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CONSISTENCY AND OPTIMALITY IN A DYNAMIC GAME OF POLLUTION CONTROL I: COMPETITION

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ABSTRACT

I model the interaction between a regulator and polluting firms as a Stackelberg differential game in which the regulator leads. The firms create pollution, which results in a stock externality. I analyze the intertemporal effects of alternate pollution control measures in a competitive industry. The principal issue here concerns the dynamic inconsistency of the optimal solution. *Inter alia*, I compare the steady state levels of pollution under optimal and under time consistent policies.

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CONSISTENCY AND OPTIMALITY IN A DYNAMIC GAME OF POLLUTION CONTROL I: COMPETITION

1. Introduction[1]

In environmental economics, until very recently, most analyses of the regulation of externality generating firms have been conducted in a static context. In such a context, the externality problem is typically solved by setting a corrective tax equal in magnitude to the marginal social damage caused by the externality being regulated. This "Pigouvian" approach has a long history—see Meade (1952), Newbery (1980), and Cropper and Oates (1992)—and the Pigouvian tradition is well established in environmental economics. Unfortunately, however, this tradition ignores a fundamental aspect of most contemporary regulatory settings, namely, the interaction between the regulator and the regulated firm *over time*. Indeed, time is a key element in the analysis of regulatory problems in environmental economics. This means that any reasonable analysis of environmental regulation must explicitly account for four features which are germane owing to the dynamic nature of the underlying problem. The first feature concerns the inherent *conflict* in the objectives of the regulating agent and the regulated agent. The second feature pertains to the *ongoing* nature of the interaction between the regulator and the regulated party. Third, the question of the dynamic effects of *alternate* regulatory instruments is relevant. Finally, because the interaction between the regulator and the regulated party is ongoing, the parties in the interaction are forward looking, i.e., the future affects the present. Thus, analyses of environmental regulation in such a context must address the problem of *dynamic inconsistency*

[1]I thank Larry Karp and Maury Obstfeld for comments, and the USDA and the Giannini Foundation for financial support. I have benefitted from the comments of seminar participants at the University of Massachusetts, Amherst. The usual disclaimer applies.

of adopted regulatory policies. While the significance of the first feature is generally well understood, analyses of environmental regulatory problems, which explicitly incorporate *all* four of the above features, have been few and far between.[2]

Given this state of affairs, in this paper I study environmental regulation in a dynamic context, explicitly incorporating in my analysis, all four features mentioned in the above paragraph. An important part of my analysis will consist of studying the effects of alternate price control instruments. While there exists a substantial literature on the effects of price versus quantity control instruments, there has been almost no research on the comparative properties of alternate price control measures. In focusing on prices versus prices, I hope to extend the vast extant literature on prices versus quantities.[3]

I model the interaction between a regulator and a competitive,[4] polluting industry as a deterministic Stackelberg differential game in which the regulator leads.[5] The differential game incorporates two important aspects of the regulator/polluter interaction: first, it explicitly considers the dynamic nature of the interaction; and, second, it recognizes that the game being played by the parties at each instance in time is different owing to the evolution of the state.

The first strand of the analysis in this paper considers dynamically inconsistent policies in a game in which the state, i.e., the stock of pollution, evolves in a manner known to all the

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[2]See Batabyal (1995a, pp. 33-37) for a sampling of papers which involve some study of dynamics, although not of dynamic inconsistency.

[3]The modern literature on prices versus quantities begins with Weitzman (1974). See Batabyal (1995a) for a recent survey of many of the important issues in this literature.

[4]In this paper, competitive means price taking.

[5]A companion paper, Batabyal (1995b), focuses on the monopolistic industry case.

players. A single regulatory regime and two kinds of price control instruments are considered. In every case analyzed, the *production* of a certain good causes pollution. The informational costs of taxing pollution directly are assumed to be prohibitive. Further, in setting policy, the regulator is constrained by the dynamic optimizing behavior of the polluting firms. As such, the regulator taxes the production of the polluting good. The regulator's objective is to maximize the sum of net benefit and tax revenues.[6] The two kinds of policies available to the regulator include a unit tax and an *ad valorem* tax. In comparison with a unit tax, an *ad valorem* tax often results in different: (a) levels of revenue, and (b) welfare effects. Further, depending on the industry structure, these two taxes can have very different effects. For these reasons, I have chosen to analyze the dynamic effects of these two policy instruments. This analysis will involve a comparison of the outcomes of the different games resulting from the use of these two price-control instruments.

As I shall show, an important part of this comparative exercise will turn on the intertemporal consistency of the policies employed by the regulator. Further, practical considerations may favor the use of *ad valorem* taxes over unit taxes. Finally, the efficacy of regulatory action will depend fundamentally on whether firm production costs are related to the stock of pollution.

In what follows, section 2 describes the Stackelberg differential game. Section 3 derives and compares the various open loop policies. In section 4, I derive dynamically consistent

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[6] This kind of objective is fairly standard in environmental economics. See van der Ploeg and de Zeeuw (1992, p. 121) for a similar objective.

policies and compare them to the open loop policies of section 3. Section 5 offers concluding comments and discusses directions for future research.

2. The Stackelberg Differential Game

My model is a variant of one studied by Karp (1984). I shall work with a representative firm which maximizes profits. $P(q)$ is the twice differentiable inverse demand function faced by the firm. I assume that $P'(q) < 0$, where q is the production rate of the firm. I shall think of the firm as facing two kinds of costs associated with production at rate q. The first kind of cost depends on the current stock of pollution. Only a portion of this cost is assumed to be internalized by the firm. As an example of such stock-dependent costs, consider the case of groundwater-based irrigation farming in the San Joaquin Valley of California. Since groundwater is used for irrigation, the cost of farming for any single farmer depends on the pumping activities of other farmers. That is, a single farmer's cost depends on the stock of groundwater. Let $c(x)$ be the internalized average cost of producing one unit of output at time t when the stock of pollution is $x(t)$. Then $c(x)q$ represents the instantaneous, internalized, pollution dependent cost of producing at rate q. I assume that $c'(x) > 0$, $c''(x) > 0$, and that $c(0) = 0$. This stock-dependent cost function is very important, and its properties will have a profound bearing on virtually all my subsequent results.

The second kind of cost is independent of the level of pollution. Let w denote the constant marginal cost of producing at rate q; thus, wq represents the pollution-independent cost of producing at rate q. Let τ_u and ζ_a denote the unit and the *ad valorem* tax, respectively. The firm's payoff in an infinite horizon game in which the regulator uses a unit tax and where r denotes the interest rate is

$$J_F = \int_0^\infty e^{-rt}\{P(q)q - wq - \tau_u q - c(x)q\}\,dt\,. \tag{1}$$

When the regulator uses an *ad valorem* tax, the corresponding firm payoff is

$$J_F = \int_0^\infty e^{-rt}\{\tau_a P(q)q - wq - c(x)q\}\,dt\,, \tag{2}$$

where $\tau_a = 1/(1+\zeta_a)$.

There are three components to the regulator's payoff. A twice differentiable function *B(q)* represents social benefit from production at rate q. *D(x)* is a differentiable function which measures the damage from pollution. In other words, firms create pollution; the level of this pollution at time t is $x(t)$. The function $D(\bullet)$ maps this pollution to a measure of environmental damage for society.[7] I assume that $B'(q) > 0$, $B''(q) < 0$, $D'(x) > 0$, and that $D''(x) > 0$. When the regulator uses a unit tax to control pollution, his payoff is

$$J_R = \int_0^\infty e^{-rt}\{B(q) + \tau_u q - D(x)\}\,dt\,. \tag{3}$$

When he uses an *ad valorem* tax, his payoff is

$$J_R = \int_0^\infty e^{-rt}\{B(q) + (1-\tau_a)P(q)q - D(x)\}\,dt\,. \tag{4}$$

The regulator controls $\tau_u(t)$ and $\tau_a(t)$ and the firm controls *q(t)*. As the leader, the regulator announces a time path for the tax which the firm treats parametrically. Both the regulator and the firm are constrained by the evolution of the stock of pollution which is given by

[7] Also see van der Ploeg and de Zeeuw (1992, p. 121).

$$dx/dt = \dot{x} = q(t), \quad \textbf{(5)}$$

where $x(0) = x_0 > 0$ is given. Equation (5) tells us that the evolution of the stock of pollution is a function of the flow of output. The environment is assumed to be unable to regenerate itself.[8]

Depending on the policy employed by the regulator, different levels of steady state pollution emerge. One can think of these levels as the outcomes of different games. One way of comparing these outcomes is to compare the levels of output and pollution. In this connection, I shall say that game 1 results in less pollution than game 2 iff $x_1^* < x_2^*$, where x_i^*, $i = 1,2$ is the steady state level of pollution in game i, $i = 1,2$. Similarly, I shall say that game 1 results in less output than game 2 iff $q_1^* < q_2^*$, where q_i^*, $i = 1,2$ is the steady state level of output in game i, $i = 1,2$. I can compare the different trajectories of output by deriving a differential equation which the equilibrium $q(t)$ in each game must satisfy. When I am able to compare the trajectories of output without resorting to additional assumptions, I shall do so. Typically though, all my comparisons of pollution and output levels will take place in the steady state. In many cases it will not be possible to obtain general results. In such cases the analysis concentrates on special functional forms.

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[8]A more general state equation of the type $\dot{x} = q(t) - f(x)$, where $f(x)$ is the "regenerative capacity" of the environment, complicates the analysis in two ways. First, the method used in section 4 to obtain dynamically consistent policies fails when the state equation has this additional term. Second, other methods of obtaining dynamically consistent policies, such as the method used in Karp (1991), result in multiple equilibria. Since equilibrium selection is *not* the principal focus of this paper, I have imposed a uniqueness requirement on the equilibrium under study; this requirement is $f(x) \equiv 0$. The imposition of this uniqueness requirement on the above state equation yields (5).

3. The Competitive Industry and Open Loop Taxes

In this section I shall derive the optimal open loop unit and *ad valorem* taxes for the regulatory objectives discussed above. In the class of Stackelberg games considered in this paper, these taxes are dynamically inconsistent except when the pollution dependent cost function is constant. In other words, if the stock dependent cost function is not constant and the regulator is able—at some time $t > 0$—to alter the time path of taxes he committed to at $t = 0$, he would choose to do so. This means that open loop taxes will fail to achieve their policy objectives because a regulator who uses such taxes will not be deemed credible by polluting firms. This is an extremely important fact, and I shall have more to say about dynamic consistency in section 4.

3.1 The Open Loop Unit Tax

I shall solve the regulator's problem using a method due to Chen and Cruz (1972) and Simaan and Cruz (1973a, 1973b). This method solves the regulator's problem when this problem has one control and one state variable. The method shows the dependence of the obtained solution on the initial condition and hence the dynamic inconsistency of this solution. The basic idea is as follows. The regulator treats the firm's first order condition as an ordinary constraint and the firm's costate variable as a state variable. These two conditions along with the requirement that the optimal solution converge to a steady state converts the differential game into a control problem for the regulator. I now illustrate the use of this method.

The first-order necessary conditions for the firm's problem, when the firm treats $\tau_u(t)$ as a parameter, are given by

$$P(q) - w - \tau_u - c(x) + \lambda(t) = 0, \tag{6}$$

and

$$\dot{\lambda} = r\lambda + c^{/}(x)q, \tag{7}$$

where $\lambda(t)$ is the costate variable. The reader should note that (7) represents a jump state constraint.[9] That is, the initial value of λ, $\lambda(0)$ is free and the value of this jump state variable at any arbitrary point in time is determined by current and/or future events. In other words, (7) is *not* a fixed initial state constraint for the regulator. This makes the regulator's problem a nonstandard control problem. Solving for τ_u from (6) and substituting in (3), I get

$$J_R = \int_0^{\infty} e^{-rt}\{B(q) + P(q)q - wq - c(x)q - D(x) + \lambda q\}dt. \tag{8}$$

Equation (8) gives the regulator's payoff as the present discounted stream of benefits and revenues less the sum of costs, pollution damage, and λq. Since λ is the shadow value of the stock of pollution to the firm, λq is the firm's implicit value of polluted air gained by production at rate q. I now want to eliminate λ from (8) by using (7). Integrating (7) and assuming that $\lambda(\infty)$ is finite,[10] I get

$$\lambda(t) = -e^{rt}\int_t^{\infty} e^{-rm}c^{/}(x)q(m)dm. \tag{9}$$

Substituting this value of λ from (9) into (8) I get

$$J_R = \int_0^{\infty} e^{-rt}[B(q) + P(q)q - wq - c(x)q - D(x)]dt - \int_0^{\infty} e^{-rt}[e^{rt}\int_t^{\infty} e^{-rm}c^{/}(x)q(m)dm]q(t)dt. \tag{10}$$

[9]For more on jump state constraints, see Karp and Newbery (1993).

[10]If this condition does not hold, the present method of obtaining open loop controls fails, and alternate methods, such as the one employed in Batabyal (1995b), will have to be used.

Now reversing the order of integration of the last integral in (10), I get

$$-\int_0^\infty e^{-rt}[e^{rt}\int_t^\infty e^{-rm}c'(x)q(m)dm]q(t)dt = \int_0^\infty e^{-rt}c'(x)q(t)\{x_0 - x(t)\}dt. \qquad \textbf{(11)}$$

Using (11), I can now write (10) as

$$J_R = \int_0^\infty e^{-rt}[B(q) + P(q)q - wq - c(x)q + c'(x)q\{x_0 - x(t)\} - D(x)]dt\ . \qquad \textbf{(12)}$$

I have now converted the regulator's problem from one of maximizing (3) over τ_u subject to (5) to one of maximizing (12) over $q(t)$ subject to (5). The first-order necessary conditions to this problem are

$$B'(q) + P'(q)q + P(q) - w - c(x) + c'(x)\{x_0 - x(t)\} + \sigma = 0\,, \qquad \textbf{(13)}$$

and

$$\dot{\sigma}(t) = r\sigma + 2c'(x)q + D'(x) - qc''(x)\{x_0 - x(t)\}\,, \qquad \textbf{(14)}$$

where $\sigma(t)$ is the costate variable. Equation (13) tells us that the solution to the regulator's problem depends on the initial stock of pollution, x_0. Further, (13) also tells us that if the regulator were able to alter his announced time path for the unit tax at any time $\epsilon\epsilon(0, \infty)$, then x_0 in (13) would have to be replaced with $x(\epsilon)$. As such, the ensuing solution $\forall_t > \epsilon$ would not be (13). Hence, this solution is dynamically inconsistent. Before deriving a differential equation which the optimal $q(t)$ satisfies, a comment on the significance of (7) is in order. The reader should note that (7) acts as a rational expectations constraint for the regulator. The rational expectations nature of this constraint stems from the fact that the firm's problem in this Stackelberg game is dynamic.

To find the differential equation which is satisfied by the optimal $q(t)$ with the imposition of the unit tax, I shall differentiate (13) w.r.t. time and then use (14) for simplification. I get

$$\{B^{//}(q)+P^{//}(q)q+2P^{/}(q)\}\dot{q}+r\{w+c(x)-B^{/}(q)-P(q)-P^{/}(q)q-c^{/}(x)(x_0-x)\}+D^{/}(x) = 0, \quad \textbf{(15)}$$

where $q^* = 0$ is the boundary condition for q.[11] From (15) we see that the open loop unit tax is dynamically consistent iff the pollution dependent cost function is constant. When this stock dependent cost function is constant, three results follow. First, the reason for the inconsistency, i.e., the dependence of the solution on the initial condition, disappears. Second, (7) implies that $\lambda(t) \equiv 0$. Third, (6) tells us that the optimal unit tax is now given by the price less the sum of the wage and the average pollution dependent cost, i.e., $\tau_u = P(q) - \{w+c(x)\}$.

To find the equation for the optimal open loop unit tax, I shall differentiate (6) w.r.t. time. I get

$$\dot{\tau}_u - r\tau_u = \{P^{/}(q)\}\dot{q} - r\{P(q) - w - c(x)\}, \quad \textbf{(16)}$$

where $\tau_u^* = P(0) - w - c(x^*)$. In (16), $\dot{q}$ is given by (15). x^* can be obtained as indicated in section 3.3 below. Let $\tau_u^{PV} = e^{-rt}\tau_u$ denote the present value of the unit tax. Substituting $\dot{q}$ from (15) into (16), we see that the present value of the unit tax is increasing as long as $P^{/}(q)\dot{q} > r\{P(q) - w - c(x)\}$.

3.2 The Open Loop Ad Valorem Tax

I can now derive the solution for the open loop *ad valorem* tax with the regulatory objective described in section 2.

[11]* denotes a steady state value.

To maximize (2) subject to (5), I form the firm's current value Hamiltonian. The resulting first-order necessary conditions are

$$\tau_a P(q) - w - c(x) + \lambda(t) = 0 , \tag{17}$$

and (7). Solving for τ_a from (17), substituting in (4), and simplifying the resulting expression, I get

$$J_R = \int_0^\infty e^{-rt}\{B(q) + P(q)q - wq - c(x)q - D(x) + \lambda q\}dt . \tag{18}$$

At this time, a comparison of (8) and (18) is in order. Note that the regulator's payoff is policy invariant. Further, the constraint in both cases is (5). Thus, we can conclude that the optimal unit and *ad valorem* taxes are equivalent, and, hence, they will both give rise to the same time profile of output and pollution. The firm's price with the unit tax is $P(q)-\tau_u$ and with the *ad valorem* tax it is $\tau_a P(q)$. It is easy to verify that $P(q) - \tau_u = \tau_a P(q) = w + c(x) - \lambda(t)$.

3.3 Analysis

Denoting steady state values by "*", (5) tells us that $q^* = 0$, (14) tells us that $\sigma^* = -D'(x^*)/r$, and (7) tells us that $\lambda^* = 0$. Using these values for q^*, σ^*, λ^*, in (13), I find that the steady state level of pollution, x^* solves

$$B'(0) + P(0) - w - c(x^*) - \{D'(x^*)/r\} + c'(x^*)(x_0 - x^*) = 0 . \tag{19}$$

Observe that x^* is *always* dependent on x_0 as long as $c'(x) \neq 0$. That is, in an optimal program, if the stock-dependent cost function is nonconstant, then the steady state level of pollution is a function of the initial level of pollution. I can now state

Proposition 1: The optimal open loop unit and *ad valorem* taxes are both positive at $t = 0$ and at $t = \infty$ as long as $P(\bullet) \neq 0$.

Proof: Using $\lambda(t) \leq 0$, I can write $\tau_a(0) = [\{c(x_0) + w - \lambda(0)\}/P\{q(0)\}] > 0$, and $\tau_a^* = [\{w + c(x^*)\}/P(0)] > 0$. In this last expression, x^* is given by (19). Further, since the unit tax and the *ad valorem* tax are equivalent, I have $\tau_u(0) > 0$ and $\tau_u^* > 0$. ■

Proposition 1 tells us that when the regulator uses policies which display perfect commitment on his part, an optimal program involves setting positive taxes at the beginning and at the end of the game. At the beginning of the game, the regulator knows that he will be able to use the tax trajectory to affect the firm's future behavior. One way to affect the firm's future behavior is to promise that future taxes will be low as long as current pollution is low. This means that the regulator would like to keep the firm's shadow value of pollution, λ, high in the beginning of the game.[12] However, as (13) and (15) showed us, at some $t > 0$, the regulator would like to deviate from his announced policy trajectory at $t = 0$ and *decrease* the valuation of pollution by the firm. One way to do this is to set higher taxes than those announced at the beginning of the game. Further, inspection of (6) and (17) tells us that in general—for any $t \in (0,\infty)$—it is not optimal to set zero taxes.

Since the optimal unit and *ad valorem* taxes are equivalent in a competitive industry, a comparative exercise is not relevant. I state the following general result which is of some interest, especially when compared with the corresponding result for a monopolistic industry, contained in Batabyal (1995b).

Proposition 2: When the regulator uses both taxes simultaneously, one is redundant.

[12]Recall that $\lambda \leq 0$. So keeping λ high means keeping it low in absolute value.

Proof (Outline): This follows from the fact that in a competitive market, the two taxes are equivalent. ■

Proposition 2 tells us that when the regulator chooses to use both taxes simultaneously, it is optimal for him to set either $\tau_u = 0$ or $\zeta_a = 0$. I now discuss the implications of using dynamically consistent taxes when the underlying market structure is competitive.

4. The Competitive Industry and Dynamically Consistent Taxes

I begin with a brief synopsis of dynamically consistent policies. The problem with inconsistent policies, i.e., open loop policies, is that such policies are not credible. In other words, forward-looking firms will recognize that at $t = 0$, the regulator will set a policy trajectory from which he will later want to deviate. Thus, such a policy trajectory will not be believed by firms, and, hence, the original policy will fail to achieve its objectives. This lack of credibility of open loop policies provides a rationale for the study of dynamically consistent policies.

I shall obtain consistent controls by using a method employed in Karp (1984, pp. 87-88).[13] While other methods—see Karp (1991)—for obtaining consistent controls do exist, there are two basic advantages to the Karp (1984) method. First, it makes the logic of the solution transparent. Second, this method facilitates the comparison of results obtained in section 3 with the results to be derived in this section. The basic idea of Karp's procedure is as follows. In a Stackelberg game, it must be possible to use the follower's first-order condition to eliminate the leader's

[13]This method is essentially identical to a "loss of leadership" method proposed by W. Buiter. For more details, see Buiter (1989) and the references cited therein.

control from his objective functional. When this has been done and the leader's problem has the form

$$J_R = \max_{q(t)}\int_0^{\infty}[e^{-rt}\{g(q,x)\} + hq(t)\lambda(t)]dt\,,\ h\in\mathbb{R}\,, \qquad \textbf{(20)}$$

$$\dot{x} = q(t)\,,\ x(0) = x_0 > 0\,, \qquad \textbf{(21)}$$

$$\dot{\lambda} = r\lambda + c'(x)q(t)\,,\ \lambda(t) \le 0\,, \qquad \textbf{(22)}$$

where $g(q, x)$, in my case, is a linear combination of the derivatives of the benefit, damage, inverse demand, and stock dependent cost functions, one can obtain consistent controls by using

Theorem 1: When the leader's problem has the form given by (20)-(22), dynamically consistent controls can be found by solving

$$\hat{J}_R = \max_{q(t)}\int_0^{\infty}e^{-rt}g(q,x)dt\,, \qquad \textbf{(23)}$$

subject to (21). Theorem 1 can be proved as in Karp (1984, pp. 94-96). Note that while the proof requires that the function multiplying the follower's costate variable be linear in the leader's control, the proof does not depend on h or the follower's costate variable being nonnegative.

Put differently, in the class of problems that can be stated as (20)-(22), the leader obtains dynamically consistent controls by disregarding the effect that the follower's marginal value of the state has on his own payoff. The logical basis of this procedure is as follows. One way to eliminate the inconsistency of the open loop solutions of section 3 lies in eliminating the term which makes the solution dependent on x_0 This can be done in two ways. The first approach is to posit that the pollution-dependent cost function is constant. Then $c'(x)=0$ and the source of the inconsistency disappears. However, this is a strong and, *a priori*, unrealistic restriction. The second approach lies in making $(x_0 - x)$ vanish. This is exactly what the above described procedure does "... by treating the [regulator's] problem as [a] sequence of short open loop problems, which in the limit becomes an infinite sequence of static optimization problems" (Karp, 1982, p. 117).

Intuitively, we can think of a regulator who revises his tax policy whenever air quality declines by some predetermined amount. The idea is to let this predetermined amount and, hence, the time interval between successive revisions approach zero. When the regulator does not commit to a specific tax trajectory at the beginning of the game but continuously revises his tax policy, x_0 in $(x_0 - x)$ must be replaced by $x(t)$. When this is done, $(x_0 - x)$ vanishes and the resulting solution is dynamically consistent.

It should be noted that dynamically consistent controls always result in a lower payoff to the leader than do open loop controls except when the two kinds of controls coincide. This stems from the fact that forcing the controls to satisfy the principle of optimality completely eliminates any gain accruing to the leader from setting policy once at the beginning of the game. Alternately put, when the leader uses consistent controls, his "... period of commitment [shrinks] to zero ..." (Buiter, 1989, p. 244). In a manner analogous to Karp (1984, p. 88). the claim in this paragraph can be verified formally by observing that

$$\max_{q(t)}[\int_0^{\infty}\{e^{-rt}g(q,x) + \lambda q\}dt] \;\geq\; \{\max_{q(t)}[\int_0^{\infty}e^{-rt}g(q,x)dt]\} + \int_0^{\infty}\hat{q}\hat{\lambda}dt\,, \qquad \textbf{(24)}$$

where $\hat{q}$ and $\hat{\lambda}$ are the optimized values of the output rate and the follower's marginal value of the state which arise from the solution to the maximization problem on the RHS of (24). The constraints for both problems are the same and are given by (21) and (22). Equality in (24) holds iff $\hat{\lambda} = 0$, a condition which holds when the pollution-dependent cost function is constant. When this last condition holds, the open loop and the dynamically consistent controls coincide. I now obtain dynamically consistent controls, in turn, when the regulator uses a unit tax and then when the regulator uses an *ad valorem* tax.

4.1 The Dynamically Consistent Unit Tax

When the regulator uses an unit tax, his problem is to solve

$$\max_{q(t)} \int_0^\infty e^{-rt}\{B(q) + P(q)q - wq - c(x)q - D(x)\}dt \,, \tag{25}$$

subject to (5). The first-order necessary conditions to this problem are

$$B'(q) + P'(q)q + P(q) - w - c(x) + \sigma = 0 \,, \tag{26}$$

and

$$\dot{\sigma} = r\sigma + c'(x)q + D'(x) \,, \tag{27}$$

where σ is the costate variable associated with (5). The maximizing rate of output solves

$$\{B''(q) + P''(q)q + 2P'(q)\}\dot{q} + r\{w + c(x) - B'(q) - P'(q)q - P(q)\} + D'(x) = 0 \,, \tag{28}$$

with boundary condition $q^* = 0$.

4.2 The Dynamically Consistent Ad Valorem Tax

The solution for the consistent *ad valorem* tax can be obtained in an analogous manner. The firm's first-order necessary condition is

$$\tau_a P(q) = w + c(x) - \lambda(t) \,. \tag{29}$$

The regulator's problem now is to solve

$$\max_{q(t)} \int_0^\infty e^{-rt}\{B(q) + P(q)q - wq - c(x)q - D(x)\}dt \,. \tag{30}$$

As in section 3, I note that (25) and (30) are identical; in both problems, the constraint is (5). I conclude that in a competitive industry, the consistent unit and *ad valorem* taxes are equivalent. The maximizing rate of output—when the regulator uses an *ad valorem* tax—solves (28). Using (26) and the relevant steady state values, I find that the steady state level of pollution, x^* solves

$$B'(0) + P(0) - w - c(x^*) - D'(x^*)/r = 0\,. \qquad \textbf{(31)}$$

A comparison of (31) with (19) reveals the essential differences in x^* stemming from the use of the consistent tax as opposed to the open loop tax.

4.3 Analysis

I can now compare the steady state pollution and output effects of the two policy instruments. Recall that $d^* = 0$ in every case. I shall first compare the open loop policies with the dynamically consistent policies.

The results of this comparative exercise are summarized in Table 1. The two equations that I shall use to compare the pollution levels with the open loop unit tax and with the dynamically consistent unit tax are (19) and (31). The subsequent analysis concentrates on special functional forms. If the relevant functions in (19) and (31) are arbitrary but $c'(x)=0$, then a comparison of (19) and (31) tells us that the open loop unit tax and the dynamically consistent unit tax both give rise to the same level of pollution. Using $B(q)=\gamma q-(1/2)q^2$, $D(x)=(1/2)\delta x^2$, $P(q)=a-bq$, $c(x)=\alpha_1 x$ in (19) and (31) and assuming that $\gamma>w$, I find that the open loop unit tax leads to a lower (higher) level of pollution as compared to the dynamically consistent unit tax depending on whether $\alpha_1 x_0/\{2\alpha_1 + \delta/r\}$ <(>) $[[\{\gamma + a - w\}/\{\alpha_1 + \delta/r\}] - [\{\gamma + a - w\}/\{2\alpha_1 + \delta/r\}]]$. Now consider

$B(q) = \gamma q - (1/2)q^2$, $D(x) = \delta x$, $P(q) = a - bq$, $c(x) = \alpha_1 x$. If $(\gamma + a) > \{w + (\delta/r)\}$, then, as compared to the consistent unit tax, the open loop unit tax leads to a lower (higher) level of pollution as $\alpha_1 x_0/2\alpha_1 <(>) [[\{\gamma + a - w - (\delta/r)\}/\alpha_1] - [\{\gamma + a - w - (\delta/r)\}/2\alpha_1]]$. Finally, consider $B(q) = \gamma q - (1/2)q^2$, $D(x) = \delta x$, $P(q) = q^{-\alpha}$, $\alpha \in (0,1)$, $c(x) = \alpha_1 x$ and let $\gamma > \{w + (\delta/r)\}$. Then as opposed to the consistent unit tax, the open loop unit tax leads to a lower (higher) level of pollution as $\alpha_1 x_0/2\alpha_1 <(>) [[\{\gamma - w - (\delta/r)\}/\alpha_1] - [\{\gamma - w - (\delta/r)\}/2\alpha_1]]$. Inspection of (19) and (31) tells us that the effects of these two taxes essentially depend on the properties of the pollution-dependent cost function. The other functions affect both the equations in a similar manner. A comparison of the open loop *ad valorem* tax and the dynamically consistent *ad valorem* tax is not germane since these two taxes are equivalent, in turn, to the open loop unit tax and the dynamically consistent unit tax.

I shall now use Table 1 and specific parameter values to: (a) illustrate the analysis of the previous paragraph, and (b) determine whether the consistent unit tax always leads to a higher steady state pollution. Consider the functional forms in the first row of Table 1. When $a = \alpha_1 = \delta = 1$, $x_0 = 10$, $w = 4$, $\gamma = 5$, and $r = 5\%$, we see that, as compared to the consistent unit tax, the open loop unit tax leads to higher steady state pollution. Holding fixed the values of the other parameters, let us increase the magnitude of the cost and damage parameters to $\alpha_1 = \delta = 4$. Once again the open loop unit tax leads to higher steady state pollution. Now consider an altered configuration of parameters. Let $a = \alpha_1 = \delta = 1$, $x_0 = 3$, $w = 4$, $\gamma = 100$, and let $r = 5\%$. We see that when the benefit parameter is relatively high and the initial level of pollution relatively low, the

Table 1

Steady State Pollution Effects of the Open Loop Unit Tax versus the Dynamically Consistent Unit Tax

Functional Forms	Restrictions on Parameters	Open Loop Unit Tax
$B(q)=\gamma q-(1/2)q^2$, $D(x)=(1/2)\delta x^2$ $P(q)=a-bq,\ c(x)=\alpha_1 x$	$\gamma>w$ and $\alpha_1 x_0/\{2\alpha_1+\delta/r\}<$ $[[\{\gamma+a-w\}/\{\alpha_1+\delta/r\}]-$ $[\{\gamma+a-w\}/\{2\alpha_1+\delta/r\}]]$	Lower Pollution
$B(q)=\gamma q-(1/2)q^2$ $D(x)=\delta x$ $P(q)=a-bq,\ c(x)=\alpha_1 x$	$\gamma+a>\{w+(\delta/r)\}$ and $\alpha_1 x_0/2\alpha_1<$ $[[\{\gamma+a-w-(\delta/r)\}/\alpha_1]-$ $[\{\gamma+a-w-(\delta/r)\}/2\alpha_1]]$	Lower Pollution
$B(q)=\gamma q-(1/2)q^2$ $D(x)=\delta x$ $P(q)=q^{-\alpha},\ \alpha\in(0,1)$ $c(x)=\alpha_1 x$	$\gamma>\{w+(\delta/r)\}$ and $\alpha_1 x_0/2\alpha_1<$ $[[\{\gamma-w-(\delta/r)\}/\alpha_1]-$ $[\{\gamma-w-(\delta/r)\}/2\alpha_1]]$	Lower pollution

consistent unit tax leads to higher steady state pollution. Next consider the functional forms in the second row of Table 1. Let $a = \alpha_1 = \delta = 1$, $x_0 = 10$, $w = 4$, $\gamma = 90$, and let $r = 5\%$. Now the consistent unit tax leads to higher steady state pollution. However, when we keep the other parameter values fixed and increase the cost and damage parameters to $\alpha_1 = \delta = 4$, we see that now it is the open loop unit tax which results in higher steady state pollution. Finally, consider the functional forms in the third row of Table 1. Let $\alpha_1 = \delta = 1$, $x_0 = 10$, $w = 4$, $\gamma = 90$, and let $r = 5\%$. In this case, the consistent unit tax leads to higher steady state pollution. However, when we leave the values of the other parameters unchanged and increase the cost and damage parameters to $\alpha_1 = \delta = 4$, we find that on this occasion it is the open loop unit tax which results in higher steady state pollution. This analysis demonstrates that the consistent unit tax *does not always* lead to higher steady state pollution. Indeed, there are parametric configurations for which the consistent unit tax actually leads to lower steady state pollution.

The analysis of this section tells us that the regulator's payoff is lower with continuous policy revision; further, there exist circumstances in which the use of dynamically consistent taxes leads to a higher level of pollution. As compared to the open loop tax, the higher pollution implies a higher level of social damage because $D'(x) > 0$. On the other hand $q^* = 0$ in all the cases analyzed. Thus, higher pollution implies lower social welfare—as embodied in the regulator's objective functional—in the steady state.[14] While the use of other functional forms also leads to interpretable results, the analysis of this section suffices to demonstrate the sensitivity of the results to the choice of functional form.

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[14]This result does not hold in models with state equations more complicated than (5).

In a competitive market, owing to the equivalence of the consistent unit and *ad valorem* taxes, we see the same time profile for output and pollution. As such, a comparative exercise is not necessary.[15]

My next task in this section is to rank the steady state pollution levels with all four taxes. Table 2 summarizes the results of this paragraph. In the rest of this paragraph, $a > b$ means that a leads to a higher level of pollution than does b, and $a \Leftrightarrow b$ means that a and b both give rise to the same level of pollution. Denote the open loop unit tax, the open loop *ad valorem* tax, the dynamically consistent unit tax, and the dynamically consistent *ad valorem* tax by τ_u^{OL}, τ_a^{OL}, τ_u^{DC}, and τ_a^{DC}, respectively. Recall that in a competitive industry, $\tau_u^{OL} \Leftrightarrow \tau_a^{OL}$ and $\tau_u^{DC} \Leftrightarrow \tau_a^{DC}$ hold. Let $c(x) = \alpha_1$. Then using this functional form and comparing (19) and (31), we see that $\tau_u^{OL} \Leftrightarrow \tau_a^{OL} \Leftrightarrow \tau_u^{DC} \Leftrightarrow \tau_a^{DC}$ holds. Using $B(q) = \gamma q - (1/2)q^2$, $D(x) = \delta x$, $c(x) = \alpha_1 x$, $P(q) = a - bq$ in (19) and (31) and assuming that $(\gamma + a) > \{w + (\delta/r)\}$, then $\tau_u^{OL} \Leftrightarrow \tau_a^{OL} <(>) \tau_u^{DC} \Leftrightarrow \tau_a^{DC}$ as $(\alpha_1 x_0/2\alpha_1) <(>) [[\{\gamma + a - w - (\delta/r)\}/\alpha_1] - [\{\gamma + a - w - (\delta/r)\}/2\alpha_1]] \equiv \Omega$. Next, let $B(q) = \gamma q - (1/2)q^2$, $D(x) = \delta x$, $c(x) = \alpha_1 x$, and let $P(q) = q^{-\alpha}$, $\alpha \in (0, 1)$. If $\gamma > \{w + (\delta/r)\}$, then $\tau_u^{OL} \Leftrightarrow \tau_a^{OL} <(>) \tau_u^{DC} \Leftrightarrow \tau_a^{DC}$ as $(\alpha_1 x_0/2\alpha_1) <(>) [[\{\gamma - w - (\delta/r)\}/\alpha_1] - [\{\gamma - w - (\delta/r)\}/2\alpha_1]] \equiv \Pi$. This analysis once again clearly brings out the sensitivity of the qualitative results to the choice of functional form and in particular to the properties of the stock dependent cost function.

I close this section by asking at what level the two taxes should be set when the regulator chooses to use both dynamically consistent taxes simultaneously. The answer is contained in

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[15] In a monopolistic industry, this equivalence breaks down. See Batabyal (1995b) for details.

Table 2

Steady State Pollution Rankings of the Alternate Policy Instruments

Market Structure	Functional Forms	Restrictions on Parameters	Ranking of Instruments
Competition	$B(q)=\gamma q-(1/2)q^{2}$ $D(x)=\delta x$ $P(q)=a-bq$ $c(x)=\alpha_1 x$	$\gamma+a>\{w+(\delta/r)\}$ and $\alpha_1 x_0/2\alpha_1<\Omega$	$\tau_u^{OL}\Leftrightarrow\tau_a^{OL}<$ $\tau_u^{DC}\Leftrightarrow\tau_a^{DC}$
Competition	$B(q)=\gamma q-(1/2)q^{2}$ $D(x)=\delta x$ $P(q)=q^{-\alpha},\ \alpha\in(0,\ 1)$ $c(x)=\alpha_1 x$	$\gamma>\{w+(\delta/r)\}$ and $\alpha_1 x_0/2\alpha_1<\Pi$	$\tau_u^{OL}\Leftrightarrow\tau_a^{OL}<$ $\tau_u^{DC}\Leftrightarrow\tau_a^{DC}$

Proposition 3: When the industry is competitive and the regulator uses both taxes simultaneously, one of the two taxes is redundant.

Proof (Outline): This follows from the fact that in a competitive industry, the two taxes are equivalent.■

The reader will note that while continuous revision of the tax by the regulator alters the solution to his optimization problem, it does not alter his optimal course of action when he chooses to use both taxes simultaneously.

5. Conclusions

In this paper I formulated and analyzed the interaction between polluting firms and a regulator as a Stackelberg differential game in which the regulator leads. I analyzed the impact of unit and *ad valorem* taxes in a competitive industry. Specifically, I derived open loop and dynamically consistent policies for the regulator. I illustrated the dynamic inconsistency of open loop policies, and I pointed out the equivalence of open loop and consistent policies when production costs are unrelated to the stock of pollution. Further, I demonstrated the equivalence of the unit tax and the *ad valorem* tax in a competitive market.

By means of numerous steady state examples, I showed how one might interpret the general results, and then I ranked the four taxes in terms of their ability to control pollution. These examples demonstrate the sensitivity of the qualitative results to: (a) the choice of functional forms for inverse demand and stock dependent costs, and (b) the nature of the taxes.

Four main policy conclusions follow from the analysis of this paper. First, owing to the sensitivity of the results to the choice of functional forms, in any given regulatory scenario, empirical research will be needed to estimate the parameters of the relevant functions and, hence, serve as a guide to regulatory action. Second, a practical consideration which might favor the use of *ad valorem* taxes *ceteris paribus* is that, compared to unit taxes, it is often easier to incorporate the effects of factors such as inflation in *ad valorem* taxes. Third, as far as policy credibility is concerned, the efficacy of regulatory action depends on the properties of the stock dependent cost function. If production costs are unrelated to the stock of pollution, then the open loop and the dynamically consistent solutions coincide. As such, it makes no difference whether the regulator announces a policy trajectory at the beginning of the game or whether he continuously revises his policy. Fourth, there is a basic tradeoff between policy credibility and policy payoff. Consistent policies yield a lower payoff than do open loop policies. This is a possible explanation as to why many regulators are loath to use consistent policies.

I believe that the two most promising extensions of this line of research lie in: (a) generalizing the analysis to imperfectly competitive markets, and (b) analyzing the various issues in a stochastic framework. I have largely completed the task listed in (a) above. The results of that analysis are to be found in Batabyal (1995b). I am currently at work on the task listed in (b) above, and I hope to report the results of my research shortly.

References

Batabyal, A.A. (1995a). “Leading Issues in Domestic Environmental Regulation: A Review Essay.” *Ecological Economics* 12:23-39.

Batabyal, A.A. (1995b). “Consistency and Optimality in a Dynamic Game of Pollution Control II: Monopoly.” Unpublished manuscript.

Buiter, W.H. (1989). “Policy Evaluation and Design for Continuous Time Linear Rational Expectations Models: Some Recent Developments.” In W.H. Buiter (ed.), *Macroeconomic Theory and Stabilization Policy*, University of Michigan Press.

Chen, C.I., and J.B. Cruz (1972). “Stackelberg Solutions for Two Person Games with Biased Information Patterns.” *IEEE Transactions on Automatic Control* AC-17:791-798.

Cropper, M.L., and W.E. Oates (1992). “Environmental Economics: A Survey.” *Journal of Economic Literature* 30:675-740.

Karp, L.S. (1982). “Dynamic Games in International Trade.” Ph.D. dissertation, University of California, Davis.

Karp, L.S. (1984). “Optimality and Consistency in a Differential Game with Non-Renewable Resources.” *Journal of Economic Dynamics and Control* 8:73-97.

Karp, L.S. (1991). “Monopsony Power and the Period of Commitment in Nonrenewable Resource Markets.” In L. Philips (ed.), *Commodity, Futures, and Financial Markets*, Kluwer Academic Publishers.

Karp, L.S., and D.M. Newbery (1993). “Intertemporal Consistency Issues in Depletable Resources.” In *Handbook of Natural Resource and Energy Economics, Vol. III*, Elsevier Science Publishers.

Meade, J.E. (1952). “External Economies and Diseconomies in a Competitive Situation.” *Economic Journal* 62:54-70.

Newbery, D.M.G. (1980). “Externalities: The Theory of Environmental Policy.” In G.A. Hughes and G.M. Heal (eds.), *Public Policy and the Tax System*, Boston: G. Allen and Unwin.

Simaan, M., and J.B. Cruz (1973a). “On the Stackelberg Strategy in Nonzero Sum Games.” *Journal of Optimization Theory and Applications* 11:533-555.

Simaan, M., and J.B. Cruz (1973b). "Additional Aspects of the Stackelberg Strategy in Nonzero Sum Games." *Journal of Optimization Theory and Applications* 11:613-626.

Van der Ploeg, F., and A. de Zeeuw (1992). "International Aspects of Pollution Control." *Environmental and Resource Economics* 2:117-139.

Weitzman, M.L. (1974). "Prices vs. Quantities." *Review of Economic Studies* 41:477-491.