ABSTRACT

Using Cobb-Douglas Functions in Nonlinear Programming: A Regional Sector Model*

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Combining Cobb-Douglas production functions and a programming algorithm into a single framework provides an effective tool to simultaneously determine optimal input levels and commodity production levels. A regional application of the model involving an eight-year tracking period is reviewed. Profit maximization is enforced by equating the marginal value products with input prices. Input and output prices are represented by continuous functions.

KEY WORDS: Cobb-Douglas, production functions, programming, regional, agricultural sector

Agricultural Economics Staff Paper # 77-61
Michigan State University
August 1977

*Prepared for presentation at the American Agricultural Economics Association Annual Meetings, August 2, 1977, San Diego, California.

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Economic analysis of resource allocation often uses the Cobb-Douglas production function to estimate the marginal value product of inputs and compares them to the price of inputs. The programming algorithms of operations research are used to determine optimal production levels for a specified objective function. Cobb-Douglas production functions and programming models are extensively applied for both micro and macro-economic analysis.

This paper describes the use of Cobb-Douglas production functions in a programming framework. The framework explicitly uses marginal value products, profit maximizing behavior, and decreasing marginal physical products for individual inputs. The use of a production function which has strong production theory characteristics [Lee, p. 41] in a programming framework results from the concept of production as a combination of both physical and behavioral processes. The physical processes are the activities which follow the laws of physics and chemistry. The Cobb-Douglas production function is widely used to quantify these physical relationships. The behavioral processes encompass both the valuing of economic goods and the management or control of the physical processes. Prices are the accepted method of valuing economic goods and maximization of an objective function, usually the profit equation is the common decision rule for regulating production. If second order conditions for maximization are present, the equating of marginal value product with marginal input cost insures profit maximization. When product prices are a monotonicly decreasing function of output and input prices are a monotonicly increasing function of each input, the constraint of Cobb-Douglas production functions to homogeneity of less than or equal to one is sufficient for ensuring the second order conditions.

The Model

The Cobb-Douglas programming model integrates several important characteristics of economic production: a) supply functions of inputs used in production, b) the allocation of inputs to the separate production activities, c) input-output relationships quantified by Cobb-Douglas production functions, and d) demand functions for the products of each activity. Cobb-Douglas production functions are incorporated into a multi-product, multi-input programming model which can be simultaneous or dynamic. The production functions are represented by:

$$1) \quad Y_i = a_i * \prod_{j=1}^{n} X_{ij}^{b_{ij}}$$

where:

i = 1,2,3, ...,m and indicate commodity enterprises,

j = 1,2,3, ...,n and indicate input category,

a_i and b_{ij} are production function coefficients,

Y_i is expected output of the i^{th} commodity, and

X_{ij} is an amount of the j^{th} input used in the i^{th} commodity enterprise.

Constraining the sum of the exponential values of each production function to be less than or equal to one ($\Sigma\ b \leq 1.0$) will be necessary in most models to ensure a unique solution.

Total use of each input (TDEM) is calculated by summing the quantities used in the production of all commodities. This calculation is:

$$2) \quad TDEM_j = \sum_{i=1}^{m} X_{ij} \qquad (j = 1,2,3, \ldots,m).$$

The price of each input (PRINP) can be derived by using an estimated supply equation. For simplicity in this example, it is entered as a monotonically increasing continuous function of the quantity of the total demand for the input (TDEM) or:

$$3) \quad PRINP_j = f(TDEM_j) \qquad (j = 1,2,3, \ldots,m).$$

The expected price for each output (EPY) can also be an estimated equation for output demand. It is important that a realistic planning price equation be used instead of a final price received equation since an assumption of perfect knowledge of future prices would significantly alter behavior both in the model and in the world it is reflecting. In this example, the expected price function explicitly includes only the effect of the quantity of the expected output from the production function equation. The relationship is a monotonically decreasing continuous function represented as:

4) $EPY_i = f(Y_i)$.

The behavioral assumption of the model equates input price (PRINP) with its value marginal product (VMP) is implemented by:

5) $PRINP_j = VMP_{ij}$

and:

6) $VMP_{ij} = (b_{ij} * Y_i * EPY_i)/X_{ij}$.

The simultaneous solution of this set of equations maximizes the implicit objective function:

$$7)\quad \Pi = \max\left[\sum_{i=1}^{m} Y_i EPY_i - \sum_{j=1}^{n} TDEM_j PRINP_j\right].$$

An Example: Michigan's Agricultural Sector

Data available [Michigan Agricultural Experiment Station, Knoblauch, Kost, Lerohl] for Michigan provided the information used in the model. Enterprise budgets estimated from data gathered from operating firms were used in the derivation of the Cobb-Douglas production function exponential coefficients. The coefficients were estimated using factor shares, so the enterprise budgets were primarily used only for the variable inputs. This step recognized that a full time farmer would charge full time labor regardless of the hours worked and bias the labor coefficient. Also, the expenses experienced which are attributible to fixed inputs to agriculture from this source were not used, since they are generally charged off to one activity or another even when the costs were inappropriate for the current year of production. Estimates of costs for these inputs were developed using farm management guidelines to estimate normal input quantities and prices of rental, wage, and custom rates to generate a normalized synthetic budget for each commodity. Factor shares were calculated from those budgets by dividing the cost of each input by total cost. This calculation yields production functions that are linearly homogeneous. Standard statistical methods of regression using both time series and cross sectional data were used to first estimate production function coefficients. These estimates were not used in the model because they had the often experienced problem [Chowdhury] of some coefficients being negative and therefore unrealistic. The unrealistic coefficients can be the result of many factors. Some suspected factors are measurement erros which are correlated with the dependent variable, the accuracy of the statistical algorithm, and the computer used [Boehm].

The expected prices for U.S. agricultural commodities [Lerohl] and Michigan's historical average production of each commodity were used to determine a single point in each demand function and an elasticity was used to extend that point to the demand function for each commodity. The elasticities used were all greater than unitary and the elasticity by crop was negatively correlated with the percent of U.S. production for the specific commodity period. The USDA input price indexes were used to produce annual shifts in supply from the beginning year's prices. The variable inputs were entered as being highly elastic. The elasticity of supply of cropland used was close to zero. The solution quantity of each input from the previous year was used with the price index adjusted value to determine an initial point on each of the input categories except cropland, dairy cows, and labor. These were determined within other components of a larger model of Michigan agricultural production which included Cobb-Douglas production function programming model described here.

The simultaneity of the Cobb-Douglas programming equations was relaxed to allow a dynamic programming problem with each year's allocation of land to crop activities being determined prior to solving for quantities of the remaining inputs. In the first step, machinery and capital items usable only in one particular crop activity were held constant. For machinery used in the production of more than one commodity, quantities were held constant but their use was allowed to shift between activities. The per acre quantities of the more variable inputs were assumed to be held at the rate used the preceeding year. After the land allocation step was completed, the land allocation was maintained while solution values for the remaining input allocations and production levels were found. This dynamic Cobb-Douglas program was developed to reflect the behavior of decision makers and the information available to them.

Table one presents model results and corresponding observations of Michigan's agricultural production. The RMS (root-mean-square) percent error [Pindyck and Rubinfeld] is included for each product modeled.

Conclusions

The model sufficiently reflects Michigan's agricultural production to indicate that it captures the causal factors which determine sector performance.

TABLE 1. ACTUAL AND SIMULATED PRODUCTION LEVELS

	CORN RMS % Error = 4.1% (000 bu.)		WHEAT RMS % Error = 4.6% (000 bu.)		DRY BEANS RMS % Error = 3.5% (000 cwt.)		SOY BEANS RMS % Error = 4.9% (000 bu.)		POTATOES RMS % Error = 8.0% (000 cwt.)	
	Actual	Simulated	Actual	Simulated	Actual	Simulated	Actual	Simulated	Actual	Simulated
1955	92.302	101.600	27.966	33.910	4,536.	4,158.	3.036	3.104	5.540	5.642
1956	100.215	113.100	31.290	31.170	5,389.	5,086.	4.326	3.593	7.834	8.345
1957	88.506	105.700	28.739	30.010	3,507.	3,574.	5.412	4.274	6.260	6.452
1958	101.136	104.500	41.420	37.060	5,226.	5,231.	6.394	4.676	8.208	8.181
1959	115.311	104.700	35.584	30.720	6,413.	7,291.	5.782	5.484	7.350	6.174
1960	111.402	100.100	33.642	28.680	6,247.	7,075.	4.420	4.270	7,452	11.970
1961	118.470	106.100	39.996	33.330	7,357.	8,562.	7.410	7.338	9.264	8.993
1962	111.951	99.800	30.063	33.300	7,391.	7,990.	7.695	7.366	8.865	9.699

	SUGAR BEETS RMS % Error = 1.6% (000 ton)		HAY RMS % Error = 3.8% (000 ton)		MILK RMS % Error = 2.5% (000 cwt.)		BEEF RMS % Error = 5.9% (000 cwt.)		HOGS RMS % Error = 2.3% (000 cwt.)	
	Actual	Simulated	Actual	Simulated	Actual	Simulated	Actual	Simulated	Actual	Simulated
1955	882.	857.	3.118	3.074	53.960	52.130	4.720	6.199	2.840	2.765
1956	693.	702.	3.587	3.578	53.650	54.680	4.500	5.563	2.660	2.737
1957	910.	913.	3.343	3.548	52.910	54.900	4.060	4.804	2.370	2.371
1958	1,107.	1,166.	2.927	3.254	52.160	53.830	3.890	4.427	2.460	2.749
1959	1,295.	1,373.	3.491	3.954	50.900	52.650	4.360	4.480	2.740	2.973
1960	945.	990.	3.373	3.828	51.730	51.270	4.410	4.878	2.530	2.605
1961	1,174.	1,237.	3.227	3.769	52.970	49.370	4.540	4.786	2.560	2.657
1962	1,076.	1,145.	3.286	3.615	56.060	46.340	4.670	4.484	2.660	2.918

	EGGS RMS % Error = 4.1% (000 doz.)		HORSES RMS % Error = 0.6% (000 head)		OTHER RMS % Error = 2.4% (Index)	
	Actual	Simulated	Actual	Simulated	Actual	Simulated
1955	1,408.	1,125.	44.	43.	95.000	88.410
1956	1,388.	1,238.	42.	42.	98.000	90.800
1957	1,365.	1,398.	40.	40.	97.000	93.940
1958	1,305.	1,453.	38.	39.	100.000	104.500
1959	1,308.	1,439.	40.	41.	103.000	107.100
1960	1,199.	1,348.	45.	45.	102.000	108.800
1961	1,162.	1,320.	50.	49.	107.000	113.600
1962	1,154.	1,157.	55.	54.	103.000	115.100

Actual production in Michigan is as recorded in Michigan Agricultural Statistics for the years 1955 through 1962, or in the case of horses is an estimate of Michigan horse population. The information listed in the first column for the commodity "other" is the index numbers of farm output for the Great Lakes States [U.S.D.A., p. 15].

Some of the advantages of Cobb-Douglas programming are highlighted by the following comparisons with linear programming. The use of Cobb-Douglas production functions allows an infinite variety in input mix per unit of output. This greatly reduces the need for multiple activities as designated in linear programming to adjust or vary input mix. The Cobb-Douglas production function assumes unitary elasticity of substitution. The linear production function assumes that no substitution of inputs can take place. The linear production function is homogeneous to degree one while the Cobb-Douglas production function is homogeneous to the degree equal to the summation of the exponential values within the production function. Since the exponential values in the Cobb-Douglas production function are equal to the factor shares, a linear programming problem can be shown to be a special case of programming using Cobb-Douglas production functions where all functions are homogeneous to degree one and all prices are either infinitely elastic or infinitely inelastic.

Cobb-Douglas programming can be used for dynamic programming, when this would better reflect actual behavior. The information needed by Cobb-Douglas programming does not involve as many numbers as those required to solve a comparable problem in linear programming.

Cobb-Douglas programming was used here to meet a specific need in a complex model of Michigan's agricultural sector. The generality of the Cobb-Douglas production function and the concept of programming as a management aid combined with the increased flexibility and applications of this formulation imply that its profitable applications are at least as broad as those of linear programming. Further testing of the model framework in additional problem solving applications is necessary before this broad use can be advocated.

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