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Efficiency analysis under uncertainty: a simulation study*

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We model production technology in a state-contingent framework assuming that the firms maximise *ex ante* their preference function subject to stochastic technology constraint; in other words, firms are assumed to act rationally. We show that rational producers who face the same stochastic technology can make significantly different production choices. Further, we develop an econometric methodology to estimate the risk-neutral probabilities, efficiency scores and the parameters of stochastic technology when there are two states of nature and only one of which is observed. Finally, we simulate noiseless data based on our state-contingent specification of technology. Our state-contingent estimator recovers technology parameters and other economic quantities of interest without any error. But, when we apply conventional efficiency estimators to the simulated data, we obtain biased estimates of technical efficiency.

Key words: data envelopment analysis, risk-neutral, state-contingent, stochastic frontier analysis.

1. Introduction

The main difficulty in identifying changes in efficiency is to be able to differentiate between real changes in efficiency and changes due to stochastic shocks, such as interruptions in production due to natural calamity such as drought or flood. In existing empirical work, these two sources of variation cannot be separated out without making highly restrictive distributional assumptions (Bhattacharyya *et al*. 1995). A major shortcoming of conventional (see Just and Pope 1978; Love and Buccola 1999; Kumbhakar 2002) efficiency analysis is that it does not explicitly take into account the substitutability of inputs (and the outputs) between the potential states of nature. This shortcoming can be overcome by modelling uncertainty in a state-contingent framework. The main theory behind state-contingent production has been explained in a series of papers (Chambers and Quiggin 1998, 2002; Quiggin and Chambers 2006) and a monograph by Chambers and Quiggin (2000).

O'Donnell and Griffiths (2006) econometrically estimate production frontiers of rice farmers in the Philippines in a state-contingent framework. They find that around three-quarters of average estimated output shortfalls were due to unfavourable seasonal conditions and only one-quarter to inefficiency.

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Stochastic frontier approaches, on the other hand, suggest what would seem to be unreasonably high levels of inefficiency. However, a limitation of their approach is that they assume the technology to be output cubicle, thereby ruling out substitution of inputs between the various states of nature. This limitation has been overcome by Nauges *et al.* (2011), who estimate a technology that is state-general in nature. Using farm data for Finland, they estimate state-general specification of technology that permits substitutability between state-contingent outputs. The estimated technical efficiency scores using their model were higher than those estimated using conventional frontier models.

In efficiency studies, the efficient frontier is generally specified by either a primal or dual representation of a nonstochastic technology, even though most production technologies are stochastic in nature. Efficiency is then measured relative to an estimate of this deterministic frontier. Data envelopment analysis (DEA[1]) and stochastic frontier (SFA[2]) analysis are two most commonly used estimation techniques to measure efficiency.

O'Donnell *et al.* (2010) and Shankar and Quiggin (2013) show that if the decision-making environment is inherently uncertain, then the conventional estimators provide us with biased estimates of production technology and the elasticity of scale. In this paper, using the same specification of technology as in Shankar (2013), we show that the conventional techniques used to analyse efficiency are found to be deficient in modelling production under uncertainty.

The paper is organised as follows. Section 2 develops an estimation methodology to recover efficiency levels, the parameters of a two-state stochastic technology and the risk-neutral probabilities of every firm in the sample, when only one of two state-contingent outputs is observed. Section 3 uses noiseless simulated data to demonstrate that our estimation methodology can be used to recover unknown parameters and other economic quantities of interest without error. Then, we apply conventional efficiency measurement techniques such as DEA, SFA, state-dependent DEA (DEAS) and output-cubical (OC) frontier estimators to the simulated data and discover that it gives us biased estimates of efficiency levels and the parameters of the production technology. Further, in Section 3, we simulate several data sets for a group of rational firms each of which use just one input to produce a single output and evaluate the performance of conventional DEA, SFA, DEAS and OC frontier estimators. Finally, we offer some concluding comments in Section 4.

2. Estimation methodology for the two-state case

We model production using a constant elasticity of substitution (CES) specification of technology, where the relationship between the total input used *ex ante* across various states of nature and the *ex post* realisation of stochastic output is given by

[1] For example, see Seiford and Thrall (1990).
[2] For example, see Aigner *et al.* (1977).

$$x = (a_1 z_1{}^b + a_2 z_2{}^b)^{\gamma/b} \tag{1}$$

where $a_s \geq 0$ can be either interpreted as a technology parameter related to production of output in state of nature $\{s\}$ or it can be conceived as a realisation of an unobserved random variable determined by nature *ex post*. The parameter b is a transformation of elasticity of substitution and is referred to as substitution parameter, and the parameter γ represents economy of scale.

A key element of the estimation methodology is the relationship between the risk-neutral probabilities and the observable variables, and in Section 2.1, we derive this relationship. In Section 2.2, we specify the econometric model and the likelihood function.

2.1. Deriving risk-neutral probabilities

We assume that the firms seek to maximise their utility function $W(y)$ where $\mathbf{y} = (y_1, y_2)$ and $y_s = z_s - wx$, $s \in \Omega$ is the *ex post* net return in the state of nature $\{s\}$. The utility function W is continuously differentiable, nondecreasing and quasi-concave in its arguments.

Following Shankar (2013) (see p. 144), for the CES specification, the risk-neutral probability π_s of a firm in state s can be written as

$$\pi_s - \gamma w a_s z_s{}^{b-1} x^{\frac{\gamma-b}{\gamma}} = 0 \quad s \in \Omega = \{1, 2\} \tag{2}$$

where the risk-neutral probability π_s of a firm in state s is given by

$$\pi_s \equiv \frac{W_s(\mathbf{y})}{\sum\limits_{j \in \Omega} W_j(\mathbf{y})} \in (0, 1) \tag{3}$$

π_s is referred to as risk-neutral probability in state of nature $\{s\}$, as it represents the subjective probability that a risk-neutral firm would require in order to make the same production choices as a rational firm with preferences W.

We can write (2) in terms of state-contingent output as

$$z_s = \left(\frac{\pi_s}{\gamma w a_s}\right)^{\frac{1}{b-1}} x^{\frac{b-\gamma}{\gamma(b-1)}} \quad s \in \Omega = \{1, 2\} \tag{4}$$

The risk-neutral probabilities in states of nature $\{1\}$ and $\{2\}$ can be written in a compact form as

$$\pi_1 = e_1[\gamma w a_1 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] + e_2[1 - \gamma w a_2 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] \tag{5}$$

$$\pi_2 = e_1[1 - \gamma w a_1 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] + e_2[\gamma w a_2 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] \tag{6}$$

where $q = e_1 z_1 + e_2 z_2$ is the observed output; $e_s = 1$ if state of nature $\{s\}$ is realised *ex post*, $s \in \Omega = \{1, 2\}$ (and 0 otherwise).

2.2. Estimating productivity and efficiency

In a real world, many firms are not fully efficient. Therefore, we accommodate for inefficiency by using an input distance function given by

$$D_I(x, z_1, z_2, a_1, a_2, b, \gamma) = \frac{x}{(a_1 {z_1}^b + a_2 {z_2}^b)^{\gamma/b}} \geq 1 \tag{7}$$

Substituting for z_1 and z_2 using (4) in (7) we have

$$x = \frac{\{\sum_{s\in\Omega} a_s (\frac{\pi_s}{\gamma w a_s})^{\frac{b}{b-1}}\}^{\frac{\gamma(b-1)}{b(\gamma-1)}}}{\mathrm{TE}} \quad s \in \Omega = \{1, 2\} \tag{8}$$

where TE represents technical efficiency. Taking a logarithm on both sides of (8) and substituting for risk-neutral probabilities in (8) using (2) and $\pi_s = 1 - \sum_{j\in\Omega\backslash\{s\}} \pi_j$; $s \in \Omega = \{1, 2\}$, we have

$$\ln q - \frac{1}{\gamma} \ln x + e_1 \left\{ \frac{1}{b} \ln a_1 + r_1(q, w, x, \boldsymbol{\beta}) \right\} + e_2 \left\{ \frac{1}{b} \ln a_2 + r_2(q, w, x, \boldsymbol{\beta}) \right\} = -u \tag{9}$$

where

$$r_1(q, w, x, \boldsymbol{\beta}) = \frac{1}{b} \ln \left(1 + \frac{a_2 [\frac{1 - w a_1 \gamma q^{b-1} x^{\frac{\gamma-b}{\gamma}}}{w a_2 \gamma}]^{\frac{b}{b-1}} x^{\frac{b(b-\gamma)}{\gamma(b-1)}}}{a_1 q^b} \right), \tag{10}$$

$$r_2(q, w, x, \boldsymbol{\beta}) = \frac{1}{b} \ln \left(1 + \frac{a_1 [\frac{1 - w a_2 \gamma q^{b-1} x^{\frac{\gamma-b}{\gamma}}}{w a_1 \gamma}]^{\frac{b}{b-1}} x^{\frac{b(b-\gamma)}{\gamma(b-1)}}}{a_2 q^b} \right), \tag{11}$$

$$\boldsymbol{\beta} = (\gamma, b, a_1, a_2)' \tag{12}$$

and $u \geq 0$ is a measure of *technical inefficiency*. Further, $q = e_1 z_1 + e_2 z_2$ is the observed ouput; $e_s = 1$ if state of nature $\{s\}$ is realised *ex post*, $s \in \Omega = \{1, 2\}$ (and 0 otherwise).

Since the risk-neutral probabilities must lie on a unit interval, we have the following restriction on parameters in Equation (9):

$$0 \leq e_1 [\gamma w a_1 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] + e_2 [1 - \gamma w a_2 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] \leq 1 \tag{13}$$

An associated econometric estimating equation is as follows:

$$\begin{aligned} \ln q_{nt} - \frac{1}{\gamma} \ln x_{nt} + e_{1nt} \{ \frac{1}{b} \ln a_1 + r_{1nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta}) \} \\ + e_{2nt} \{ \frac{1}{b} \ln a_2 + r_{2nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta}) \} = v_{nt} - u_{nt} \end{aligned} \tag{14}$$

where the subscripts n and t represent firms and time periods, respectively, (n = 1,…, N; t = 1, …, T), and v_{nt} is a random variable representing statistical noise.

Equation (14) can be rewritten in a compact form as follows:

$$r_{nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta}) = \varepsilon_{nt} = v_{nt} - u_{nt} \tag{15}$$

where using (10) and (11), $r_{nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})$ is given by

$$\begin{aligned} r_{nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta}) = \ln q_{nt} - \frac{1}{\gamma}\ln x_{nt} + e_{1nt}\left\{\frac{1}{b}\ln a_1 + r_{1nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})\right\} \\ + e_{2nt}\left\{\frac{1}{b}\ln a_2 + r_{2nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})\right\} \end{aligned} \tag{16}$$

We make the following assumptions about the error terms v_{nt} and u_{nt} in (15):

1. v_{nt} is a symmetrically distributed random variable; and
2. v_{nt} and u_{nt} are independently and identically distributed across observations.

Following the methodology outlined in Aigner *et al.* (1977), the density function of the compound error $\boldsymbol{\varepsilon}_{nt}$ can be written as follows:

$$f_\varepsilon(\varepsilon_{nt}) = \frac{2}{\sqrt{2\pi(\sigma_u^2 + \sigma_v^2)}}\left[\Phi\left(\frac{-\varepsilon_{nt}(\sigma_u/\sigma_v)}{\sqrt{(\sigma_u^2 + \sigma_v^2)}}\right)\right]\exp\left(\frac{-\varepsilon_{nt}^2}{2(\sigma_u^2 + \sigma_v^2)}\right) \tag{17}$$

where $\boldsymbol{\varepsilon}_{nt} = v_{nt} - u_{nt}$, v_{nt} is normally distributed with mean 0 and variance σ_v^2, that is $v_{nt} \sim N(0, \sigma_v^2)$, and u_{nt} is half-normally distributed, that is $u_{nt} \sim \left|N(0, \sigma_u^2)\right|$.

In order to provide intuitive interpretation, the variances in the above equation are re-parameterised such that $\sigma^2 = \sigma_u^2 + \sigma_v^2$ and $\lambda = \sigma_u/\sigma_v$. Then, (17) can be rewritten as follows:

$$f_\varepsilon(\varepsilon_{nt}) = \frac{2}{\sigma\sqrt{2\pi}}\left[\Phi\left(\frac{-\varepsilon_{nt}\lambda}{\sigma}\right)\right]\phi\left(\frac{\varepsilon_{nt}}{\sigma}\right) \tag{18}$$

where ϕ and $\Phi(\cdot)$ are the standard normal probability density function (pdf) and cumulative distribution function (cdf), respectively.

For the entire sample (n = 1, …, N; t = 1, …, T), the log-likelihood function is given by

$$\begin{aligned} \ln L(\mathbf{q}, \mathbf{w}, \mathbf{x}|\boldsymbol{\beta}) = -\frac{NT}{2}\ln\frac{\pi\sigma^2}{2} + \sum_{t=1}^{T}\sum_{n=1}^{N}\ln\left[\Phi(\frac{-r_{nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})\lambda}{\sigma})\right] \\ -\frac{1}{2\sigma^2}\sum_{t=1}^{T}\sum_{n=1}^{N} r_{nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})^2 \end{aligned} \tag{19}$$

where $\mathbf{q} = (q_{11}, q_{12}, \ldots, q_{NT})'$, $\mathbf{w} = (w_{11}, w_{12}, \ldots, w_{NT})'$, $\mathbf{x} = (x_{11}, x_{12}, \ldots, x_{NT})'$ and $\boldsymbol{\beta} = (\gamma, a_1, a_2, b)'$.

And the corresponding restriction on each observation in the sample is given by as follows:

$$0 \leq e_{1nt}[\gamma w a_1 q_{nt}{}^{b-1} x_{nt}{}^{\frac{\gamma-b}{\gamma}}] + e_{2nt}[1 - \gamma w a_2 q_{nt}{}^{b-1} x_{nt}{}^{\frac{\gamma-b}{\gamma}}] \leq 1 \tag{20}$$

Technical efficiency is then calculated by separating the composite error term into two components using the transformation suggested in Jondrow *et al.* (1982).

3. Numerical simulations

In Section 3.1, we validate our estimation methodology using simulated data and compare the performance of our state-contingent estimator with conventional estimators. In Section 3.2, we perform a simulation experiment to show that conventional estimators provide us with biased estimates of efficiency scores.

3.1. Numerical examples using simulated data

The input demand x is simulated by substituting Equation (4) into Equation (1), and the state-contingent outputs z_1 and z_2 are generated using Equation (4). The following equation expresses input demand in terms of the risk-neutral probabilities and the technology parameters:

$$x = [a_1(\frac{\pi_1}{a_1 w\gamma})^{\frac{b}{b-1}} + a_2(\frac{\pi_2}{a_2 w\gamma})^{\frac{b}{b-1}}]^{\frac{\gamma(b-1)}{b(\gamma-1)}} \tag{21}$$

Therefore, in Table 1, the input demand x is simulated using (21), and state-contingent outputs (z_1, z_2) are simulated using Equation (4). In our simulation, we assigned equal probabilities to each state of nature. The realised state of nature and the output corresponding to this state of nature are listed in columns 6 and 7, respectively, in Table 1. Finally, the values of the parameters used to generate this table were $b = 2$, $a_1 = 1.5$, $a_2 = 0.5$, $w = 0.5$ and $\gamma = 1.25$. Furthermore, all producers in Table 1 are assumed to be technically efficient.

Given that the state of nature $\{s\}$ is observed, the econometric equation for the conventional OC frontier estimator with Cobb–Douglas functional can be written as follows:

$$\ln(q_{nt}) = e_{1nt}[-\alpha \ln(a_1)] + e_{2nt}[-\alpha \ln(a_2)] + \alpha \ln(x_{nt}) + v_{nt} - u_{nt} \tag{22}$$

where $e_{jnt} = 1$ if $j = s \in \{1, 2\}$ (and 0 otherwise) and $q_{nt} = e_{1nt}z_{1nt} + e_{2nt}z_{2nt}$. The subscripts n and t represent firms and time periods, respectively

Table 1 Simulated data: $(a_1,a_2) = (1.5, 0.5)$, $b = 2$, $\gamma = 1.25$, $w = 0.5$

Firm	π_1	x	z_1	z_2	s	z_s	DEA	SFA	DEAS	OC
1	0.030	50.978	0.339	32.836	2	32.836	1.000	0.938	1.000	0.9925
2	0.042	47.940	0.457	31.256	1	0.457	0.015	0.014	0.238	0.0327
3	0.147	27.455	1.144	19.919	1	1.144	0.059	0.056	0.596	0.1218
4	0.244	15.953	1.371	12.746	1	1.371	0.108	0.107	0.714	0.2146
5	0.246	15.772	1.373	12.626	2	12.626	1.000	0.992	1.000	0.8773
6	0.306	11.172	1.389	9.449	2	9.449	1.000	1.000	1.000	0.8386
7	0.320	10.305	1.384	8.820	1	1.384	0.157	0.157	0.721	0.2952
8	0.369	7.772	1.347	6.910	2	6.910	1.000	1.000	1.000	0.7933
9	0.380	7.298	1.336	6.538	1	1.336	0.205	0.204	0.696	0.3641
10	0.418	5.889	1.292	5.396	1	1.292	0.242	0.237	0.673	0.4100
11	0.479	4.235	1.215	3.964	2	3.964	1.000	0.968	1.000	0.7001
12	0.500	3.805	1.189	3.567	2	3.567	1.000	0.956	1.000	0.6798
13	0.504	3.730	1.184	3.497	1	1.184	0.339	0.323	0.617	0.5197
14	0.546	3.060	1.139	2.842	2	2.842	1.000	0.919	1.000	0.6322
15	0.548	3.033	1.137	2.814	2	2.814	1.000	0.917	1.000	0.6300
16	0.549	3.019	1.136	2.801	2	2.801	1.000	0.916	1.000	0.6289
17	0.566	2.807	1.121	2.580	1	1.122	0.439	0.391	0.608	0.6021
18	0.595	2.506	1.101	2.249	1	1.101	0.500	0.423	0.661	0.6409
19	0.657	2.075	1.086	1.701	2	1.701	1.000	0.769	1.000	0.4984
20	0.704	1.906	1.106	1.395	2	1.395	1.000	0.678	1.000	0.4341
21	0.750	1.854	1.159	1.159	2	1.159	1.000	0.577	1.000	0.3678
22	0.791	1.895	1.238	0.982	1	1.238	0.920	0.605	1.000	0.8784
23	0.864	2.192	1.476	0.697	1	1.476	0.803	0.636	1.000	0.9444
24	0.944	2.928	1.919	0.341	1	1.919	0.712	0.644	1.000	0.9996
25	0.979	3.434	2.189	0.141	2	0.141	0.044	0.041	0.044	0.0289
Mean							0.662	0.579	0.823	0.5650

DEA, data envelopment analysis; DEAS, state-dependent DEA; OC, output-cubical; SFA, stochastic frontier analysis.

$(n = 1, \ldots, 25;\ t = 1)$, v_{nt} is a two-sided random variable representing statistical noise, and u_{nt} is a one-sided random variable representing inefficiency.

We apply a conventional OC frontier estimator to the simulated data shown in Table 1. Table 1 shows that the OC stochastic frontier estimator provides biased estimates of efficiency scores. But the ML estimator that assumes a CES specification of technology renders every firm to be fully efficient. The associated risk-neutral probabilities and unobserved state-contingent outputs were also recovered without error.

We draw the risk-neutral probabilities from a triangular[3] distribution. Including the risk-neutral probability corresponding to risk-free production plan, that is $\pi_1 = 0.75$, and the mode of the triangular distribution, that is $\pi_1 = 0.5$, there are a total of 25 observations in the sample. Columns 2, 3, 4, 5, 6 and 7 in Table 1 report risk-neutral probabilities, inputs used, outputs in state of nature {1} and {2}, realised states of nature and output in the realised state of nature, respectively.

[3] The risk-neutral probabilities in the tables below are chosen in way that the histogram of π_1 (and π_2) is triangular in shape.

An output oriented model with variable returns to scale (VRS) was used to compute efficiency scores for the DEA technique. In case of the SFA, we used a random-effect stochastic frontier model with Cobb–Douglas functional form to compute efficiency scores. The SFA specification can be written as follows:

$$\ln z_s = \alpha_0 + \alpha_1 \ln x + v - u \quad s \in \{1, 2\} \tag{23}$$

where v is the symmetric two-sided normally distributed error term representing statistical noise, and u is the one-sided half-normally distributed error term representing technical inefficiency.

The technology is not risky if $\frac{\pi_2}{\pi_1} = \frac{a_2}{a_1} = \frac{1}{3}$, that is, if $(\pi_1, \pi_2) = (0.75, 0.25)$. Any firm that has a risk-neutral probability >0.75 in state of nature {1} produces more output in state of nature {1} than in state of nature {2}. For example, in Table 1, firm 24 has risk-neutral probability $\pi_1 = 0.944$ in state of nature {1}, and it produces $z_1 = 1.919$ amount of output in state of nature {1} and $z_2 = 0.341$ quantity of output in state of nature {2}. Similarly, firms that have a risk-neutral probability <0.75 in state of nature {1} produce less output in state of nature {1} than in state of nature {2}. For example, in Table 1, firm 12 assigns equal probabilities to both states of nature, that is $\pi_1 = \pi_2 = 0.5$, and it produces $z_1 = 1.189$ amount of output in state of nature {1} and $z_2 = 3.567$ amount of output in state of nature {2}. This implies that, for firms that have a risk-neutral probability >0.75 in state of nature {1}, state of nature {1} corresponds to a 'favourable' state of nature, and for firms that have a risk-neutral probability less than 0.75 in state of nature {1}, state of nature {2} corresponds to a 'favourable' state of nature.

While DEA automatically imposes a monotonicity (to make sure that output is nondecreasing in input) constraint on the efficient frontier, this is not the case with SFA. Hence, in most efficiency studies (for example, see Terrell 1996), in order to ensure a monotonic relationship between input and output on the production frontier, α_1 in (23) is constrained to be non-negative ($\alpha_1, \geq 0$). For example, the dashed line in Figure 1b below shows that when a group of firms use a technology that exhibits decreasing returns ($\gamma = 1.25$) to scale and high degree of output substitutability ($b = 1.1$) between the two states of nature, the frontier (without the constraint $\alpha_1 \geq 0$) is downward sloping when the monotonicity constraint is not imposed.

Figure 1b shows that for this particular technology, for all the firms (dashed line) that experience state of nature {1}, there is negative relationship between the total input used and the observed output. Therefore, SFA-based efficiency estimates with monotonicity (see Figure 2b) constraint would be incorrect as all the firms that face state of nature {1} will be classified to be very inefficient.

Imposing the monotonicity constraint involves fixing one end (left) of the frontier to be on the ordinate and the other end (right) of the frontier to be the point which represents the maximum output. This means that,

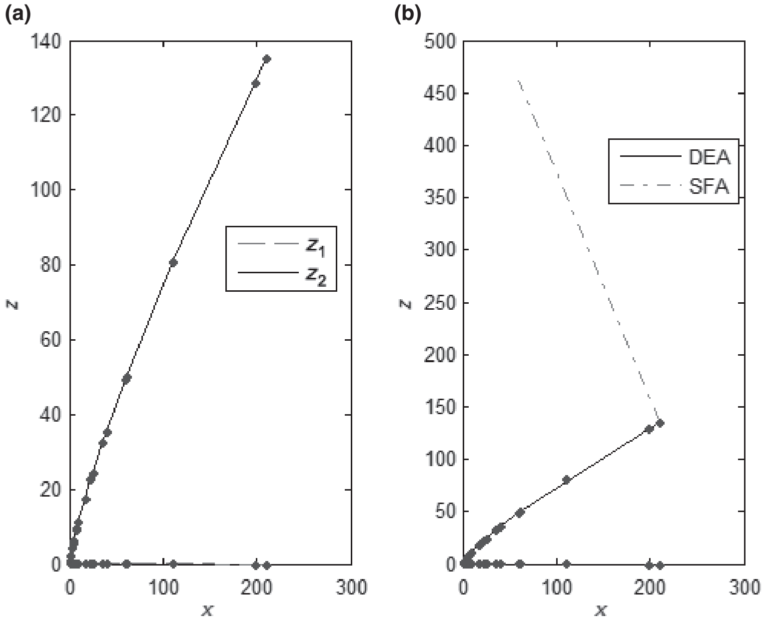

Figure 1 Efficient production plan and frontiers: decreasing returns to scale and high substitutability.

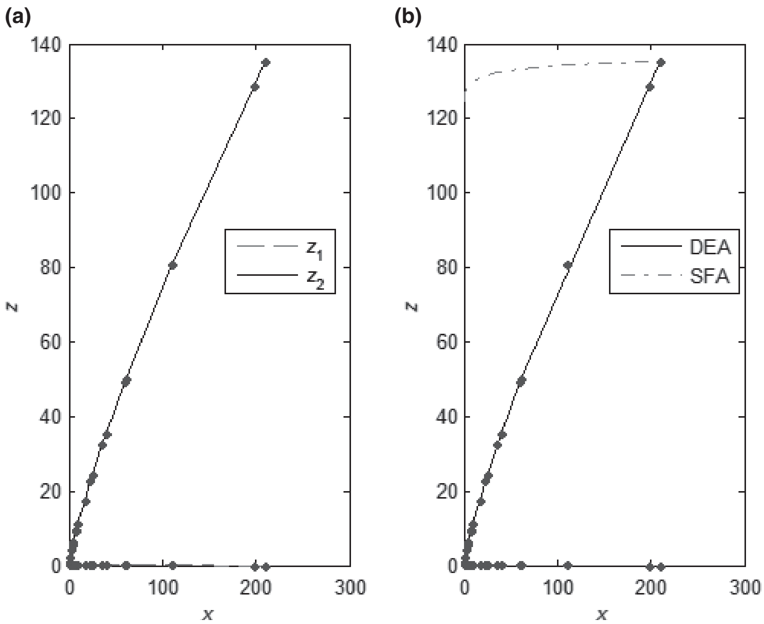

Figure 2 Efficient production plan and frontiers: decreasing returns to scale and high substitutability with monotonicity constraint.

except the firm with highest output, every other firm will have an efficiency score <1 irrespective of the state of nature experienced by them *ex post*. This can be clearly seen in Figure 2b, where the DEA frontier (solid line) contains many firms, but the SFA frontier (dashed line) passes through just one firm.

We observe in Table 1 that for the technology given by (1), the DEA technique ensures that the riskless production plan (firm 21 in this table) is on the VRS production frontier (with an efficiency score of 1), because a riskless production plan uses the least amount of input (or is least costly). We also observe in Table 1 that all the firms which encounter an 'unfavourable' state of nature have an output oriented DEA efficiency score <1.

Figure 3a plots the input–output combination chosen by rational and efficient firms using the technology given by $(a_1, a_2) = (1.5, 0.5)$, $b = 2$, $\gamma = 1.25$ and $w = 0.5$. While the dashed line represents input–output pairs (x, z_1) in state of nature $\{1\}$, the solid line represents the input–output combination (x, z_2) in state of nature $\{2\}$.

Figure 3b plots both DEA and SFA production frontiers alongside the input–output combination chosen by efficient and rational firms in two possible states of nature. Figure 3 and Table 1 show that even when the firms experience a 'favourable' state of nature, the DEA- and SFA- based efficiency estimates are <1. For example, in Table 1, firms 22, 23 and 24 experience a favourable state of nature, but they are found not be fully efficient using the DEA and SFA estimation techniques.

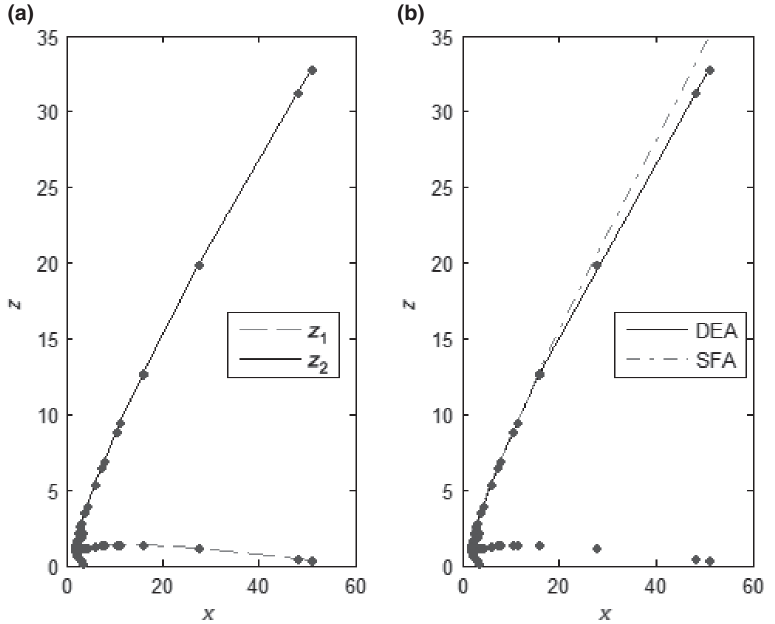

Figure 3 Efficient production plan and frontiers: decreasing returns to scale and moderate substitutability.

Efficiency estimates from SFA are always less (see Table 1) than those from DEA because unlike the DEA technique, the SFA model makes allowance for statistical noise. This is the case even when the producer chooses a riskless production plan (firm 21 in Table 1); while the DEA reports an efficiency score of one, the SFA estimates indicate that the riskless firm is not fully efficient.

Figure 4a shows a efficient production plan chosen by rational firms whose technology exhibits increasing ($\gamma = 0.8$) returns to scale and moderate ($b = 2$) degree of substitutability between *ex post* state-contingent outputs. It is important to notice that unlike a conventional production frontier, the output in both the states of nature is not monotonically increasing in input, hence the production function is multivalued. This means that, for the same amount of input used, it is possible that the output can actually take two different values.

We observe in Figure 4a that in both states of nature, the output initially increases with an increase in input until the risk-neutral probability in state of nature {1} is <0.75, that is $\pi_1 < 0.75$, and then, the output in both states of nature decreases with a decrease in input used in the production process. Again, Figure 4a shows that, when $\pi_1 < 0.75$, the efficient production plan representing output in state {2} contains the efficient production plan representing state of nature {1}, and when $\pi_1 > 0.75$, the opposite is true, that is, the efficient production plan in state of nature {1} contains the efficient production plan in state of nature {2}.

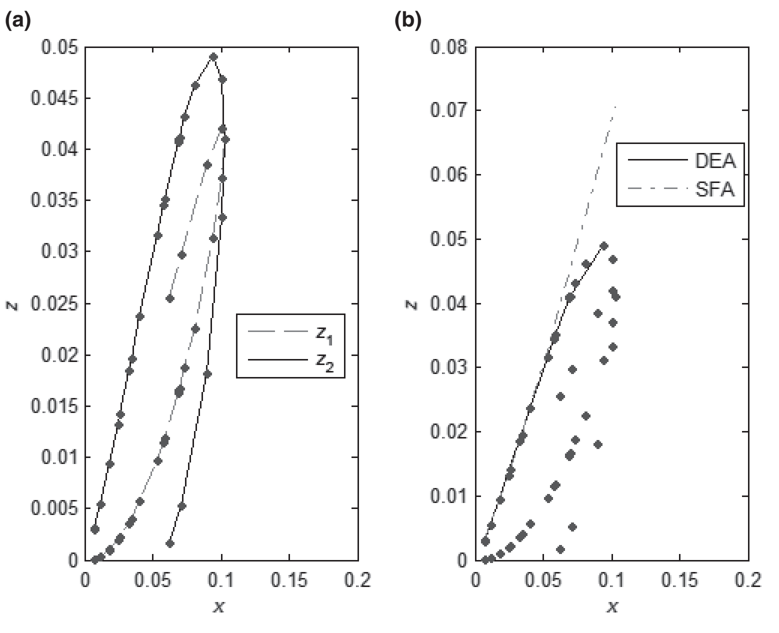

Figure 4 Efficient production plan and frontiers: increasing returns to scale and moderate substitutability.

Since in our simulation exercise, all the firms are actually operating on the frontier, they are in fact technically efficient. Therefore, performance of the firms is completely dependent on their evaluation of risk-neutral probabilities. The fact that different firms choose different production plans and some firms appear to perform 'better' than others is because some firms may have access to better information about future states of nature. Therefore, the apparent inefficiency is allocative rather than technical in nature.

Column 10 in Table 1 reports efficiency estimates (DEAS) when the DEA estimator is applied separately to firms that experience state of nature {1} and {2}, respectively. We note that from Table 1 that firms that face a 'favourable' state of nature are found to be fully efficient when the DEAS estimator is used. For example, state of nature {2} is the 'favourable' state of nature for firms 14, 15 and 16, and these firms are estimated to be fully efficient by DEAS. Similarly, state of nature {1} is the 'favourable' state of nature for firms 22, 23 and 24, and the DEAS estimator reports an efficiency score of 1 for these firms. Irrespective of whether firms face 'favourable' or 'unfavourable' states of nature, efficiency estimates are always higher with the DEAS estimator compared to the DEA estimator. This can be verified by observing Table 1, which shows that the efficiency estimates with the DEAS (column 10) estimator are always greater than or equal to the DEA (column 8) estimator.

The OC frontier estimator which recognises the state-contingent nature of the production process does not always estimate efficiency levels to be greater or equal to conventional stochastic frontier (SFA) estimator. For example, for firm 5, while the conventional SFA estimator computes the efficiency score to be 0.992, OC frontier estimator provides a lower efficiency score of 0.8773. On the contrary, for firm 24, estimates from the OC (0.9996) frontier estimator are higher than the conventional SFA (0.644) estimator. In Table 1, we observe that except for firm 25, all other firms that face 'unfavourable' state of nature have a higher efficiency level estimates using OC frontier estimator compared to conventional stochastic frontier (SFA) estimator. However, mean efficiency estimates of the OC frontier estimator (0.5650) and the conventional stochastic frontier estimator (0.579) are similar in magnitude.

While the DEAS estimator classifies the risk-less firm (21) to be fully efficient, the OC frontier estimator (0.3678) reports efficiency level estimates to be even less than the conventional stochastic frontier estimator (0.577). Finally, even though both DEAS and OC frontier estimators recognise the stochastic nature of the underlying production technology, the DEAS estimator always provides higher efficiency level estimates than the OC frontier estimator. Again, this is because unlike the OC frontier estimator, the DEAS estimator makes no allowance for statistical noise.

In Figure 5a, the curve drawn using a solid line and a dashed line represents the actual CES frontier in state of nature {1} and {2}, respectively.

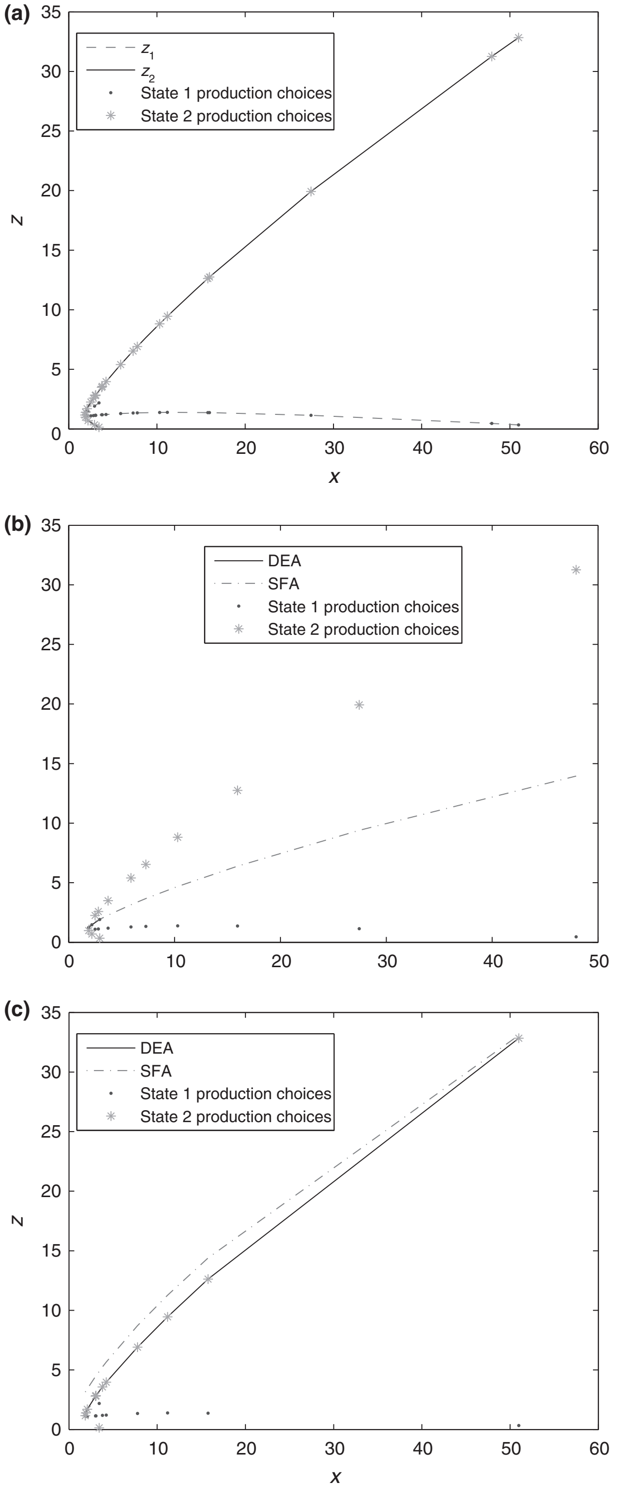

Figure 5 Efficient production plan and frontiers in state of nature {1} and {2}, respectively: decreasing returns to scale and moderate substitutability.

Figure 5b,c shows the efficient frontier using DEAS (solid line) and OC (dashed) frontier estimators in state of nature {1} and {2}, respectively. Again, in Figure 5b,c, the stars and dots represent production choices in state of nature {1} and {2}, respectively.

Unlike the DEA and SFA estimators, both the DEAS and OC frontier estimators recognise the state-contingent nature of production technology. Therefore, both the DEAS and OC frontier estimator predict two separate frontiers, one in each state of nature {1} and {2}, respectively. Again, efficiency estimates from DEAS are higher than those from the OC estimator. For example, when technology exhibits decreasing ($\gamma = 1.25$) returns to scale and moderate ($b = 2$) substitutability between state-contingent outputs, all the firms that experience a 'favourable' state of nature are classified to be fully efficient by the DEAS estimator. This can be seen (also from Table 1) in Figure 5b, where firms 22, 23 and 24 lie on the solid line representing the DEAS frontier. But only firm 24 is found to lie on the OC frontier when state of nature {1} is realised.

In Figure 5, the OC frontier lies above the DEAS frontier, indicating that when technology exhibits decreasing ($\gamma = 1.25$) returns to scale and moderate ($b = 2$) degree of substitutability between state-contingent outputs, the DEAS estimator provides us with higher efficiency level estimates compared to the OC frontier estimator. This is not true when technology exhibits increasing returns to scale. For example, in Figure 6b where state of nature {1} is realised and technology exhibits increasing ($\gamma = 0.8$) returns to scale and moderate ($b = 2$) substitutability between state-contingent outputs, a portion of the OC frontier can be observed to be below the DEAS frontier. This implies that firms that are closer to this region where the OC frontier is below the DEAS frontier will have higher efficiency level estimates from the OC frontier estimator compared to the DEAS estimator.

3.2. Simulation experiments

To further evaluate the performance of conventional DEA and SFA estimators, we perform a simulation experiment. In the simulation experiment, we fix the risk-neutral probabilities shown in the second column in Table 2 in each of the $N = 1000$ replications, but we allow each of the 25 firms to experience any of the two possible states of nature *ex post* with probability 0.5.

Table 2 reports the descriptive statistics of the estimated efficiency scores obtained by the DEA and SFA estimators, respectively, for a technology that exhibits decreasing returns to scale and moderate degree of substitutability between state-contingent outputs. In Table 2, we observe that except for firm 21 (using DEA), all the firms have mean efficiency scores <1, which indicates that both the DEA and SFA estimators are biased. When the technology exhibits decreasing returns to scale, even firm 21 which chooses riskless output combinations is found to be less than fully efficient by the SFA

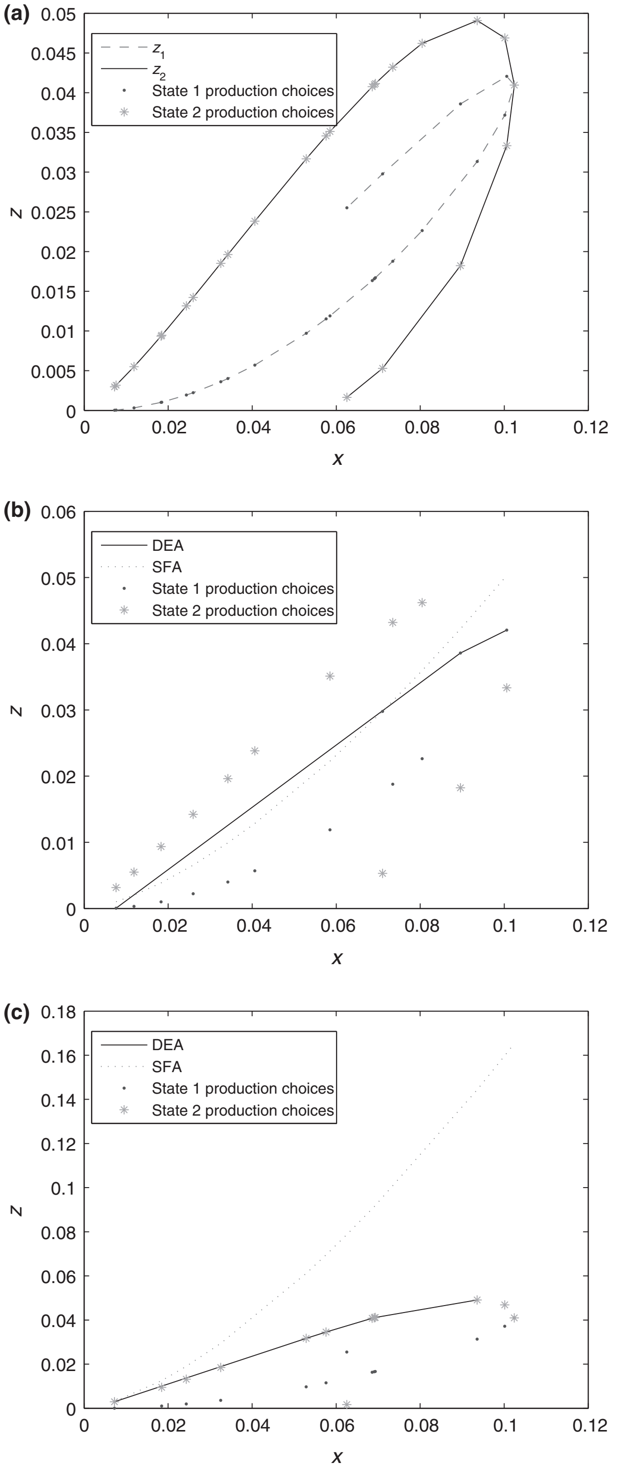

Figure 6 Efficient production plan and frontiers in state of nature {1} and {2}, respectively: increasing returns to scale and moderate substitutability.

Table 2 Sample statistics for decreasing returns to scale and moderate output substitutability: $(a_1, a_2) = (1.5, 0.5)$, $b = 2$, $\gamma = 1.25$, $w = 0.5$

Firm	π_1	DEA				SFA			
		Mean	St. Dev	Min	Max	Mean	St. Dev	Min	Max
1	0.030	0.520	0.492	0.011	1.000	0.518	0.494	0.011	1.000
2	0.042	0.506	0.481	0.014	1.000	0.506	0.486	0.014	1.000
3	0.147	0.430	0.386	0.043	1.000	0.457	0.412	0.047	1.000
4	0.244	0.383	0.317	0.063	1.000	0.443	0.355	0.077	1.000
5	0.246	0.394	0.312	0.064	1.000	0.457	0.351	0.078	1.000
6	0.306	0.323	0.268	0.076	1.000	0.399	0.317	0.099	1.000
7	0.320	0.328	0.257	0.079	1.000	0.413	0.309	0.105	1.000
8	0.369	0.328	0.227	0.091	1.000	0.438	0.287	0.128	1.000
9	0.380	0.296	0.223	0.094	1.000	0.398	0.284	0.134	0.999
10	0.418	0.318	0.210	0.106	0.927	0.446	0.275	0.158	0.942
11	0.479	0.317	0.186	0.134	0.892	0.474	0.259	0.213	0.927
12	0.500	0.336	0.181	0.148	0.884	0.509	0.256	0.239	0.946
13	0.504	0.336	0.183	0.151	0.868	0.510	0.256	0.244	0.939
14	0.546	0.369	0.171	0.192	0.890	0.569	0.246	0.312	0.983
15	0.548	0.381	0.175	0.194	0.890	0.586	0.247	0.316	0.978
16	0.549	0.381	0.178	0.195	0.891	0.585	0.249	0.318	0.993
17	0.566	0.398	0.173	0.219	0.901	0.609	0.242	0.354	1.000
18	0.595	0.450	0.166	0.273	0.927	0.671	0.231	0.421	1.000
19	0.657	0.654	0.153	0.486	1.000	0.827	0.166	0.592	1.000
20	0.704	0.884	0.098	0.775	1.000	0.916	0.085	0.426	1.000
21	0.750	1.000	0.000	1.000	1.000	0.924	0.047	0.463	1.000
22	0.791	0.827	0.095	0.716	0.984	0.842	0.105	0.481	1.000
23	0.864	0.422	0.159	0.253	0.774	0.575	0.207	0.322	0.882
24	0.944	0.234	0.168	0.062	0.623	0.358	0.245	0.087	0.710
25	0.979	0.187	0.168	0.020	0.602	0.287	0.250	0.033	0.656

DEA, data envelopment analysis; SFA, stochastic frontier analysis.

estimator. Some firms in our simulated experiment have a SFA-based efficiency estimate that is very low. When firms experience an 'unfavourable' state of nature, the monotonicity constraint imposed by the SFA estimator shifts the estimated frontier to the level of the maximum output that is feasible using the technology as shown in Figure 2b, which results in low efficiency scores for firms that are far away from the estimated frontier.

Table 3 reports the descriptive statistics of the estimated efficiency scores obtained by DEAS and OC frontier estimators, respectively, for a technology that exhibits decreasing returns to scale and moderate degree of substitutability between state-contingent outputs. Again, in Table 3, we observe that except for firm 21 (using DEA), all the firms have a mean efficiency score <1, which indicates that both the DEAS and OC frontier estimators are biased. Comparing Table 3 with Table 2, we find that, while the mean DEAS efficiency estimates of every firm are higher than the corresponding DEA efficiency estimates, the OC frontier estimator does not always provide us with higher mean efficiency level estimates compared to the conventional stochastic frontier (SFA) estimator. Again, when technology exhibits, decreasing returns to scale firm 21 is always classified to be fully efficient

Table 3 Sample statistics for decreasing returns to scale and moderate output substitutability: $(a_1, a_2) = (1.5, 0.5)$, $b = 2$, $\gamma = 1.25$, $w = 0.5$

Firm	π_1	DEAS				OC			
		Mean	St. Dev	Min	Max	Mean	St. Dev	Min	Max
1	0.030	0.550	0.408	0.155	1.000	0.521	0.471	0.015	0.999
2	0.042	0.599	0.379	0.209	1.000	0.528	0.465	0.021	0.999
3	0.147	0.797	0.211	0.523	1.000	0.584	0.414	0.084	0.999
4	0.244	0.863	0.168	0.626	1.000	0.611	0.362	0.154	1.000
5	0.246	0.866	0.166	0.627	1.000	0.598	0.363	0.154	1.000
6	0.306	0.869	0.162	0.634	1.000	0.621	0.316	0.214	1.000
7	0.320	0.863	0.166	0.632	1.000	0.616	0.308	0.227	1.000
8	0.369	0.856	0.173	0.615	1.000	0.615	0.269	0.281	0.998
9	0.380	0.855	0.174	0.610	1.000	0.612	0.258	0.296	1.000
10	0.418	0.841	0.185	0.590	1.000	0.636	0.230	0.351	0.999
11	0.479	0.829	0.200	0.555	1.000	0.635	0.182	0.343	0.998
12	0.500	0.822	0.205	0.543	1.000	0.645	0.171	0.254	0.999
13	0.504	0.822	0.207	0.541	1.000	0.640	0.168	0.274	0.998
14	0.546	0.824	0.197	0.573	1.000	0.650	0.144	0.211	1.000
15	0.548	0.817	0.197	0.576	1.000	0.643	0.140	0.221	0.997
16	0.549	0.827	0.194	0.578	1.000	0.644	0.140	0.208	0.997
17	0.566	0.842	0.181	0.608	1.000	0.652	0.134	0.230	0.997
18	0.595	0.854	0.155	0.662	1.000	0.650	0.134	0.173	1.000
19	0.657	0.924	0.094	0.786	1.000	0.648	0.149	0.133	0.997
20	0.704	0.964	0.050	0.887	1.000	0.643	0.183	0.137	0.997
21	0.750	1.000	0.000	1.000	1.000	0.635	0.227	0.131	0.999
22	0.791	0.939	0.106	0.729	1.000	0.624	0.279	0.096	0.999
23	0.864	0.716	0.305	0.377	1.000	0.564	0.359	0.080	1.000
24	0.944	0.573	0.437	0.126	1.000	0.566	0.446	0.031	1.000
25	0.979	0.533	0.478	0.044	1.000	0.487	0.481	0.014	1.000

DEAS, state-dependent data envelopment analysis; OC, output-cubical.

by the DEAS estimator. The OC frontier estimator, like conventional frontier estimator, does not always classify the risk-less firm (21) to be fully efficient.

4. Conclusion

In conventional efficiency analysis, the functional specification involves extreme *a priori* restrictions that render the interaction between random factors and controllable inputs to be empirically invalid. Econometrically convenient stochastic errors are introduced to capture the effect of random inputs on an otherwise deterministic production process. We perform a simulation experiment where the data are generated using our state-contingent specification of the stochastic technology. The simulation experiment clearly indicates that conventional efficiency estimators such as DEA, DEAS, SFA and OC are systematically biased. It is important to note that the problem is not with the DEA, DEAS, SFA and OC frontier estimators themselves, but it is due to mis-specification of the stochastic technology. In our simulation data, when we applied a maximum likelihood estimator on our CES specification of technology, all the firms were estimated to be fully efficient.

Almost all conventional frontier models are OC. If this restrictive representation of technology is incorrect, then the results of the simulation experiment in this paper have important policy implications. Discovering that a substantial number of firms lie inside the 'efficiency frontier', one may mistakenly conclude that there are potential opportunities for beneficial policy interventions to improve efficiency, when in fact devoting resources in a manner that allows producers to expand production and manage risk would lead to a better outcome. Analysis in the state-contingent framework therefore suggests that utmost care must be taken when drawing policy conclusions.

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