

*Economic Research Institute Study Paper*

*ERI #96-13*

**ON SOME ASPECTS OF THE MANAGEMENT OF A  
STOCHASTICALLY DEVELOPING FOREST**

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**May 1996**

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**Amitrajeet A. Batabyal**

**ABSTRACT**

In this paper I focus on some important biological aspects of the forest management problem. I model a stochastically developing forest as a multidimensional, continuous-time Markov chain. Next, I pose three questions concerning the long-run characteristics of a stationary forest, the stochastic process followed by dying species, and the age of an arbitrary species in the forest. I then (a) characterize a stationary forest probabilistically, (b) describe the stochastic process governing the demise of species in this forest, and (c) provide a method for determining the age of an arbitrary species in the forest. Finally, I discuss the forest management implications of the issues raised in this paper.

*JEL* Classification:

Key words: forest, management, continuous-time Markov chain

# ON SOME ASPECTS OF THE MANAGEMENT OF A STOCHASTICALLY DEVELOPING FOREST<sup>1</sup>

## 1. Introduction

Many interesting questions in stochastic capital theory concern the determination of the optimal time at which to terminate an aging process subject to biological and economic uncertainties. The optimal forest management problem is such a question. Until very recently, this management question has been analyzed in a deterministic context because the "... introduction of [uncertainty] complicates the analysis considerably" (Dasgupta, 1982, p. 182). While economists such as Brock and Rothschild (1984), and Brock, Rothschild, and Stiglitz (1988) have analyzed aspects of the forest management question in a stochastic setting, they have done so in a very particular manner. Specifically, they have posed the "tree cutting" question as an optimal stopping time problem. Even the more general analysis of Clarke and Reed (1989) analyzes the management problem by positing that the price of timber follows a geometric Brownian motion process and that the logarithm of timber size follows a diffusion process whose local behavior resembles that of a Wiener process with drift.

Similarly, ecologists and foresters have also studied different aspects of this forest management problem. For instance, Hobbs and Legg (1983) and Shugart, Crow, and Hett (1973) have applied Markov models to study plant by plant replacement processes and the dynamics of vegetation stands, respectively. Moore (1990) has, *inter alia*, studied nonequilibrium age structures in Montana forests using a semi-Markov model of vegetation dynamics.

While these analyses have undeniably advanced our understanding of the tree cutting problem in a stochastic context, they have omitted three key aspects of the biological uncertainty affecting many large and long standing forests, particularly those in developing countries. These aspects pertain to the stochastic

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<sup>1</sup>Dedicated to the memory of Sutapa Batabyal.

development of trees in long standing forests, the stochastic demise of trees<sup>2</sup> and the uncertain introduction of new or previously known species into the forest. These features characterize large areas of forest land in many developing countries. The Amazon Basin in Brazil, parts of the Chambal Valley in India, and portions of the Irrawady Basin in Myanmar are but three examples of forests in which the above-described features play an important role in the evolution of the forest. Further, national forest managers who typically “inherit” such long standing forests with a multitude of species have a difficult time attempting to formulate forest policy. Before embarking on any policy—which, from an economic standpoint, would involve the formulation and solution of an appropriate optimization problem—such managers would like to acquire information about questions which are inherently stochastic in nature. What kinds of species and how many trees within a particular species can one expect to see in a mature forest? Can one say anything about the stochastic process governing the demise of species? What can one say about the likelihood that a given species with a specific number of trees is the oldest—in the sense that it originated earliest—species in the forest?

To the best of my knowledge, these questions have not been addressed in the ecological literature previously. As such, in this paper I propose to answer these three questions. In order to do so, I shall adapt and apply stochastic population models to the forest management problem [see Bailey (1964, pp. 117-136), Bartlett (1960, pp. 17-44), and, in particular, Ross (1983, pp. 156-164)]. In this paper, I shall be concerned with the biological and not the economic aspects of the management problem.<sup>3</sup> More specifically, I shall model the stochastically developing forest with the biological characteristics alluded to above as a multidimensional, continuous-time Markov chain (CTMC).<sup>4</sup> As far as I am aware, the use of this

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<sup>2</sup>Due to various reasons such as insect infestation, natural disasters, and old age.

<sup>3</sup>Research on the economic aspects of the problem is ongoing, and I expect to report those results separately.

<sup>4</sup>See Bailey (1964, pp. 117-136) or Ross (1983, pp. 141-183) for more details.

modelling technique to study the above-described questions is novel.<sup>5</sup> I wish to point out that this model bears considerable resemblance to the  $M/G/\infty$ <sup>6</sup> of queuing theory, and thus I will point out some of the similarities in due course. In order to provide closed-form answers to the three questions that I have posed, it will be necessary to make some distributional assumptions. Without such assumptions, the problem becomes intractable and closed-form answers to my questions cannot be provided.

## 2. The Modeling Framework

In order to answer the three outstanding questions, I will analyze a mature, i.e., stationary, forest. Suppose that new trees are “born” from existing trees in a forest in a statistically independent manner at an exponential rate  $\gamma$ . Due to a variety of possible reasons—some of which are listed in footnote 2—trees are assumed to “die” independently at an exponential rate  $\epsilon$ . In a queuing context, one would say that trees “depart” at an exponential rate  $\epsilon$ . Finally, new individuals “enter” the forest due to reasons such as plantings, pollination by natural agents, etc., according to a Poisson process with rate  $\delta$ . In a queuing framework, one would say that new individuals “arrive” at rate  $\delta$ . I shall assume that  $\gamma < \epsilon$ . This assumption will ensure ergodicity and, hence, the existence of stationary probabilities for my CTMC.

If I let  $C_k(t)$  be the *number* of species at time  $t$  which have  $k$  trees,  $d \geq 0$ , and if I let  $\vec{C}(t) = \{C_1(t), C_2(t), C_3(t), \dots\}$  be the *vector* of all possible numbers of species in the forest under consideration,<sup>7</sup> then the vector stochastic process,  $\{\vec{C}(t): t \geq 0\}$ , is a CTMC because along every dimension  $\{C_k(t): t \geq 0\}$ , for all  $s \geq 0$ ,  $t \geq 0$ , and for nonnegative integers  $m, n, c(v), v \in [0, s]$ ,

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<sup>5</sup>It is important to stress that I am *not* claiming that the use of Markov models in ecology is novel. What I am claiming is that the use of the multidimensional CTMC to study the questions of this paper is novel.

<sup>6</sup>In this three-letter designation, the first letter refers to the fact that the interarrival times of customers in a queue has the Markovian property. The second letter refers to the fact that the server's service times have a general distribution. Finally,  $\infty$  refers to the case in which there are an infinite number of servers. For more details, see Wolff (1989, pp. 75-81).

<sup>7</sup>As the reader will note, I have used a countable state space to ease the mathematical exposition. If I constrain the state space to be finite, all of the subsequent analysis will carry through, albeit with greater algebraic clutter.

the property that

$$s) = m, C_k(v) = c(v), 0 \leq v \leq s\} = Pr\{C_k(t+ \quad (1)$$

holds. With respect to the state vector,  $\vec{c} = \{c_1, c_2, c_3, \dots\}$ ,  $c_k > 0$ , I can now define the following four states of the CTMC.<sup>8</sup> Let

$$A_k(t)\vec{c} = \{c_1, c_2, c_3, \dots, c_{k-1}, c_k - 1, c_{k+1} + 1, c_{k+2}, \dots\}, k \geq 1, \quad (2)$$

$$D_k(t)\vec{c} = \{c_1, c_2, c_3, \dots, c_{k-1} + 1, c_k - 1, c_{k+1}, c_{k+2}, \dots\}, k \geq 2, \quad (3)$$

$$A_0(t)\vec{c} = \{c_1 + 1, c_2, c_3, \dots\}, \quad (4)$$

and let

$$D_1(t)\vec{c} = \{c_1 - 1, c_2, c_3, \dots\}. \quad (5)$$

Equation (2) denotes the state that the CTMC is in after  $\vec{c}$  if a new tree is “born” in a species with  $k$  trees,  $k \geq 1$ . Equation (3) denotes the state after  $\vec{c}$  if a tree in a species with  $k$  trees “dies,”  $k \geq 2$ . Equation (4) denotes the state after  $\vec{c}$  when a new tree “enters” due to plantings or pollination. Finally, (5) represents the state after  $\vec{c}$  when a species consisting of one tree loses that tree.

Let  $x\{\vec{c}, \vec{c}'\}$  denote the CTMC's instantaneous transition rates. For my purposes, the relevant transitions are

$$x\{\vec{c}, A_0(t)\vec{c}\} = \delta, \quad (6)$$

$$x\{\vec{c}, A_k(t)\vec{c}\} = \gamma k c_k, k \geq 1, \quad (7)$$

and

$$x\{\vec{c}, D_k(t)\vec{c}\} = \epsilon k c_k, k \geq 1. \quad (8)$$

Before I proceed to analyze the above-described forest, which I have modeled as a CTMC, I note the following. In what follows, I shall assume that the forest under consideration once consisted of a bare tract of land, i.e., at time  $t = 0$ ,  $\vec{C}(0) = \vec{0}$ . Further, I shall call an entering tree, a type  $k$  tree, if the species of which this tree is a member will consist of  $k$  trees at time  $t$ . Finally, I shall say that a particular species is in state  $k$  if this species consists of  $k$  trees. I am now in a position to analyze the above-described forest.

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<sup>8</sup>Appropriate state space definition is a crucial aspect of the modelling exercise. As van Hulst (1979) has noted in another context, if this is not done carefully, the state space can blow out to enormous size.

### 3. Analysis and Results

The first question of interest to a forest manager concerns the characteristics of a mature (stationary) forest. To answer this question, I now determine the stationary probabilities for this CTMC.

By generalizing Theorem 4 of Wolff (1989, p. 74), I note that the  $\{\vec{C}_k(t): t \geq 0, k \geq 1\}$  are independent Poisson distributed random variables with mean  $\delta \int_0^t P_k(w) dw$ , where  $P_k(w)$  is the probability that a species which originated at time  $w$  will consist of  $k$  trees by time  $t$ . Denote the stationary probabilities by  $P(\vec{c})$ . I can now state

*Theorem 1:* The CTMC  $\{\vec{C}(t): t \geq 0\}$  has stationary probabilities given by

$$P(\vec{c}) = \prod_{k=1}^{\infty} \exp(-\beta_k) \frac{\beta_k^{c_k}}{c_k!}, \text{ where}$$

$\beta_k = [1/k][\delta/\epsilon][\gamma/\epsilon]^{k-1}$ ,  $k \geq 2$ . *Proof:* In order to prove the theorem, I shall use (a) the time reversibility of  $\{\vec{C}(t): t \geq 0\}$ , (b) the assumption that  $\vec{C}(0) = \vec{0}$ , and (c) the fact that the  $\{\vec{C}_k(t): t \geq 0, k \geq 1\}$  are independent Poisson random variables. The independence suggests that the form of the stationary probabilities should be multiplicative. As such, let the relevant stationary probabilities be given by

$$P(\vec{c}) = \prod_{k=1}^{\infty} \exp(-\beta_k) \frac{\beta_k^{c_k}}{c_k!}, \quad (9)$$

where the set  $\{\beta_k\}$ ,  $k \in [1, \infty]$  has to be determined. To determine these constants, I now show that  $\{\vec{C}(t): t \geq 0\}$

is time reversible. For  $P(\vec{c})$  given by (9), to show time reversibility, I have to show that

$$P(\vec{c}) \cdot x\{\vec{c}, A_0(t)\vec{c}\} = P\{A_0(t)\vec{c}\} \cdot x\{A_0(t)\vec{c}, \vec{c}\}, \quad (10)$$

and

$$P(\vec{c}) \cdot x\{\vec{c}, D_k(t)\vec{c}\} = P\{D_k(t)\vec{c}\} \cdot x\{D_k(t)\vec{c}, \vec{c}\} \quad (11)$$

hold. Simple algebra tells us that (10) requires that

$$\prod_{k=1}^{\infty} \exp(-\beta_k) \cdot \frac{\beta_k^{c_k}}{c_k!} \cdot \delta = \frac{\exp(-\beta_1) \beta_1^{c_1+1}}{(c_1+1)!} \cdot \prod_{k=2}^{\infty} \exp(-\beta_k) \cdot \frac{\beta_k^{c_k}}{c_k!} \cdot (c_1+1) \cdot \epsilon \quad (12)$$



hold and (11) requires that

$$\frac{\beta_{k-1}^{c_{k-1}}}{c_{k-1}!} \cdot \exp(-\beta_k) \cdot \frac{\beta_k^{c_k}}{c_k!} \cdot \epsilon \cdot k \cdot c_k = \exp(-\beta_{k-1}) \cdot \frac{\beta_{k-1}^{c_{k-1}+1}}{(c_{k-1}+1)!} \cdot \exp(-\beta_k) \cdot \frac{\beta_k^{c_k-1}}{(c_k-1)!} \cdot \gamma \cdot (c_{k-1}-1) \quad (13)$$

hold. In writing (13), I have cancelled all the common terms. Solving (13) for  $\beta_1$ , I get  $\beta_1 = \delta/\epsilon$ .

Solving (13) for  $\beta_k$  recursively by using  $\beta_1 = \delta/\epsilon$ , I get

$$\beta_k = \frac{1}{k} \cdot \frac{\delta}{\epsilon} \cdot \left[\frac{\gamma}{\epsilon}\right]^{k-1}, \quad k \geq 2. \quad (14)$$

I have now shown that  $\{\vec{C}(t): t \geq 0\}$  is time reversible. Since the stationary probabilities given by (9) do, in fact, solve the reversibility equations, (9) does indeed give us the requisite stationary probabilities. ■

*Remark:* The time reversibility of the above described CTMC can also be demonstrated by solving the equation

$$P(\vec{c}) \cdot x \{ \vec{c}, A_k(t)\vec{c} \} = P\{A_k(t)\vec{c}\} \cdot x \{ A_k(t)\vec{c}, \vec{c} \}.$$

I have now answered the first of my three questions. That is, I have determined the characteristics of the forest under study in steady state. Equation (9) should be of considerable help to a forest manager for planning purposes. First, inspecting (9), the manager will know that in the kind of forest under study, the limiting number of species with  $k$  trees are independent Poisson random variables. In other words, there are no interaction effects between the various limiting number of species.<sup>9</sup> Second, the manager will be able to infer directly, the mean number of species with  $k$  trees. This information is given by the individual  $\beta'_k$ 's. This information should be of considerable use in planning logging policies, particularly those policies which are designed to *selectively* log certain species but not others.

The second management question that I have posed concerns the determination of the stochastic process which governs the demise of species. I have already argued that a key aspect of large forests is that trees die out due to a variety of reasons in an uncertain manner. Before embarking on a specific planting and/or logging policy, clearly, the forest manager would like to know the nature of the stochastic process

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<sup>9</sup>Note that this is a steady state result. It does not mean that there will never be any interaction effects.

governing species death. Access to this information will enable the forest manager to decide, *inter alia*, at what rate he/she should plant new trees and at what rate he/she should log presently standing trees. I now answer this question. Let  $F(t)$  be the number of species which die out in the interval  $(0, t)$ . What kind of stochastic process is  $\{F(t): t \geq 0\}$ ? The answer is contained in

*Theorem 2:* If  $\{F(t): t \geq 0\}$  is stationary at time  $t = 0$ , then  $\{F(t): t \geq 0\}$  is a homogeneous Poisson process with mean  $\delta$ . If the forest is bare at  $t = 0$ , then  $\{F(t): t \geq 0\}$  is a nonhomogeneous Poisson process with intensity function  $\delta(t)$ .

*Proof:* To prove the theorem, I shall exploit the similarities between my CTMC model and the  $M/G/\infty$  model of queuing theory. The key step lies in recognizing that my task is equivalent to determining the stochastic process followed by the *output* process of an infinite server queue with Poisson arrivals.<sup>10</sup> The solution to this problem is well known in queuing theory—see Ross (1985, p. 224)—and, hence, the claimed result follows. ■

Theorem 2 tells us that the stochastic process governing the demise of species depends fundamentally on whether the forest under study is in steady state at time  $t = 0$  or not. If the forest is in steady state, then  $F(t)$  is a homogeneous Poisson process. The more interesting case is when the forest is nonstationary at  $t = 0$ . In this case, the theorem tells us that  $F(t)$  is still a Poisson process, but a nonhomogeneous one. The nonhomogeneity tells us that because this stochastic process does not possess stationary increments, deaths are more likely to occur at certain times than at others. Intuitively, this is what one would expect in this latter situation.

I shall now answer the third question that I posed in section 1. This question concerns the probabilistic determination of the age of the various species. Specifically, I am interested in determining the likelihood that a given species with  $k$  trees is the oldest species in the forest. The answer to this question is contained in

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<sup>10</sup>Also see the discussion at the beginning of section 2.

*Theorem 3:* If in a stationary forest there exists  $c_k$ ,  $k > 0$ , species with  $k$  trees, then the probability that a given species with  $k$  trees is the oldest—in the sense that it originated earliest—species in the forest is  $k/\sum_{j=1}^{\infty} j c_j$ .

*Proof:* This result follows from Corollary 5.6.7 of Ross (1983, p. 164). ■

*Remark:* The technique involved in demonstrating the above theorem involves first truncating the countable state space of the CTMC to a finite state space. It is important to note that this truncation involves keeping track of the various species as time progresses. In other words, keeping track of the age of the various species is an important part of the proof. For a similar application in another context, see Batabyal (1994).

The significance of Theorem 3 lies in the fact that it provides a probabilistic method of dating species. This provides the forest manager with information on the basis of which he/she can make conservation decisions.<sup>11</sup> The provision of such information would appear to be particularly necessary in resolving “What to preserve?” debates that are ongoing in many parts of the developing world.

#### 4. Conclusions

In contrast with previous stochastic models of the forest management problem, in this paper I have focused on three biological aspects of the problem which have not been addressed previously by ecologists or economists. I showed how to model a stochastically developing forest with many species of trees as a multidimensional CTMC. Further, I posed and answered three questions concerning: (a) the long-run characteristics of a stationary forest, (b) the stochastic process followed by dying species, and (c) the age of an arbitrary species in a stationary forest.

As noted by Horn (1975), Markov models have considerable predictive power. As such, the questions addressed and the answers provided in this paper have a direct bearing on the general forest

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<sup>11</sup>Moore (1990) has also addressed an aspect of this conservation issue, albeit in a very different way. His approach is to maximize the mean time to local extinction for the relevant vulnerable species.

management problem. In combination with the economic aspects—on which research is ongoing—the answers provided here should offer the forest manager substantial policy guidance regarding some important but hitherto unanswered questions.

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