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ABSTRACT

Adapting Portfolio Theory for Asset Indivisibility: A Conceptual Framework

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Portfolio theory is used to explore the kinds of risk-return effects induced by investments in indivisible assets. It is shown that risk-return economies require higher levels of return and risk as the size of the fixed investment increases and portfolio choices can be limited using Baumol's EL criterion.

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Adapting Portfolio Theory for Asset Indivisibility: A Conceptual Framework*

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*The views expressed in this paper are those of the authors and do not necessarily reflect those of the U.S. Department of Agriculture.

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ADAPTING PORTFOLIO THEORY FOR ASSET INDIVISIBILITY: A CONCEPTUAL FRAMEWORK

A firm's acquisition and use of durable assets are often influenced by high degrees of asset indivisibility. Equipment, machinery, building and land all tend to be available for sale only in discrete size units. Most operating inputs exhibit greater divisibility although even hired labor is often characterized by increasing reliance on annual contracts that give it a quality of indivisibility. Asset use is generally characterized by greater divisibility although specialization and timing in use may also induce discrete characteristics.

The indivisibility feature poses numerous kinds of effects on producing firms and, in turn, introduces substantial challenges in modeling the firm to properly reflect its decision environment. First, discontinuities in the supply function for indivisible assets indicate that large price changes may be required to warrant resource reallocations. Second, problems of generating sufficient cash and/or credit to meet financing terms for purchasing the indivisible asset (s) may influence the rate of investment and lead to discontinuous changes in farm size (Barry). These indivisibilities can cause serious problems in matching resources with different productive capacities. Managerial capacity in particular must be matched with the managerial requirements of investments in new technology or increases in size of business. Lags in managerial adjustment can severely jeopardize the operating efficiency of the firm at the very time when levered growth has reduced its liquidity and exposed its equity to adverse fluctuations in asset prices. Finally the adjustment in capital structure and production organization that commonly accompanies investment in indivisible assets can alter substantially the degree of risk faced by the firm.

In this paper we utilize portfolio theory to explore the kinds of risk-return effects induced by investment in indivisible assets. We distinguish between the portfolio effects of (1) indivisibility in asset acquisition and use and (2) indivisibility in asset acquisition and divisibility in use. We first illustrate the effects of asset indivisibility on the set of EV efficient portfolios facing the firm. Then we identify the effects on the range of portfolio choices using EV efficiency criteria, including Baumol's expected gain-lower confidence limit criterion.

Asset Indivisibility and EV Efficient Frontiers

We begin by using Tobin's separation property to illustrate the derivation of EV efficient frontiers for a three asset model with two risky assets (x_1,x_2) and one safe asset (x_3). Let the firm be constrained by initial wealth W_o and let the safe asset yield a rate of return (r_{3-1}). Further let expected returns and variances on x_1 and x_2 be (r_1-1, r_2-1) and (σ_1^2, σ_2^2) respectively with covariance σ_{12}.

Assume initially that all asset choices are completely divisible in acquisition and use. The firm's expected wealth (W) is defined as $W = r_1x_1 + r_2x_2 + r_3x_3$ and the variance of expected wealth is $\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12}x_1x_2$. The initial wealth constraint is introduced by substituting for $x_3 = W_o - x_1 - x_2$ in the expected wealth equation.

Solving for x_2 in terms of x_1 for a given level of expected wealth and variance yields equations (1) and (2)

$$(1) \quad x_2 = \frac{(W - r_3 W_o) - (r_1 - r_3)x}{(r_2 - r_3)}$$

$$(2) \quad x_2 = \frac{-x_1 \sigma_{12} + \sqrt{x_1^2 \sigma_{12}^2 - \sigma_2^2 (\sigma^2 + \sigma_1^2 x_1^2)}}{\sigma_2^2}$$

When W and σ^2 are held constant, equations (1) and (2) describe isowealth and isovariance lines--loci of alternative combinations of x_1 and x_2 yielding the same values of W and σ^2. Changes in W and σ^2 provide a set of such insovalue lines which in the case of isovariance are reflected as a series of concentric ellipses. The concentric ellipses are represented by curve segments $\sigma_1, \sigma_2, \ldots$ in Figure 1 while the set of isowealth lines is reflected by lines $W_1, W_2, \ldots$. Line DC limits investments in the risky assets to W_o.

The derivation of an E-V efficiency frontier seeks the asset combination (portfolio) that provides minimum variance for alternative levels of expected returns or wealth. Variance minimization is indicated by tangency points between successively higher isovariance and isowealth curves. The path of minimum variance points is followed until the constraint set (W_o) is reached whereupon the optimal asset combination adjusts to follow the constraint set as still higher levels of wealth are sought. This path begins along expansion path AB. At point B all resources are invested in the risky assets and expected wealth can increase only by moving along line segments BD or BC depending on relative expected rates of return on x_1 and

x_2. In either case portfolio variance increases as well. If r_2 is greater than r_1, then the optimal expansion path is represented by ABD which when plotted in EV space yields efficiency frontier ABD in Figure 2.

Assume next that x_1 is a fixed, indivisible asset that is not initially included as a portfolio choice. The optimal expansion path then consists of line segment AD on the vertical axis of Figure 1. In constrast to the case above, larger variances are required along the vertical axis to achieve the same level of expected wealth; however, the beginning and terminal portfolios are identical. In EV space the efficiency frontier for expansion path AD lies interior to that of ABD and is therefore a less desirable set of portfolios.

Now assume that a risky asset is indivisible in acquisition and divisible in use. This assumption means that once the asset is acquired it may be used in any amount up to its full capacity. If the required acquisition of the indivisible asset is F, then the initial wealth constraint can be modified to reflect this fixity by substituting for x_1 in the initial wealth constraint F. Similarly, if the variable use of F is x_1 and the difference between F and x_1 is unused resources U, then to the expected wealth equation we add U to reflect the fact that expected wealth equals investments and expected returns in assets x_1, x_2 and x_3 plus unused resources. Thus expected wealth is $W = r_1 x_1 + r_2 x_2 + r_3 x_3 + U$ and the initial wealth constraint is $W_o = F + x_2 + x_3$.

To find the isowealth equation in terms of x_1 and x_2, we solve for x_3 in the initial wealth equation ($x_3 = W_o - F - x_2$), set $U = F - x_1$ and then substitute for x_3 and U in the expected wealth equation. The resulting isowealth equation after collecting terms is:

$$(3) \quad x_2 = (W - r_3 W_o - F(1 - r_3) - (r_1 - 1)x_1)/(r_2 - r_3)$$

The EV efficient expansion path is modified to account for these new asset characteristics. The expansion path begins at point A as before. However, since the slope in (3) is steeper in a negative direction than before (in equation (1)) the expansion path contains a larger proportion of x_1 relative to x_2 than in the case where both assets were divisible. Once the limit on the use of x_1 is reached at point H, further increases in expected wealth and variance are obtained by expanding the use of x_2 until point G on the constraint set is reached. The expansion path then continues along the constraint set as before from point G to D, increasing use of x_2 and decreasing use of x_1 until D is reached. In EV space the efficiency frontier corresponding to path AHGD in Figure 1 is represented by the dotted EV frontier AHGD in Figure 2. Only at points E and G are solutions obtained under an assumption of indivisibility in asset acquisition that is identical to the complete divisibility case outlined earlier. At other points the new (dotted) efficiency frontier lies interior to the original frontier and terminates at D' and A' respectively. The reason for the interior location and the differences in expected wealth between portfolios A and A' and D and D' is attributed to the earnings sacrificed on unused resources.

Finally, assume that asset x_1 is indivisible in both acquisition and use at level F. This assumption means that acquisition of x_1 requires its full utilization. The EV efficient expansion path now becomes line segment

FG in Figure 1 and is represented in EV space by dashed line FHEG. The portion of this efficiency frontier below point H lies interior to that of the preceding case while the efficiency frontiers are identical from H to G. However, portfolios beyond point G cannot be attained once x_1 is acquired due to the requirement that it be fully utilized. Again, only at points E and G are solutions obtained under an assumption of indivisibility in asset acquisition and use that are identical to the solutions for complete divisibility outlined earlier.

The analysis to this point clearly indicates that indivisibility in asset acquisition and use can significantly modify the risk-return characteristics of alternative portfolios comprising a producer's choice set. The degree of modification depends on the degree of indivisibility and on wheather it affects both asset acquisition and use. The greater the degree of indivisibility, the greater the fixed costs associated with assets that may be idle in use. The modification of variance depends on the relative variances of divisible and indivisible assets and on their correlation.

These features reflect a phenomenon that can be termed economies of size in risk. If we define economies of size in risk as the EV frontier obtained when all assets variable in use and in acquisition, then economies of size in risk with portfolios including fixed assets can be obtained in only a limited number of asset combinations. In our example, only at points E and G in Figure 1. However, increases in the size of fixed assets (i.e., F to F') permit a shift in the points of economies of size in risk (E and G to E' and G') accompanied by increases in expected wealth and variances of the portfolios. Thus the larger the level of expected wealth and variance that are needed to achieve economies of size in risk.

The particular configuration of expansion paths in Figure 1 and efficiency frontiers in Figure 2 rest on an assumption that x_2 provides higher rate of expected returns than x_1. For an alternative assumption that ($r_1 < r_2$) the analysis would be modified only slightly and can be traced through the two diagrams without much difficulty.

Asset Indivisibility and Optimal Portfolio Choices

Figure 3 contains reproductions of two of the efficiency frontiers derived in Figure 2--curves AD and A'HEGD' represented by curves AA' and BB' respectively. Hence, in Figure 3 curve AA' reflects an EV efficient set that does not contain investment in an indivisible asset while curve BB' reflects an EV efficient set that does exhibit investment in an indivisible asset. The two curves intersect in a manner that produces a bubble or scallop as reflected by segment CB" B'" D. This configuration of efficiency frontiers is likely faced by a decision maker considering the inclusion of a fixed asset. What are the effects of ___ configuration of efficiency frontiers on the range of optimal portfolio choices?

Optimal portfolio choice depends upon the decision maker's valuation in utility terms of the risk (variance), expected wealth and other relevant characteristics of the portfolios comprising the choice set. To simplify the utility maximizing choice, several efficiency criteria have been developed to order portfolio choices into efficient and inefficient sets. We have already utilized the mean-variance (EV) efficiency criterion to restrict the utility maximizing choice to those portfolios providing minimum variance for various levels of expected wealth--the EV efficient set. EV efficiency requires either that the ordinal utility function be quadratic or that the density function of risky choices be normal. In terms of EV efficiency, the efficient portfolio choices in Figure 3 will always be from the lower boundary of the two frontiers--boundary A' C B'' B''' DA--since it provides minimum variance for alternative levels of expected returns.

Baumol's E-L criterion can be used to further restrict the EV efficient set and to explore the influences of investment in indivisible assets on the ranges of portfolio choices. Baumol's criterion for reducing the set of efficient portfolios is based on deriving the lower conficence limits (L) for alternative portfolios and is expressed as

(4) $L = W - k\sigma$

where W = expected wealth, σ = standard deviation of wealth and k = a constant determining the probability of E exceeding L. Hence, the probability of wealth exceeding the lower confidence limit (L) is the same for each portfolio. Portfolios are considered to be inefficient by the EL criterion if another portfdio (s) exists with a greater expected value and a greater than or equal lower confidence limit. Therefore, the efficient EL set will include all portfolios to the right of the maximum point of an L function (Figure 4). The maximum of the L function is found by specifying the relationship among members of the EV efficient set as

(5) $\sigma = f(W)$

Substituting equation (5) into equation (4) and differentiating with respect to E, yields the solution $f'(W) = 1/k$, a portfolio in the E-V set which maximizes the L function. The decision maker would then restrict his choice to those portfolios lying on the portion of the EV frontier with slope greater than 1/k.

The decision maker's preferred portfolio on the EV frontier is found by maximizing his expected utility function. The expected utility function identifies a preferred trade-off (λ) between increases in variance and expected wealth on the EV frontier. A tangent line with slope λ which identifies the preferred portfolio can be written as:

$$(6) \quad U(W, \sigma^2) = \lambda W - \sigma^2 \quad .$$

If the initial utility maximizing solution has slope λ, then the k value which defines an L function for that particular EV frontier is found by multiplying (6) by $1/\lambda$ (a linear transformation which does not affect the orderability of the equation) and factoring σ^2. The result is

$$(7) \quad U(W,\sigma^2) = W - (\sigma/\lambda)\ (\sigma) = W - k\sigma$$

where $k = \sigma/\lambda$.

The maxims (L) of the Baumol EL sets corresponding to EV sets AA' and BB' in Figure 3 are equal if the k_a value associated with EV set AA' and k_b value associated with EV set BB' are equal to σ_a/λ and σ_b/λ respectively, where σ_a and σ_b are standard deviations associated with two EV efficient portfolios having the same slope (e.g., A" and B" in Figure 3). However, in order for portfolios A" and B" with similar slopes to have the same L value k_b must exceed k_a since $k = \sigma/\lambda$ and $\sigma_a < \sigma_b$. Thus, if the probability of returns exceeding L is equated for the two EV sets either by reducing k_b or increasing $k_{a'}$, the L value associated with B'' will exceed the L value associated with A'' and portfolio B'' will exceed the L value associated with A'' and portfolio B'' will dominate A''. Hence, the indivisible asset will be adapted. A smilar deduction can also be made if the original portfolio lies in the range A''C--the portfolios including the indivisible asset will be dominant.

If the original, divisible solution were below point A'', the L values associates with EV set A''A' would exceed the L values for EV set CB''D at points of equal slope (λ). And, equating the k values implied by λ at points along A''A' indicates that these portions of A''A' cannot be excluded as inefficient by Baumol's EL criterion.

A similar analysis can be made if the original, divisible portfolio lies on the AD portion of EV set AA'. Portfolios in the range A'''A will always dominate portfolios in the range B'''D since equating k values at points of equal slope implies that L values associated with the A'''A portion of EV set AA' will exceed the L values associated with the B'''D portion of EV set BB'. And, since the expected wealth of portfolios in A'''A always exceed the corresponding expected wealth of portfolios in B'''D, Baumol's criterion is satisfied. Dominance conditions for the DA''' portion of EV set AA' are indeterminant since it neither dominates nor is dominated by the B'''B'' portion of EV frontier BB''. However, for large fixed assets or in the case where expansion follows the ABC path in Figure 1, the portion of the EV frontier lying above point D is truncated. And for non EL efficient portfolios, adapting the indivisible asset increases variance and expected wealth.

Concluding Comments

The conceptual framework developed in this paper is designed to facilitate analysis of investments in indivisible assets under conditions of uncertainty. The theoretical results indicate how asset indivisibility can modify the set of portfolio choices confronting the decision maker and influence his expected utility maximizing choice. These results support the need for empirical methods which incorporate asset indivisibility as needed to produce results that more meaningfully explain decision behavior.

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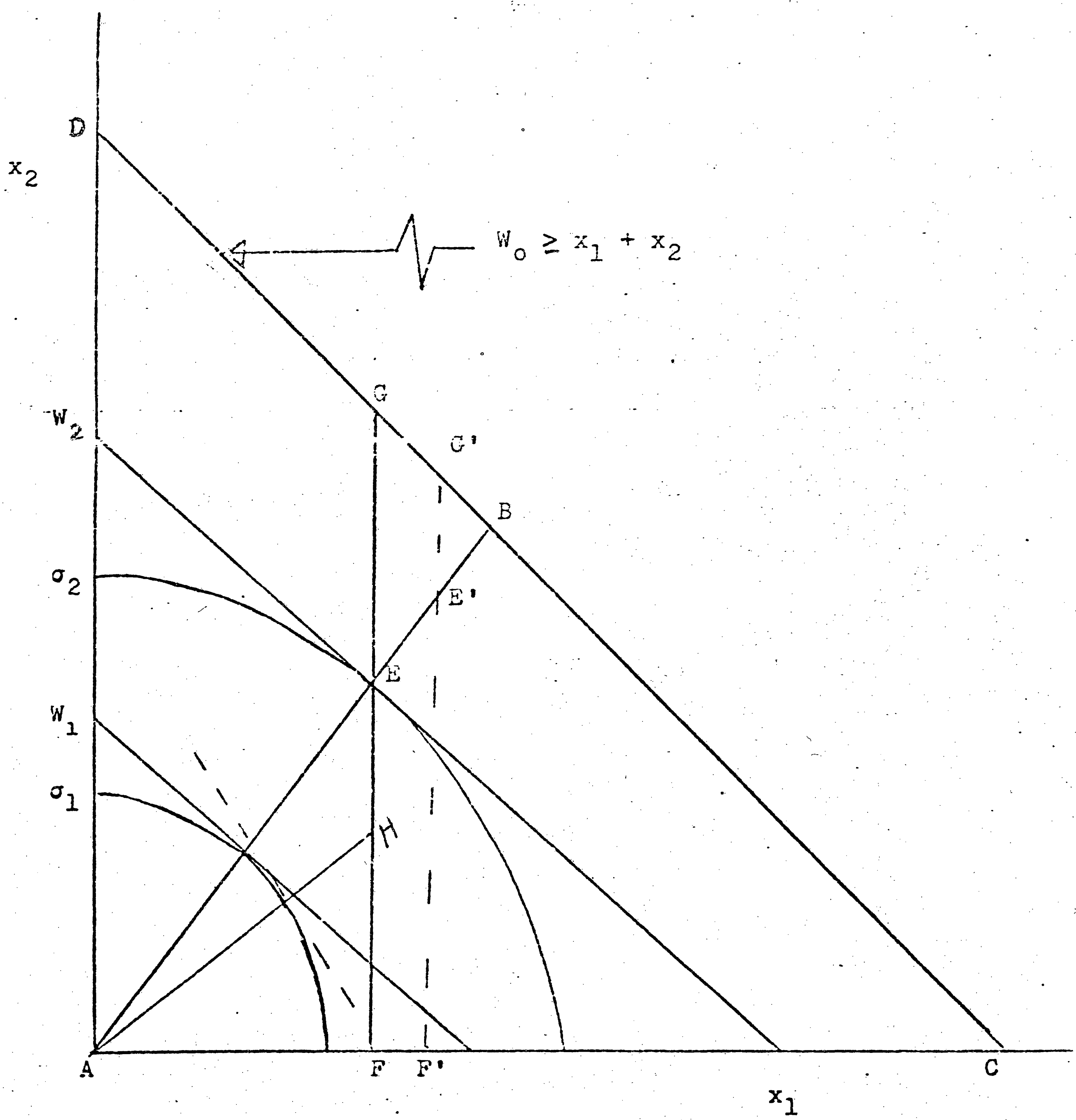

Figure 1. Expansion Paths Under Alternative Assumptions of Asset Divisibility

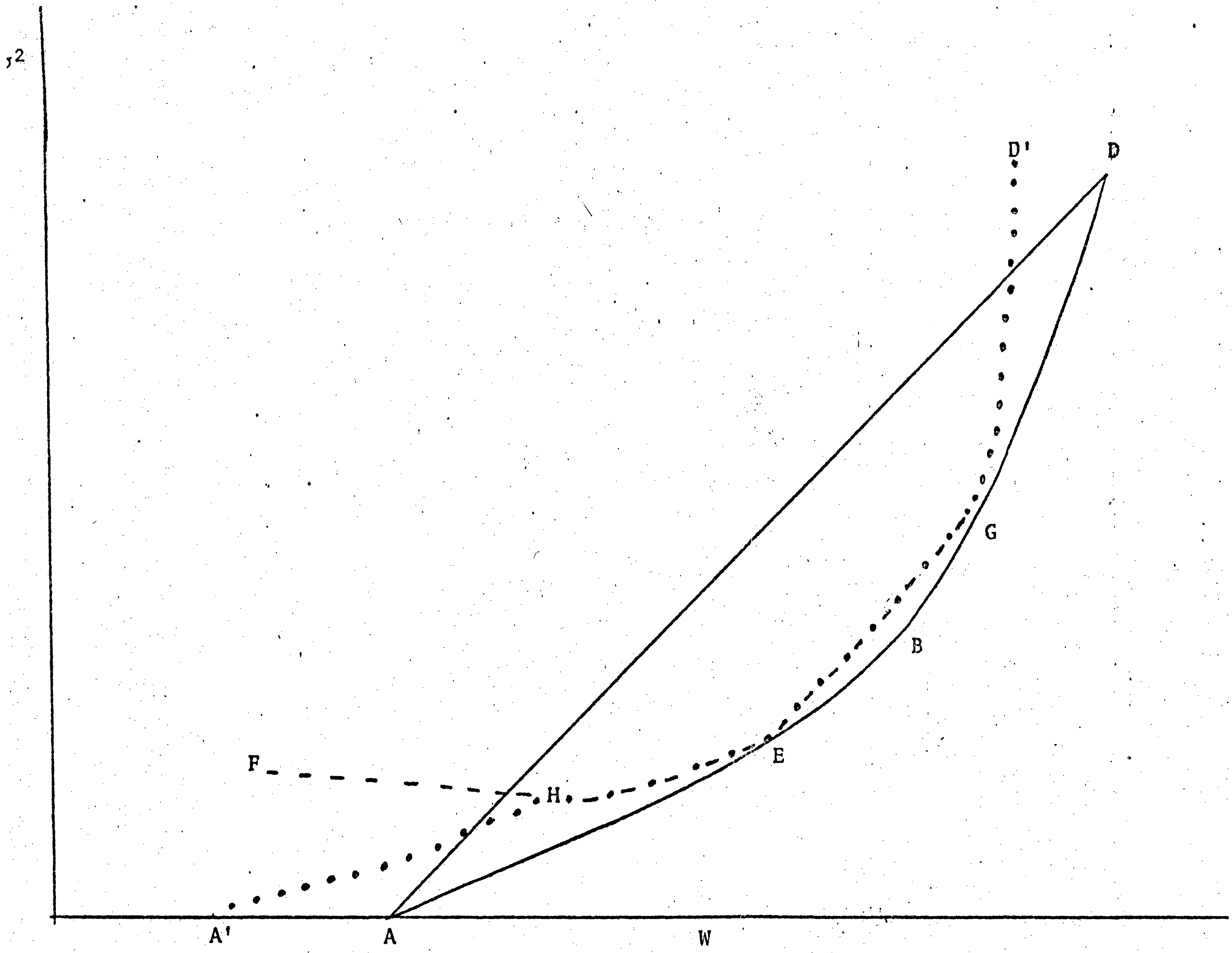

Figure 2. EV Efficient Sets

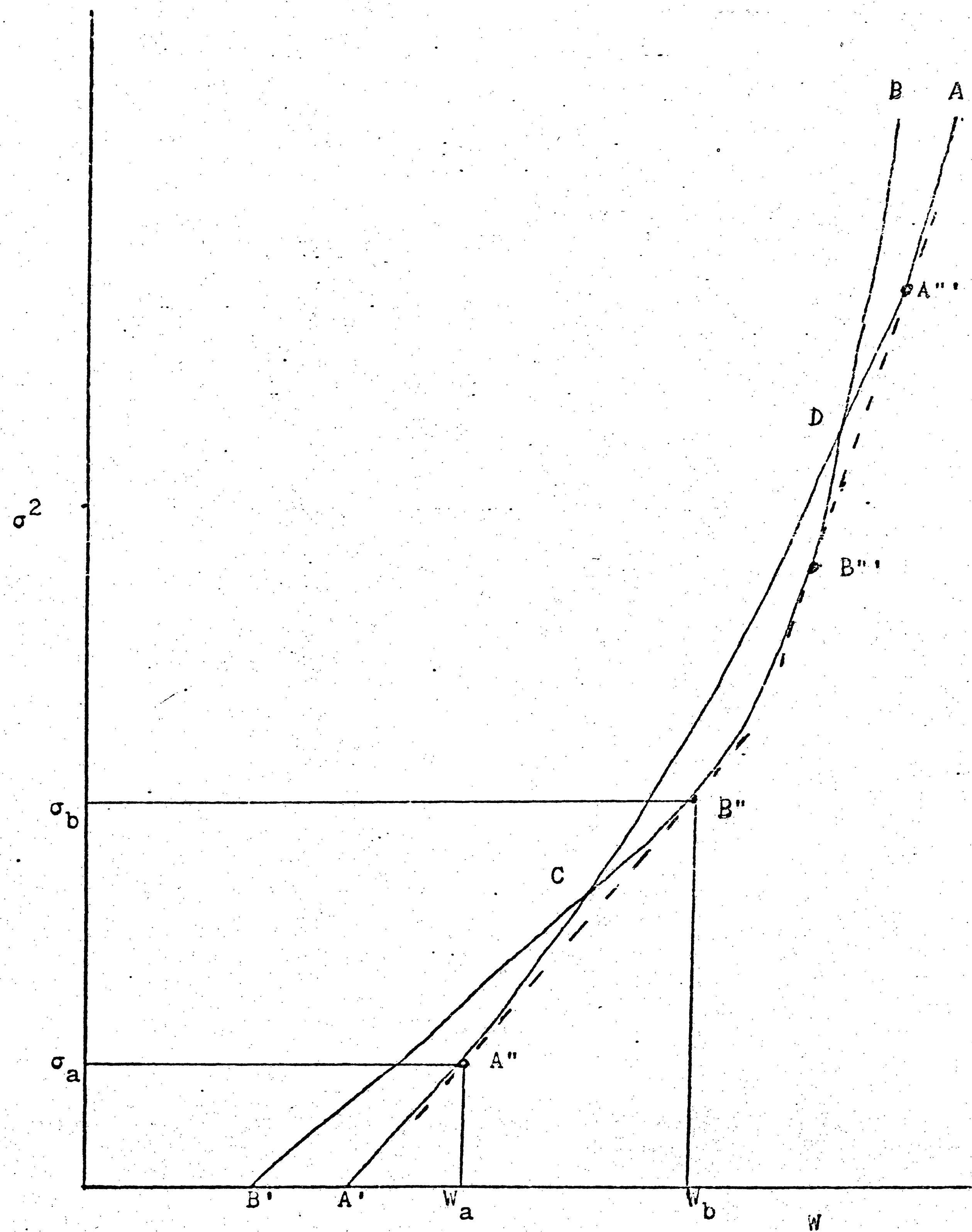

Figure 3. Baumol's EL Criterion Applied to EV Sets

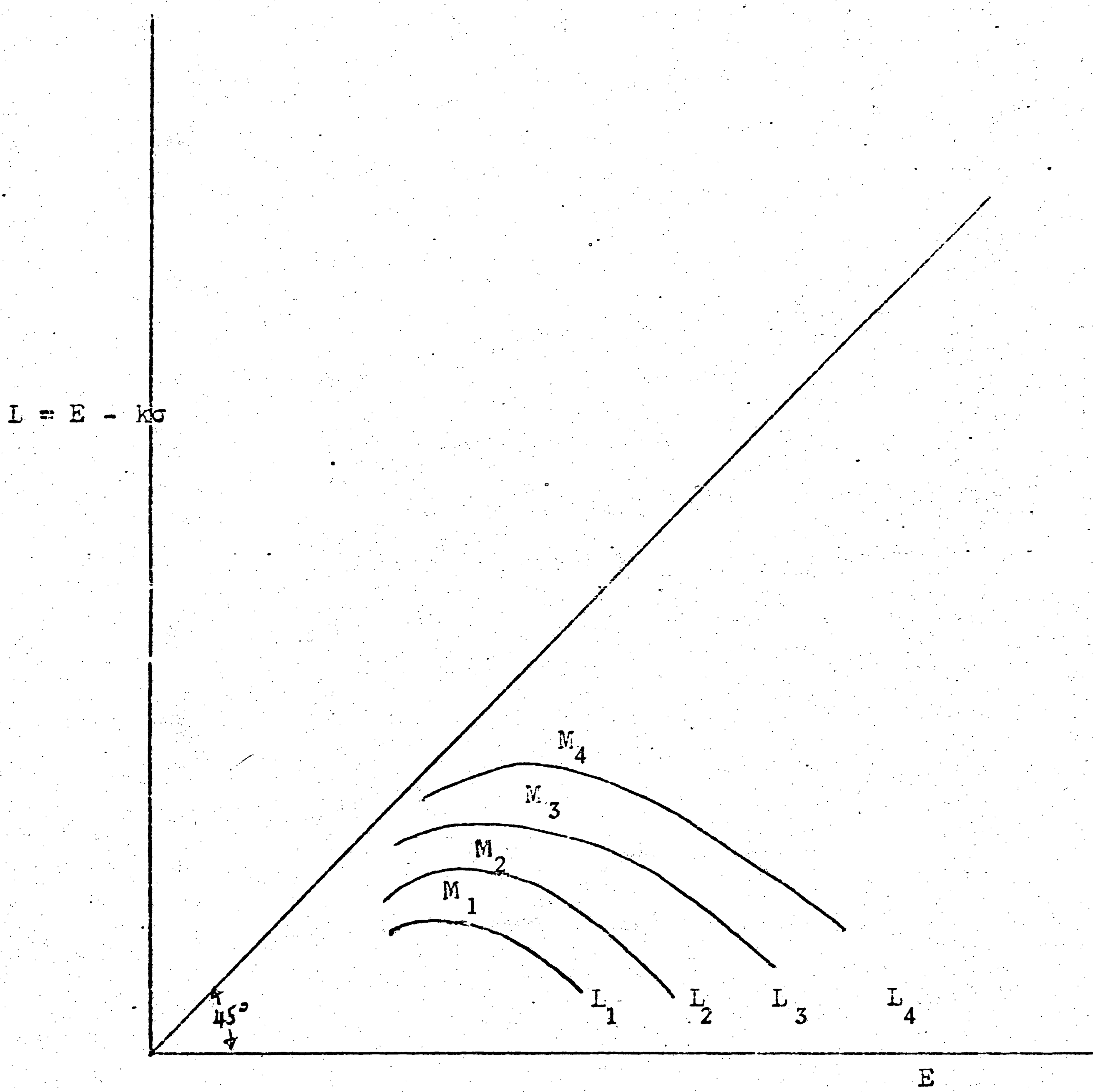

Figure 4. Baumol's Expected-Gain Confidence Limit