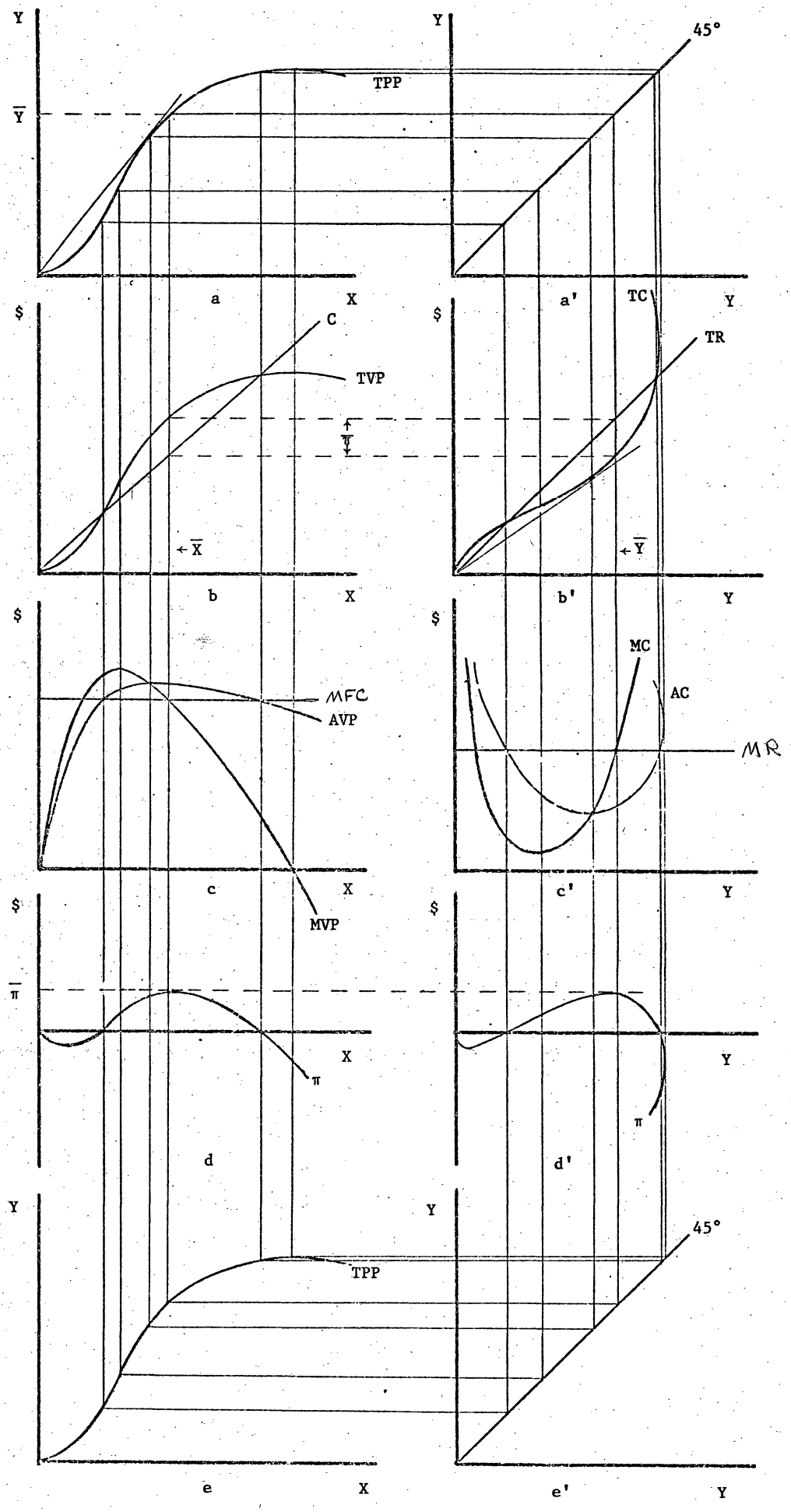

Figure 2. Linkage between production function and cost-output relationship with extensions to marginal conditions.

Following the lines in Figure 2, the correspondence of break-even points (first and fifth vertical lines), inflection and maximum and minimum marginal points (second vertical line), ray-line tangency and maximum and minimum average points (third vertical line), profit maximization point (fourth vertical line) and maximum physical product (sixth vertical line) is shown. As in the case of optimal Y, the student may verify that $\bar{\pi}$ in panels b and b' equals $\bar{\pi}$ in panels d and d'.

Figures 1 and 2 are also useful in demonstrating the impact of improved technology (shifting TPP), changing factor price (change in slope of C) and changes in fixed cost (shifting intercept of C) on the firm's supply function (marginal cost curve) and optimum factor and product levels. Exercises of this type, using Figures 1 and 2 as well as equations (1) through (4), are extremely valuable in getting across these fundamental relationships. For this purpose we suggest duplicating and distributing several copies of the figures and asking the students to trace through several postulated changes in these parameters.

Suggested Presentation Format

We have found the following format to be a logical and useful way to present the essence of Figures 1 and 2. First, the theory is fully developed in terms of optimization from the factor side (left side of Figure 2, exclusive of panel e). In this connection we have found the addition of panel d to add significantly to student's understanding of panels b and c. (Doll, Rhodes and West do this in their text.)

The next step is then to go to Figure 1 and develop this idea slowly and in detail. This has to be one of the most important (albeit most often neglected) relationships in the theory of the firm.

We procede then to product optimization ideas and concepts of panels b', c' and d' of Figure 2 following from panel b' of Figure 1.[2] Finally the additional linkages (beyond those shown in Figure 1) are developed by adding panels e and e', thereby "closing the circle."

If we would take time in our curricula to thoroughly ingrain our students with the ideas and linkages demonstrated in Figures 1 and 2, we believe that it would vastly improve their professionalism and competence. These vertical and horizontal linkages are so fundamental that the scarcity of these developments in production economics and micro-theory texts is a serious deficiency.

Two Factor Case

The extension of concepts of the previous section to multifactor cases can be conveniently and rigorously accomplished only by using mathematics. The three dimensional nature of the two factor case precludes (at least for these authors) the geometric approach.

Although the mathematics of deriving costs as a function of product from the underlying production function is not complicated, it is sufficiently messy that it poses some difficulty for students. Certainly such sketchy treatments as that provided by Henderson and Quandt do not accomplish much by way of developing fundamental understanding and insight. To simply say that a "...system of three equations in four variables can be reduced to a single equation in which cost is stated as an explicit function of the level of output..." (p. 71)[3] is probably only of value

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[2]Concepts of product supply can, of course, be incorporated into the presentation at this point by adding emphasis (darkening) the relevant portion of MC. Factor demand can be incorporated similarly in panel c.

[3]In all fairness it should be noted that Henderson and Quandt by virtue of this statement at least draw attention to the important relationship between the factor and product sides.

to those who already understand the concept.

In this section we give first a general statement of the derivation and then provide an example using the generalized Cobb-Douglas form, keeping in mind that our objective is to make explicit the transformation from the production function or factor side to the cost-product relationships. The information needed in addition to the production function and cost equation of the single input case is that provided by the expansion path. It is the expansion path that defines for us a relationship between X_1 and X_2 exclusive of other variables (i.e., Y).

Thus, knowing the production function

(5) $Y = f(X_1, X_2)$

and the expansion path

(6) $\frac{MPP_1}{MPP_2} = \frac{r_1}{r_2}$

inverses of the production function in terms of X_1 as a function of Y only and X_2 as a function Y only may be obtained as follows. First, (6) may be expressed alternatively as

(6a) $X_2 = g(X_1)$ and as

(6b) $X_1 = h(X_2)$

Substituting (6a) into (5) and taking the inverse we obtain:

(7) $X_1 = k(Y)$

Similarly substituting (6b) into (5) and taking the inverse, we obtain

(8) $X_2 = \ell(Y)$

Plugging (7) and (8) into the cost-input relationship, it is converted to a cost-output relationship, i.e.,

(9) $C = r_1X_1 + r_2X_2 = r_1k(Y) + r_2\ell(Y) = TC$

As in the single factor case the power (Cobb-Douglas) production function provides a convenient algebraic example.[4] Given

$$(10) \quad Y = AX_1^{\alpha} X_2^{\beta}$$

the expansion path is simply

$$(11) \quad \frac{\alpha X_2}{\beta X_1} = \frac{r_1}{r_2}$$

which implies

$$(11a) \quad X_2 = \frac{\beta r_1}{\alpha r_2} X_1 \text{ and}$$

$$(11b) \quad X_1 = \frac{\alpha r_2}{\beta r_1} X_2$$

Substituting (11a) and (11b) into (10) and taking inverses we obtain

$$(12) \quad X_1 = \left[\frac{Y}{A\left(\frac{\beta r_1}{\alpha r_2}\right)^{\beta}}\right]^{\frac{1}{\alpha+\beta}} \quad \text{and}$$

$$(13) \quad X_2 = \left[\frac{Y}{A\left(\frac{\alpha r_2}{\beta r_1}\right)^{\alpha}}\right]^{\frac{1}{\alpha+\beta}}$$

Finally the cost function is obtained by substituting (12) and (13) into the cost equation, (9), yielding

$$(14) \quad TC = r_1\left[\frac{Y}{A\left(\frac{\beta r_1}{\alpha r_2}\right)^{\beta}}\right]^{\frac{1}{\alpha+\beta}} + r_2\left[\frac{Y}{A\left(\frac{\alpha r_2}{\beta r_1}\right)^{\alpha}}\right]^{\frac{1}{\alpha+\beta}} = KY^{\frac{1}{\alpha+\beta}}$$

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[4] This is so because the Cobb-Douglas expansion path is linear given constant factor prices.

$$\text{where } K = \frac{r_1}{\left[A\left(\frac{\beta r_1}{\alpha r_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}} + \frac{r_2}{\left[A\left(\frac{\alpha r_2}{\beta r_1}\right)^{\alpha}\right]^{\frac{1}{\alpha+\beta}}}$$

Conclusion

There are two most important insights in the development of the theory of the firm that cannot be overemphasized in our teaching. First, students before reaching the Ph.D. level should learn that costs of production are intimately related to and, in fact, are derived from the production function. Second, they should know that when considering the theory of the firm from the product side, implicit in the cost function is the expansion path. That is, when one draws a cost curve it is assumed that the cost-product points are based on the presumption of least-cost combination of the several factors involved.

If these two ideas and the factor- and product-side linkages were given greater attention in our teaching, professors would surely be less frustrated than they often are in students' responses to such fundamental questions as: What happens to the supply of wheat when the price of fertilizer is increased?

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