Rob King

RISK ANALYSIS FOR AGRICULTURAL PRODUCTION FIRMS: CONCEPTS, INFORMATION REQUIREMENTS AND POLICY ISSUES

Proceedings of a Seminar Sponsored by
Southern Regional Project S-180
"An Economic Analysis of Risk Management Strategies for Agricultural Production Firms"
New Orleans, Louisiana
March 25-28, 1984

Department of Agricultural Economics
Agricultural Experiment Station
College of Agriculture
University of Illinois at Urbana-Champaign

July, 1984

AE-4574

THE TRANSFORMATION APPROACH TO STOCHASTIC DOMINANCE: PRELIMINARY RESULTS

Jack Meyer

This paper reports on the initial stages of research concerning a different approach to the study of various issues discussed under the heading stochastic dominance. Emphasis is on the methodology involved and the reasons why the approach may be a valuable alternative to the standard one when addressing certain questions. A few significant and interesting results are presented in order to help illustrate the approach. The work reported on is preliminary and incomplete, but hopefully is a first step toward a useful reformulation of the various stochastic dominance definitions and their applications.

In the standard framework of analysis stochastic dominance is a definition of what it means for one random variable to be ranked higher than, or to dominate, another random variable. This usage of the term will be followed here, stochastic dominance is simply the definition of a partial order over the set of random variables of concern. Various adjectives, such as first degree or second degree, have been used to distinguish between these various definitions. In most instances, the definition proposed is claimed to be an important or valuable one by connecting the ordering over random variables in the definition to the conditions required for the one random variable to be unanimously preferred to the other for all expected utility agents with specified characteristics. This means of justifying or showing a reason for stochastic dominance definitions will also be used here.

In standard approaches, stochastic dominance definitions are given by specifying conditions on the pair of cumulative distribution functions which represent the pair of random variables being ranked or ordered. For instance, first degree stochastic dominance (FSD) is defined by comparing the magnitude of the cumulative distribution function of the one random variable, $F(x)$, with the CDF, $G(x)$, of the other. If $G(x)$ is always at least as large as $F(x)$, and larger at one or more points, then the random variable defined by $F(x)$ is said to dominate in the first degree that random variable defined by $G(x)$. While I am certain that all of you are quite familiar with this particular definition, I wish to emphasize again that the definition is a

---

Jack Meyer is Professor of Economics, Michigan State University.

condition on a pair of cumulative distribution functions. Furthermore, all forms of stochastic dominance of which I am aware use conditions on a pair of cumulative distribution functions in order to define what is meant by one random variable dominating another. It is this aspect of standard theory that is varied in the transformation approach to stochastic dominance presented here.

This research defines stochastic orderings by characterizing those transformations defined on the support of the random variable which can be said to yield a new random variable which dominates the original one. That is, the focus is on what can be done to a given random variable in order to transform it into another which dominates. The stochastic dominance definition given shortly will be in the form of conditions on an initial random variable and how it can be transformed in order to obtain, as a result of the transformation, a new random variable which is better or ranked higher.

Both the standard and this transformation approach can address the same questions, but each approach has advantages in addressing certain questions. For instance, if one is simply attempting to rank random variables it seems quite indirect and probably inefficient to do so by finding the transformation required to go from one to the other and then examining the properties of this transformation. The standard approach seems well suited to answer this question quite efficiently. On the other hand if one wishes to discuss the impact of a particular government policy which has the effect of transforming a random variable, it may well be indirect and inefficient to do so by first calculating the new random variable and then comparing it with the original using standard methods. Direct examination of the properties of the transformation involved seems to be a more direct way to address this issue. Even more pointedly, if one wishes to seek an optimal policy of transforming a given random variable among those available to the policymaker, it seems very inefficient indeed to simulate the effects of each of the various alternatives by calculating the new random variable which would obtain under each, and then comparing these using standard methods. A direct way of comparing the various alternatives certainly seems worth exploring. These uses of the transformation approach will be more fully illustrated later on in the paper.

In principle the methodology presented here can be used to translate all the existing definitions of stochastic dominance into this transformation framework as well as to develop new and interesting forms of stochastic dominance. In practice, only one of the more tractable forms of stochastic dominance will be translated in this paper, illustrating the general methodological approach.

The Transformation Approach

This section defines the notation, gives the assumptions and outlines the framework of analysis used in the transformation approach to stochastic dominance. The beginning point here, as it is in the standard approach, is an initial or given random variable described by its cumulative distribution function (CDF) which will be denoted F(x). For simplicity it is assumed that the support of this random variable lies in the interval [0,1]. All other random variables of concern, i.e., those which are to be compared with this initial random variable, are assumed obtainable from it by means of a real valued function t(x) defined on [0,1]. That is, the stochastic dominance orderings to be defined shortly will involve conditions on the original random variable, and on the transformation t(x) used to obtain the new random variable. Obviously then one can only compare random variables if the one can be obtained from the other by an allowed transformation. This restriction on the domain of the stochastic ordering is necessary if one is to define the ordering in terms of t(x).

Restrictions on transformation t(x) are central to the analysis. On the one hand, the more restrictive the assumption on t(x), the fewer the pairs of random variables which are able to be compared one with the other, and on the other hand, it is the restrictions which allow those which can be compared to be ranked unambiguously. Furthermore, certain restrictions on t(x) greatly simplify the mathematics and the presentation without making the comparisons which can be made a small or uninteresting class. Throughout, t(x) is assumed to be a nondecreasing, bounded, realvalued, continuous and piecewise differentiable function. These conditions allow reasonably simple mathematics yet allow comparisons across sets of random variables which are broad. For instance, all pairs of random variables with differentiable CDFs and bounded supports can be compared with one another even with these restrictions on t(x).

The assumption that t(x) be nondecreasing is an important and I think reasonable one. It implies that if the random variable is such that preference is monotonic in size of outcome, then the transformation does not reverse the relative rankings of the various outcomes. Certainly in order to make comparisons of random variables which are related in some way to comparisons made by expected utility decision makers this condition will be required. Also, most real world transformations seem to satisfy this monotonicity requirement possibly for moral hazard reasons. Two such examples will be discussed more fully later in the paper. The remaining assumptions on t(x) are primarily for mathematical convenience and can be relaxed somewhat at the expense of complicating the presentation.

To summarize, the approach to stochastic dominance used here assumes an initial random variable given by its CDF, F(x), which is transformed using t(x) into a new random variable with which the original is compared or ranked. The set of new random variables which are defined to stochastically dominate the original will be characterized solely on the basis of t(x) and F(x). These two functions will be the basis for stochastic dominance definitions.

Second Degree Stochastic Dominance

In order to further illustrate the approach taken here and to present the first result using this approach consider the following definition.

Definition 1: The random variable obtained from F(x) using transformation t(x) is said to dominate F(x) in the second degree if

$$\int_0^y k(x)dF(x) > 0 \text{ all } y \in [0, 1] \text{ where}$$

$$k(x) = t(x) - x.$$

This definition says that random variables obtained from F(x) using transformations with a particular property dominate F(x) in the second degree. In order to give significance to this definition and to tie the notion of dominance producing transformations to the standard stochastic dominance notions, a connection between this definition and preference by expected utility agents will be established. This is the manner in which most stochastic dominance definitions which have been proposed have been given meaning or significance.

Theorem 1: Transformation t(x) represents a second degree stochastic dominating change in random variable F(x) if and only if it increases expected utility for all concave and nondecreasing utility functions

Proof: The proof of this result is used to illustrate the workings of the approach and hence will be longer and more detailed than necessary. Central to the proof technique used here is the following function giving expected utility as a function of parameter θ.

$$Eu(\theta) = \int_0^1 u[x+\theta k(x)]dF(x)$$

This function, since $k(x) \equiv t(x)-x$, is such that Eu(1) is expected utility after transformation t(x), and Eu(0) is expected utility beforehand. Thus signing Eu(1)-Eu(0) is the problem at hand in determining if the transformation increased or decreased expected utility. Now the first two derivatives of this function are the following.

$$\frac{dEu(\theta)}{d\theta} = \int_0^1 u'[x+\theta k(x)]k(x)dF(x)$$

$$\frac{d^2Eu(\theta)}{d\theta^2} = \int_0^1 u''[x+\theta k(x)](k(x))^2dF(x)$$

Notice that assuming risk aversion, $Eu(\theta)$ is concave in θ. This has the immediate implication that if

$\left.\frac{dEu(\theta)}{d\theta}\right|_{\theta=0}$ is negative, then so is $Eu(1)-Eu(0)$. That

is, the sign of the derivative at $\theta=0$ is sufficient when negative to sign the change in expected utility. What is not so obvious nor immediate is that the same is true if

$\left.\frac{dEu(\theta)}{d\theta}\right|_{\theta=0}$ is positive. To see this integrate the first deriva-

tive by parts to obtain:

$$u'[1+\theta k(1)] \int_0^1 k(x)dF(x) - \int_0^1 [u''[x+\theta k(x)][1+\theta k'(x)]$$

$$\int_0^1 k(s)dF(s)]dx$$

Recall that risk aversion and $t'(x) = 1+k'(x) > 0$ are assumed and thus under the conditions given on the transformation, $\frac{dEu(\theta)}{d\theta}$ is of the same sign for all $\theta\varepsilon[0, 1]$. Hence signing

$\left.\frac{dEu(\theta)}{d\theta}\right|_{\theta=0}$ to be positive is sufficient to imply that

$Eu(1)-Eu(\theta)$ is positive. Of course, $\left.\frac{dEu(\theta)}{d\theta}\right|_{\theta=0} =$

$\int_0^1 u'(x)k(x)dF(x)$, a form very similar to that signed in stan-

dard stochastic dominance models. In fact, the conditions on the transformation given in the definition of second degree stochastic dominance make this expression unambiguously positive for the specified utility functions. What has been shown so far is that $Eu(\theta)$ is concave in θ and if increasing at $\theta=0$ does not turn downward before $\theta=0$ and if decreasing at $\theta=0$ it always slopes down for all $\theta>0$.

To establish the necessity of the stochastic dominance conditions most authors in standard research have relied on constructed counter-example utility functions. Hadar and Russell or Hanoch and Levy are examples. The same procedure is followed here. Assume that a transformation is given such that

$\int_0^y k(x)dF(x) < 0$ holds for some $y \varepsilon [0, 1]$. A utility function which is concave and increasing will be constructed so that expected utility falls under the given transformation. Two cases must be considered. First assume $y+k(y) \geq y$ then expected utility falls for the following utility functions.

$$u(x) = \begin{cases} x & x \leq y \\ y & x > y \end{cases}$$

To see this note that

$$\int_0^1 [u[x+k(x)]-u(x)]dF(x) = \int_0^y k(x)dF(x) + \int_y^{y+k(y)} (y-x)dF(x)$$

which is negative. For the remaining case where $y+k(y) <$ holds, the same utility function is employed, but the change in expected utility reduces to

$$\int_0^{y+k(y)} k(x)dF(x) + \int_{y+k(y)}^{y} (x+k(x)-ydF(x)$$

which is less than $\int_0^y k(x)dF(x)$ and hence is negative.

Q.E.D.

Having characterized those transformations which improve variable F(x) in the opinion of all risk averters, a couple of examples of such transformations are in order. We draw our examples from the field of insurance. Consider first an insurance policy providing full reimbursement of loss once a deductible of size d is met. Assume the insured variable has random future value normalized to fall in the interval [0, 1]. This insurance policy can then be represented by the transformation

$$t(x) = \begin{cases} 1-d-\delta & x < 1-d \\ x-\delta & x \geq 1-d \end{cases}$$

where x is the insured variable's future value and δ is the premium paid to obtain this insurance. This transformation is nondecreasing in x and piecewise differentiable and continuous. Furthermore, for this transformation, k(x) is a nonincreasing function and hence satisfies the condition in the second degree stochastic dominance definition as long as

$\int_0^1 k(x)dF(x) \geq 0$. This later condition is a requirement that the mean of the random variable not be decreased by the transformation. To see this, simply look at the risk neutral utility function in the proof of Theorem 1. Thus, full reimbursement insurance with a deductible is always preferred by a risk averse agent as long as the premium is not higher than the actuarily fair value. Certainly this is neither a surprising nor new result, but does illustrate the approach. It also leads quite naturally to questions such as what if the premium exceeds that which is actuarily fair? It appears at this stage that a form of stochastic dominance in which the lower bound on risk aversion is strictly greater than zero will be needed to deal with this case. Extensions to stochastic dominance with respect to a function are underway but incomplete as yet.

A second form of insurance requires the economic agent to share the loss in that only a fixed percentage θ is reimbursed. This insurance policy transforms the insured random variable according to $t(x) = x+\theta(1-x)-\delta$ where again δ is the premium charged for the insurance. The $k(x)$ function derived from this transformation is also a decreasing function and hence as long as δ is no larger than the actuarily fair premium, this coinsurance is beneficial to all risk averse agents. The conclusions drawn for this case are quite similar to those found in the deductible example. We go on to note however that $\delta=\theta(1-\bar{x})$ is the actuarily fair premium where $\bar{x}$ is the mean of x. Hence, this transformation at the actuarily fair premium is $t(x) = x+\theta(\bar{x}-x)$ for $\theta \in [0, 1]$. This of course is exactly the transformation intensively studied by Sandmo and others concerned with the output effects of risk changes for a competitive firm. An elaboration of this point can be found in Meyer and Ormiston.

Hopefully the results and examples of this section illustrate the transformation approach and its potential uses. Next, the paper is concluded by speculating on future results and applications of this general methodology.

Future Work

This final section will be used to return to the original and main point of this paper and that is to show what the transformation approach to stochastic dominance is, and to illustrate its potential uses. The results presented earlier concerning second degree stochastic dominance gives adequate evidence as to what the approach is and that it can be carried out, although, doing so involves more than a trivial amount of effort. Other research not presented here indicates that a variety of other forms of stochastic dominance can be translated into conditions on transformations as well. Formally doing this is one future task.

Before going further in this essentially theoretical direction however, it is worth examining some potential uses for the yet to be established results. I conclude this paper with a series of questions which seem to be interesting, and can be addressed more easily, or sometimes only, using the approach described here today. That is, I wish to speculate on, and get your reaction to, the potential uses for this theory.

One important set of questions that can be dealt with very directly using transformations is the basic question of how to accomplish an improving change in a random variable. The focus thus far in economic research has been on what is an improvement, and the how question has largely been ignored. Since the transformations identified as improving the current random variable can be imposed as policy instruments the how question is answered directly. This question can also be separated into several different versions based on the fact that the improving transformation depends on both the agent's preferences and the agent's current random variable. Thus one could, as was done in the second degree stochastic dominance case, fix the initial random variable and ask what transformations are improving for large groups of economic agents. On the other hand, one could fix the economic agent and ask what transformations are improving for a large set of initial random variables. Given our inability to know precisely the current random variable faced by an economic agent this is potentially useful application of this theory. In principle at least, one could partially specify the initial random variable and the agent's preferences and attempt to identify improving transformations.

In addition to searching for transformations, this approach is also well suited to evaluation of policies which contain or are represented by a particular transformation. Again since the formulation explicitly considers the agent's preferences and the initial random variable in evaluating the impact of the transformation, one can very directly determine who, identified by preferences and the initial random variable, would benefit from a current or proposed policy. Identifying the gainers (and losers) is at the heart of much of policy analysis and evaluation.

REFERENCES

1. Hadar, J., and W.R. Russell. "Rules for Ordering Uncertain Prospects." American Economic Review. Vol. 59, 1969, pp. 25-34.

2. Hanoch, G., and H. Levy. "The Efficiency Analysis of Choices Involving Risk." Review of Economic Studies. Vol. 36, 1969, pp. 335-346.

3. Meyer, J., and M.B. Ormiston. "The Transformation Approach to the Analysis of Risk Changes in Economic Models." Working Paper, November 1983.

4. Sandmo, A. "On the Theory of the Competitive Firm Under Price Uncertainty." American Economic Review. Vol. 61, 1971, pp. 65-73.