MIXED ESTIMATION: A NEW PERSPECTIVE

by

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Over twenty years ago Theil and Goldberger introduced the classical "mixed estimator" as a means of combining sample information with introspectively generated stochastic linear constraints on the parameters of a linear model. A voluminous literature developing mixed estimation theory beyond the seminal ideas presented by Theil and Goldberger has evolved during the last two decades (see the survey by Conway and Mittelhammer). The technique appears destined to be included as core material in modern textbook treatments of econometrics (e.g. see Judge, Griffiths, Hill and Lee; Judge, Hill, Griffiths, Lutkepohl and Lee; Theil (1971, 1978); Johnston; Kmenta; Vinod and Ullah; Madalla; Dhrymes). A number of econometric applications using mixed estimation have appeared in the literature in recent years (e.g. Conway and Yanagida; Hammig; Hammig and Mittelhammer (1980, 1982); Holloway; Mittelhammer; Mittelhammer and Price; Mittelhammer, Young, Tasanasanta and Donnelly; Paulus; Price and Mittelhammer; Theil).

The theoretical literature that followed the original article by Theil and Goldberger and on which applications were based characteristically assumes the existence of prior stochastic constraints, and then goes on to develop further results concerning mixed estimator properties. There has been little discussion of the subjective probability underpinnings of the introspective stochastic constraints that would add to the original ethereal passage from Theil and Goldberger (p. 73):

> Our idea is that such a person carries out a large number of mixed regression analyses in the course of a lifetime, so that he produces a similarly large number of errors (the difference between the a priori estimates and the corresponding true values of the parameters). Dividing these errors by the corresponding square roots of their variances, we obtain standardized errors

> which are--in the light of this subjectivist probability idea--random drawings from a parent with zero mean and unit variance.

In all applications of the technique, the stochastic constraints have been derived by an appeal to a confidence interval approach based on some nondescript stochastic mechanism, the typical procedure exemplified by the discussion of Nagar and Kakwani:

> Let us suppose that prior information is available on the first two elements of β. We may know, for example, that almost certainly, say with 95% probability, β_1 lies between 0 and 1, and β_2 lies between 1/4 and 3/4. If we use "two times the σ rule", the range of β_1 is $1/2 \pm\sqrt{1/16}$ and that of β_2 is $1/2 \pm\sqrt{1/64}$, and we can write
>
> $$1/2 = \beta_1 + V_1 \qquad E\,V_1 = 0 \qquad E\,V_1^2 = 1/16$$
>
> $$1/2 = \beta_2 + V_2 \qquad E\,V_2 = 0 \qquad E\,V_2^2 = 1/64.$$

Discussing a related example involving the prior constraint on β_1 only, Theil and Goldberger state:

> Values of β_1 outside the range zero and one would then be outside the "two-sigma" range . . . .; this gives some indication of the virtual impossibility of such values.

A continuing weakness of mixed estimation theory and practice is the lack of a definitive conceptualization of the fundamental stochastic mechanism underlying the generation of the stochastic linear constraints and a subsequent demonstration of mixed estimator properties derived within the logic of probability theory and the paradigm of classical statistics. Recent articles by Swamy and Mehta (1979, 1982) sharply criticize the mixed estimation technique for its lack of probability axiom underpinnings. In fact, Swamy and Mehta (1982, p. 6) go so far as to state that "Because of this problem, the user of Theil's model of introspection cannot ascribe any meaning to numerical probabilities."

It is the purpose of this paper to provide an approach to mixed estimation that provides a clear description of how outcomes of the linear stochastic constraints are generated and thereby present a clear probability foundation for the estimator. The approach described here will provide practitioners of mixed estimation a number of decided benefits. First, the approach removes the seemingly ethereal origins of the prior constraints. Second, the interpretation of the Theil-Goldberger mixed estimator (TGME) is clarified, and under the conceptualization adopted here, the TGME is shown to be inefficient. Third, a new mixed estimator, called the prior integrated mixed estimator (PIME), is introduced and is shown to be more efficient than the TGME.

We begin the exposition by briefly reviewing the TGME technique. We then present our conceptualization of the process generating outcomes of the stochastic constraints, discuss the interpretation of the TGME in that context, and then introduce the more efficient PIME and develop a number of its important properties. We end with a simple application of the PIME to illustrate its potential use in application.

The Mixed Estimator

We begin by denoting the classical general linear model as

$$(1) \qquad \underset{\sim}{Y} = x\,\beta + \underset{\sim}{u}$$

where subtildes denote random variables, Y is a (nxl) vector of observations on the dependent variable, x is a (nxk) matrix of n observations on the explanatory variables, β is a (kxl) vector of parameters, and u is a (nxl) vector of random disturbances. The traditional assumptions of the general linear model are maintained, i.e. x is a matrix of fixed elements with rank $k < n$, $E\,\underset{\sim}{u} = [0]$, and $E\,\underset{\sim\sim}{uu'} = \sigma^2\, I$.

Now assume that prior information on linear combinations of the elements of β is specified as

(2) $\quad \underset{\sim}{r} = R\,\beta + \underset{\sim}{V}$

where r is a (jxl) vector of prior estimates of j linear combinations of the elements of β, R is a (jxk) matrix of fixed numbers with rank $j \leq k$ representing the coefficients of the j linear combinations, and V is a (jxl) vector of random errors in the prior information. Theil and Goldberger assume that $E\,\underset{\sim}{u}\,\underset{\sim}{V}' = \left[0\right]$, $E\,\underset{\sim}{V} = \left[0\right]$, and $E\underset{\sim}{V}\underset{\sim}{V}' = \Psi$, where Ψ is known.

Under the above assumptions, a straightforward application of generalized least squares to (1) and (2) results in the estimator

(3) $\quad \hat{\underset{\sim}{\beta}} = (\sigma^{-2}x'x + R'\,\Psi^{-1}\,R)^{-1}\,(\sigma^{-2}x'\underset{\sim}{y} + R'\,\Psi^{-1}\,\underset{\sim}{r}).$

Expounding the properties of (3) under various variations on disturbance assumptions has occupied most of the literature since 1961 (see Conway and Mittelhammer). In particular if $E\,\underset{\sim}{V} \neq \left[0\right]$, $\hat{\underset{\sim}{\beta}}$ is a biased estimator of β.

Stochastic Constraint Generation

In keeping within the repeated sampling context of classical statistical analysis, how is the outcome r generated from sample to sample? From what population does the sample come? In the previous example of Nagar and Kakwani, and of Theil and Goldberger, what stochastic mechanism generates r = 1/2, and how would the "next" outcome r be generated? We address these questions, and others, in the following conceptualization.

We begin by emulating the Bayesians in specifying our prior information in a form where probability densities represent uncertainty in the prior information. We remain decidedly classical, however, in that we specify not a prior density for the unknown $R\beta$ and thereby make $R\beta$ conceptually random, but rather specify a sampling density for $\underset{\sim}{r}$ where $\underset{\sim}{r}$ is interpreted as an introspective estimator, i.e. a prior random guesstimator, for the value of the fixed unknown $R\beta$. The density of the estimator $\underset{\sim}{r}$, say, $f(r)$, has a straightforward interpretation as representing the relative frequencies with which the various potential guesses about the true value of $R\beta$ would be issued via random sampling from $f(r)$, the relative frequencies being in accordance with how likely we perceive one potential value of $R\beta$ to be relative to another. For example if β_1 were representing an income elasticity in a demand equation, and the researcher's introspection suggested that the value of β_1 was certainly in the inelastic range $[0, 1]$, but he or she was totally ignorant of which particular values were more likely than others, the density of $\underset{\sim}{r}$ could be specified as uniform,

$$(4) \quad \underset{\sim}{r} \sim 1\, I_{[0,\, 1]}(r),$$

where $I_{\{S\}}(r)$ is the indicator function taking the value 1 when $r \varepsilon S$, and 0 when $r \notin S$. The uniform density specification, implying $\text{Prob}\left[\underset{\sim}{r} \varepsilon [0,1]\right] = 1$, represents the subjective belief that the income elasticity must be in the $[0,1]$ range and that the researcher considers the particular potential guess values for $R\beta$ contained in the $[0,1]$ range to be equally likely. Reflecting this equally likely attitude, random sampling from $f(\underset{\sim}{r})$ would generate guesstimates that would assume the particular values in $[0,1]$ with equal frequency. We assume that $\underset{\sim}{r}$ is independent of $\underset{\sim}{y}$, i.e. the prior information is generated independently from the sample information.

Having specified the density of the guesstimator $\underset{\sim}{r}$, the density of the random errors $\underset{\sim}{v} = \underset{\sim}{r} - R\beta$ is, of course, derivable from the density of $\underset{\sim}{r}$. In our uniform density example above $\underset{\sim}{V} \sim 1\ I_{[-R\beta,\ 1-R\beta]}(V)$. It is recognized that the guesstimator $\underset{\sim}{r}$ is an unbiased estimator of $R\beta$ iff $E\ \underset{\sim}{V} = [0]$.

Given the development above, the outcome r used in the TGME would then be viewed as a random drawing from the conceptual sampling distribution of potential guesses for $R\beta$, i.e. a drawing from f(r). In practice, the researcher could utilize the computer in generating an outcome from the specified f(r). The outcome, r, would be used together with the outcome, y, generated by (1) to calculate the TGME (3) (in the usual case where σ^2 is unknown, $S^2 = (y - xb)'\ (y - xb)/(n-k)$, where b is the least square estimator, will be used in place of σ^2, resulting in an approximation to (3) that improves as the sample size increases) the value of Ψ in (3) representing the covariance matrix of the vector $\underset{\sim}{r}$ implied by f(r). In repeated sampling, a newly observed y and a newly generated (drawn from f(r)) r are used to calculate a new outcome of the mixed estimator.

Given the above conceptualization of stochastic constraint generation and of repeated sampling, all of the results in the literature concerning properties of the TGME hold <u>a fortiori</u>. In addition under this conceptualization, contrary to Swamy and Mehta's claim, the users of the TGME could ascribe meanings to numerical probabilities in analyzing $\hat{\beta}$ in the same way as for any classical estimator. However, the TGME uses the prior information inefficiently, and is dominated in terms of mean square error (MSE) by the PIME estimator, to be examined next.

The PIME Estimator and MSE Comparisons

The fact that the researcher knows the mean and variance of $\underset{\sim}{r}$ (since he or she specified the density of the guesstimator) makes an application of the well-known Rao-Blackwell theorem (e.g. see Hogg and Craig, p. 349) possible. Specifically, the PIME is defined as the expectation of $\hat{\beta}$ with respect to the density of r, i.e.

$$(5) \qquad \hat{\beta}^* = E_r \hat{\beta} = (\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1} (\sigma^{-2}x'y + R'\Psi^{-1} E\, r),$$

the term "prior integrated mixed estimator" referring to the integration with respect to the density of the prior estimator inherent in the expectation operation.

The PIME estimator is preferred to the TGME on the basis of mean square error. To see this, first note that

$$(6) \qquad E\,\hat{\beta}^* = E\,\hat{\beta} = \beta + (\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1}R'\Psi^{-1}\delta$$

where $\delta = EV$, so that both the PIME and TGME exhibit identical bias, if any, in estimating β. However, the covariance matrix of $\hat{\beta}^*$ is

$$(7) \qquad \text{COV}\,\hat{\beta}^* = (\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1}\,\sigma^{-2}x'x\,(\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1},$$

while the covariance matrix of $\hat{\beta}$ is

$$(8) \qquad \text{COV}\,\hat{\beta} = (\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1},$$

and thus

$$(9) \qquad \text{COV}\,\hat{\beta} - \text{COV}\,\hat{\beta}^* = (\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1}\,(R'\Psi^{-1}R)\,(\sigma^{-2}x'x + R'\Psi^{-1}R)^{-1}$$

which must be positive semidefinite (psd) since $R'\Psi^{-1}R$ is clearly psd. Thus $\hat{\beta}^*$ dominates $\hat{\beta}$ with respect to the strong mean square error (SMSE) criterion (see Wallace).

An important question in applications is whether the PIME is superior to the ordinary least square estimator, b, with respect to mean square error. The superiority of $\hat{\beta}^*$ relative to b depends on the expected errors in the prior stochastic constraints. The mean square error (MSE) matrix of $\hat{\beta}^*$ equals

$$(10)\quad \text{MSE } \hat{\underset{\sim}{\beta}}{}^{*} = \text{COV } \hat{\underset{\sim}{\beta}}{}^{*} + \Delta^{-1}R'\Psi^{-1}\delta\delta'\Psi^{-1}R\Delta^{-1}$$

where $\Delta = \sigma^{-2}x'x + R'\Psi^{-1}R$. Under the assumptions of the linear model

$$(11)\quad \text{MSE } \underset{\sim}{b} = \sigma^2(x'x)^{-1},$$

and thus the difference in MSE matrices equals

$$(12)\quad \text{MSE } \underset{\sim}{b} - \text{MSE } \hat{\underset{\sim}{\beta}}{}^{*} = \Delta^{-1}\left[\Delta\sigma^2(x'x)^{-1}\Delta - \sigma^{-2}x'x - R'\Psi^{-1}\delta\delta'\Psi^{-1}R\right]\Delta^{-1}$$

which is psd, and hence $\hat{\underset{\sim}{\beta}}{}^{*}$ is strong mean square error superior to $\underset{\sim}{b}$, iff the matrix in brackets is psd. After some considerable algebraic manipulation, the matrix in brackets in (12) can be rewritten as

$$(13)\quad \left[\cdot\right] = R'\Psi^{-1}\left[2\Psi + \sigma^2 R(x'x)^{-1}R' - \delta\delta'\right]\Psi^{-1}R$$

which will in turn be psd iff the matrix in brackets in (13) is psd. Using results in Rao (p. 60) it can be shown that positive semidefiniteness will be attained iff

$$(14)\quad \delta'\left[\sigma^2R(x'x)^{-1}R' + 2\Psi\right]^{-1}\delta \le 1$$

which is a definition of the set of bias vectors, δ, for which the PIME will be SMSE superior to the OLS estimator.

In applications, since the bias, δ, is unknown, the researcher may desire to test the hypothesis that $\hat{\underset{\sim}{\beta}}{}^{*}$ is SMSE superior to $\underset{\sim}{b}$ (such tests have been developed for the TGME; see Conway and Mittelhammer). The following approach based on confidence intervals is suggested in the case where $\underset{\sim}{u}$, and hence $\underset{\sim}{y}$, is assumed multivariate normally distributed. Begin by establishing a $(1-\alpha)\%$ lower bounded confidence interval estimate for σ^2 in the usual way using the pivotal quantity $(n-k)\,\underset{\sim}{s}^2/\sigma^2$, giving the estimate $\sigma^2 \ge \sigma_o^2 = (n-k)s^2/\chi^2_\alpha$, where χ^2_α is an α level upper tail critical point of the central χ^2 density with n-k degrees of freedom.

Next, use the random variable $\underset{\sim}{\xi} = E\underset{\sim}{r} - R\underset{\sim}{b}$, which has a normal density with mean δ and covariance matrix $\sigma^2 R(x'x)^{-1}R'$, in the usual way to form a $(1-\alpha)\%$ confidence set for the bias vector as

$$(15)\quad \Omega = \{\delta : (\xi-\delta)' \, (S^2 R(x'x)^{-1}R')^{-1} \, (\xi-\delta) \leq j\, F_\alpha\}$$

where j is the row rank of R and F_α is the α-level critical level of the central F density with j and n-k degrees of freedom. Then using Bonferroni's inequality, we can establish a hypothesis test of SMSE of $\hat{\underset{\sim}{\beta}}^*$ (i.e. a test of the condition (14)) having level of significance $\leq 2\alpha$ in two steps:

1. Solve the maximization problem

$$\max_{\delta} T(\delta) = \max_{\delta} \delta' \left[\sigma_o^2 R(x'x)^{-1}R' + 2\Psi\right]^{-1} \delta \quad \text{s.t. } \delta\varepsilon\Omega$$

2. if max $T(\delta) > 1$, reject SMSE superiority of $\hat{\underset{\sim}{\beta}}^*$ over $\underset{\sim}{b}$, otherwise accept SMSE superiority of $\hat{\underset{\sim}{\beta}}^*$ over $\underset{\sim}{b}$ with confidence $\geq 1-2\alpha$.

Note that in step 1. $T(\delta)$ is maximized for any δ by making σ^2 as small as possible, and thus the lower bound σ_o^2 is used for the σ^2 value. The problem in 1. is a nonlinear programming problem that, when δ has few entries, might be solvable by graphical techniques, and in any case can be solved by commercial algorithms such as MINOS available from Stanford University.

An Illustrative Application

As a simple illustration of the PIME, we use an example utilized by Zellner, in which he estimates the short run consumption function

$$\underset{\sim}{C} = \beta_0 + \beta_1 Y + \underset{\sim}{u}$$

where C = quarterly deflated U. S. personal consumption expenditures and Y = quarterly deflated U. S. personal disposable income. Using quarterly data from 1947 through the first quarter of 1955, except for two quarters deleted by Zellner as outliers, Zellner obtained the least square estimate

$$C = \underset{(6.64)}{38.09} + \underset{(.033)}{.747}\ Y$$

where the estimated standard errors of the parameter estimates are in parentheses.

Suppose now that the researcher feels certain that the marginal propensity to consume is in the range $[.6,\ 1]$. Suppose further that all points within the interval are considered to be equally likely, and thus all of the guesses for dC/dY contained in $[.6,\ 1]$ should be generated with equal frequency via random sampling from f(r). Then the sampling density for the guesstimator would be specified as $\underset{\sim}{r} \sim 2.5\ I_{[.6,\ 1]}(r)$, a uniform density with mean $E\underset{\sim}{r} = .8$ and variance $\underset{\sim}{r} = \Psi = \frac{1}{75}$. The PIME estimate would then be calculated as

$$(16)\quad \hat{\beta}* = (S^{-2}x'x + R'\ (\tfrac{1}{75})^{-1}R)^{-1}\ (S^{-2}x'y + R'\ (\tfrac{1}{75})^{-1}\ (.8)),$$

where $R = [0\ 1]$ in this case, and using Zellners data for X, Y, and S^2 yields $\hat{\beta}*' = [37.29\quad .751]$ giving the estimate for the consumption function

$$(17)\quad C = \underset{(6.14)}{37.29} + \underset{(.031)}{.751}\ Y.$$

Incorporation of prior information in this case raised slightly the estimate of the marginal propensity to consume, and lowered the standard deviation of the estimate.

In order to test whether $\underset{\sim}{\hat{\beta}*}$ offers a MSE improvement over $\underset{\sim}{b}$, we first establish a 95% lower bound confidence interval for σ^2 as

$$(18)\quad \sigma^2 \geq \frac{(n-k)S^2}{\chi^2_{.05}} = 6.605$$

where in Zellner's problem $(n-k) = 29$, $S^2 = 9.703$, and $\chi^2_{.05} = 42.6$

Since $\xi = E\underset{\sim}{r} - Rb = .053$, the 95% confidence set for the bias vector is defined as

(19) $\Omega = \{\delta : (.053 - \delta)' (.0011)^{-1} (.053 - \delta) \leq 4.19\}$

$= \{\delta : -.0148 \leq \delta \leq .1208\}$

where $F_{.05} = 4.19$.

Then in step 1 of the testing procedure we solve

(20) $\max_{\delta} T(\delta) = \max_{\delta} \delta' \left[.0274\right]^{-1} \delta$ s.t. $\left[-.0148 \leq \delta \leq .1208\right]$

which yields max $T(\delta) = .53$

Step 2 of the testing procedure results in the conclusion that SMSE superiority of $\hat{\underset{\sim}{\beta}}^*$ over $\underset{\sim}{b}$ can be accepted at level of significance $\leq .10$ since max $T(\delta) = .53 \leq 1$.

References

Conway, R. K. and R. C. Mittelhammer. "The Theory of Mixed Estimation in Econometric Modeling", Dept. of Agricultural Economics Staff Paper 82-7, Washington State University, Pullman, WA, 1982.

Conway, R. and J. Yanagida. "Estimating U. S. Potato Demand Using Prior Information." ESS Staff Report, U.S.D.A., May, 1981.

Dhrymes, P. Introductory Econometrics, Springer-Verlag, New York, 1978.

Hammig, M. D. "Regional Acreage Response by Quarter for Fresh Tomatoes: An Example of the Use of Mixed Estimation." Southern Journal of Agricultural Economics, 11(1979).

Hammig, M. D. and R. C. Mittelhammer. "An Evaluation of Import Tariffs in the Canada-U.S. Fresh Tomato Market", Canadian Journal of Agricultural Economics, 30(1982):133-152.

Hammig, M. D. and R. C. Mittelhammer. "An Imperfectly Competitive Market Model of the U. S. Lettuce Industry", Western Journal of Agricultural Economics, 5(1980):1-12.

Hogg, R. V. and A. T. Craig. Introduction to Mathematical Statistics, 4th edition, MacMillan, New York, 1978.

Holloway, M. "A Production Function Analysis of Water Resource Productivity in Pacific Northwest Agriculture", Ph.D. dissertation, Oregon State University, 1972.

Johnston, J. Econometric Methods, McGraw Hill, New York, 1972.

Judge, G., W. Griffiths, R. C. Hill, T. C. Lee. The Theory and Practice of Econometrics, John Wiley and Sons, New York, 1980.

Judge, G., R. C. Hill, W. Griffiths, H. Lutkepohl, T. C. Lee. Introduction to the Theory and Practice of Econometrics, John Wiley and Sons, New York, 1982.

Kmenta, J. Elements of Econometrics, MacMillan, New York, 1971.

Maddala, G. S. Econometrics, McGraw-Hill, New York, 1977.

Mittelhammer, R. C. "The Estimation of Domestic Demand for Salad Vegetables Using A Priori Information", Ph.D. dissertation, Washington State University, 1978.

Mittelhammer, R. C. and D. W. Price. "Estimating the Effects of Volume, Price, and Costs on Marketing Margins of Selected Fresh Vegetables Through Mixed Estimation", Agricultural Economic Research, 30(1978):1-7.

Mittelhammer, R., D. Young, D. Tasanasanta, J. Donnelly. "Mitigating the Effects of Multicollinearity Using Exact and Stochastic Prior Information: The Case of an Aggregate Production Function in Thailand", American Journal of Agricultural Economics, 62(1980):199-210.

Nagar, A. L. and N. C. Kakwami. "The Bias and Moment Matrix of a Mixed Regression Estimator", Econometrica, 32(1964):174-182.

Paulus, J. D. "Mixed Estimation of a Complete System of Consumer Demand Equations." Annals of Economic and Social Measurement, 4(1975):117-131.

Price, D. and R. C. Mittelhammer. "A Matrix of Demand Elasticities for Fresh Fruit." Western Journal of Agricultural Economics 4(1979):69-86.

Rao, C. R. Linear Statistical Inference and Its Application, John Wiley and Sons, New York, 1973.

Swamy, P. A. V. B. and J. S. Mehta. "Ridge Regression Estimation of the Rotterdam Model", Special Studies Paper Number 136, Board of Governors, Federal Reserve Board. Washington, D.C., 1979.

Swamy, P. A. V. B. and J. S. Mehta. "Ridge Regression Estimation of the Rotterdam Model", forthcoming in Journal of Econometrics.

Theil, H. Introduction to Econometrics, Prentice Hall, Englewood Cliffs, N.J., 1978.

Theil, H. Principles of Econometrics, John Wiley and Sons, New York, 1971.

Theil, H. Theory and Measurement of Consumer Demand, Vol. 1 and 2, North Holland, Amsterdam, 1975.

Theil, H. and A. Goldberger. "On Pure and Mixed Statistical Estimation in Economics." International Economic Review 12(1961):67-78.

Vinod, H. D. and A. Ullah. Recent Advances in Regression Methods, Marcel Dekker, New York, 1981.

Wallace, T. D. "Pretest Estimation in Regression: A Survey", American Journal of Agricultural Economics, 59(1977):431-443.

Zellner, A. "The Short Run Consumption Function", Econometrica, 25(1957): 552-567.

THE CLASSICAL REAL WORLD AND INTROSPECTION

Classical "Real World" Processes

$$\underset{\sim}{Y} = x\beta + \underset{\sim}{\mu}$$

$$\underset{\sim}{r_*} = R\beta + \underset{\sim}{V}$$

where β is a symbol standing for a <u>fixed</u>, <u>constant</u> kxl vector of numbers unknown to the researcher

$$\underset{\sim}{Y} \sim N\ (x\beta,\ \sigma^2 I)$$

$$\underset{\sim}{r_*} \sim f(r_*)$$

MODEL SPACE: $\underset{\sim}{Y} = X\omega + \underset{\sim}{\mu};\ \omega \in \Omega,\ \underset{\sim}{Y} \sim N\ (X\omega, \sigma^2 I)$

MAINTAINED HYPOTHESIS: $\exists\, \omega_* \in \Omega$ s.t. observed $\underset{\sim}{Y} \sim N\ (X\omega_*, \sigma^2 I)$

PRIOR KNOWLEDGE SPACE: $\Lambda = \{r:\ r = R\omega,\ \omega \in \Omega\}$

DEDUCTION: $\exists\, r_* \in \Lambda$ s.t. $R\omega_* = r_*$

ASSERTIONS: $r_* \in A_i$ with degree of belief $=$ $P(r_* \in A_i)$,
$\forall$ measureable $A_i \in \Lambda \rightarrow r_* \sim f(r_*)$

IMPLICATIONS: $R\omega_* \in A_i$ with degree of belief $P(r_* \in A_i)$